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# LIMIT THEORY FOR GEOMETRIC STATISTICS OF POINT PROCESSES HAVING FAST DECAY OF CORRELATIONS

BY B. BŁASZCZYSZYN, D. YOGESHWARAN \* AND J. E. YUKICH †

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Let  $\mathcal{P}$  be a simple, stationary point process on  $\mathbb{R}^d$  having fast decay of correlations, i.e., its correlation functions factorize up to an additive error decaying faster than any power of the separation distance. Let  $\mathcal{P}_n := \mathcal{P} \cap W_n$  be its restriction to windows  $W_n := [-\frac{1}{2}n^{1/d}, \frac{1}{2}n^{1/d}]^d \subset \mathbb{R}^d$ . We consider the statistic  $H_n^\xi := \sum_{x \in \mathcal{P}_n} \xi(x, \mathcal{P}_n)$  where  $\xi(x, \mathcal{P}_n)$  denotes a score function representing the interaction of  $x$  with respect to  $\mathcal{P}_n$ . When  $\xi$  depends on local data in the sense that its radius of stabilization has an exponential tail, we establish expectation asymptotics, variance asymptotics, and central limit theorems for  $H_n^\xi$  and, more generally, for statistics of the re-scaled, possibly signed,  $\xi$ -weighted point measures  $\mu_n^\xi := \sum_{x \in \mathcal{P}_n} \xi(x, \mathcal{P}_n) \delta_{n^{-1/d}x}$ , as  $W_n \uparrow \mathbb{R}^d$ . This gives the limit theory for non-linear geometric statistics (such as clique counts, the number of Morse critical points, intrinsic volumes of the Boolean model, and total edge length of the  $k$ -nearest neighbors graph) of  $\alpha$ -determinantal point processes (for  $-1/\alpha \in \mathbb{N}$ ) having fast decreasing kernels, including the  $\beta$ -Ginibre ensembles, extending the Gaussian fluctuation results of Soshnikov [72] to non-linear statistics. It also gives the limit theory for geometric U-statistics of  $\alpha$ -permanental point processes (for  $1/\alpha \in \mathbb{N}$ ) as well as the zero set of Gaussian entire functions, extending the central limit theorems of Nazarov and Sodin [53] and Shirai and Takahashi [71], which are also confined to linear statistics. The proof of the central limit theorem relies on a factorial moment expansion originating in [12, 13] to show the fast decay of the correlations of  $\xi$ -weighted point measures. The latter property is shown to imply a condition equivalent to Brillinger mixing and consequently yields the asymptotic normality of  $\mu_n^\xi$  via an extension of the cumulant method.

## CONTENTS

<b>1</b>	<b>Introduction and main results</b>	<b>2</b>
1.1	Admissible point processes having fast decay of correlations	5
1.2	Admissible score functions	7
1.3	Fast decay of correlations of the $\xi$ -weighted measures	9
1.4	Main results	10
<b>2</b>	<b>Examples and applications</b>	<b>15</b>
2.1	Moments of point processes having fast decay of correlations	15
2.2	Examples of point processes having fast decay of correlations	16
	2.2.1 Class A1 input	16
	2.2.2 Class A2 input	18

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It is natural to ask whether the limit theory of these papers extends to more general input satisfying a notion of ‘asymptotic independence’ for point processes. Recall that if  $\xi \equiv 1$  and if  $\mathcal{P}$  is an  $\alpha$ -determinantal point process with  $\alpha = -1/m$  or an  $\alpha$ -permanental point process with  $\alpha = 2/m$  for some  $m$  in the set of positive integers  $\mathbb{N}$  (respectively  $\mathcal{P}$  is the zero set of a Gaussian entire function), then remarkable results of Soshnikov [72], Shirai and Takahashi [71] (respectively Nazarov and Sodin [53]), show that the counting statistic  $\mathcal{P}_n(W_n) := \sum_{x \in \mathcal{P}_n} \mathbf{1}[x \in W_n]$  is asymptotically normal. One may ask whether asymptotic normality of  $H_n^\xi$  still holds when  $\xi$  is either a local  $U$ -statistic or an exponentially stabilizing score function. We answer these questions affirmatively. Loosely speaking, subject to a mild growth condition on  $\text{Var} H_n^\xi$ , our approach shows that  $H_n^\xi$  is asymptotically normal whenever  $\mathcal{P}$  is a point process having fast decay of correlations.

Heuristically, when the score functions depend on ‘local data’ and when the input is ‘asymptotically independent’, one might expect that the statistics  $H_n^\xi$  obey a strong law and a central limit theorem. The notion of dependency on ‘local data’ for score functions is formalized via stabilization in [7, 23, 59, 61, 64]. Here we formalize the idea of asymptotically independent input  $\mathcal{P}$  via the notion of ‘fast decay of correlation functions’. We thereby extend the limit theory of the afore-mentioned papers to input having fast decay of correlation functions. A point process  $\mathcal{P}$  on  $\mathbb{R}^d$  has fast decay of correlations if for all  $p, q \in \mathbb{N}$  and all  $x_1, \dots, x_{p+q} \in \mathbb{R}^d$ , its correlation functions  $\rho^{(p+q)}(x_1, \dots, x_{p+q})$  factorize into  $\rho^{(p)}(x_1, \dots, x_p)\rho^{(q)}(x_{p+1}, \dots, x_{p+q})$  up to an additive error decaying faster than any power of the separation distance

$$(1.3) \quad s := d(\{x_1, \dots, x_p\}, \{x_{p+1}, \dots, x_{p+q}\}) := \inf_{i \in \{1, \dots, p\}, j \in \{p+1, \dots, p+q\}} |x_i - x_j|$$

as at (1.10) below. Roughly speaking, such point processes exhibit asymptotic independence at large distances. Examples of such point processes are given in Section 2.2. Point processes with fast decay of correlations are called ‘clustering point processes’ in statistical physics [43, 45, 53]. We shall avoid this terminology since, at least from the point of view of spatial statistics, it suggests that the points of  $\mathcal{P}$  clump or aggregate together, which need not be the case.

If either  $\mathcal{P}$  has fast decay of correlations and  $\xi$  is a local  $U$ -statistic or if  $\mathcal{P}$  has exponentially fast decay of correlations and  $\xi$  is an exponentially stabilizing score function, then with  $\delta_x$  denoting the point mass at  $x$ , our main results establish expectation and variance asymptotics as  $n \rightarrow \infty$ , as well as central limit theorems for the re-scaled, possibly signed,  $\xi$ -weighted point measures

$$(1.4) \quad \mu_n^\xi := \sum_{x \in \mathcal{P}_n} \xi(x, \mathcal{P}_n) \delta_{n^{-1/d}x},$$

thereby also establishing the limit theory for the total mass of  $\mu_n^\xi$  given by the non-linear statistics  $H_n^\xi$ .

As shown in Theorems 1.12-1.15 this yields the limit theory for general non-linear statistics of  $\alpha$ -determinantal and  $\alpha$ -permanental point processes, the point process given by the zero set of a Gaussian entire function, as well as rarified Gibbsian input.

The benefit of the general approach taken here is four-fold: (i) we establish the asymptotic normality of the random measures  $\mu_n^\xi$ , with  $\mathcal{P}$  either an  $\alpha$ -permanental point process (with  $1/\alpha \in \mathbb{N}$ ), an  $\alpha$ -determinantal point process (with  $-1/\alpha \in \mathbb{N}$ ), or the zero set of a Gaussian entire function, thereby extending the work of Soshnikov [72], Shirai and Takahashi [71], and Nazarov and Sodin [53], who

restrict to linear statistics, (ii) we extend the limit theory of [7, 42, 59, 60, 61, 64], which is confined to Poisson and binomial input, to point processes having fast decay of correlations, (iii) we apply our general results to deduce asymptotic normality and variance asymptotics for geometric statistics of input having fast decay of correlations, including statistics of simplicial complexes and germ-grain models, clique counts, Morse critical points, as well of statistics of random graphs, (iv) our general proof of the asymptotic normality of  $\mu_n^\xi$  relates the fast decay of correlations of the input process  $\mathcal{P}$  to a similar fast correlation decay for the family of  $\xi$ -weighted (point) measures

$$(1.5) \quad \sum_{x \in \mathcal{P}_n} \xi(x, \mathcal{P}_n) \delta_x,$$

consequently implying Brillinger mixing of these measures, and thus directly relating the two concepts: Fast decay of correlations implies Brillinger mixing.

Given input  $\mathcal{P}$  having fast decay of correlations, an interesting feature of the measures  $\mu_n^\xi$  is that their variances are at most of order  $\text{Vol}_d(W_n)$ , the volume of the window  $W_n$  (Theorem 1.12). This holds also for the statistic  $\hat{H}_n^\xi := \sum_{x \in \mathcal{P}_n} \xi(x, \mathcal{P})$ , which involves summands having no boundary effects. An interesting feature of this statistic is that if its variance is  $o(\text{Vol}_d(W_n))$  then it has to be  $O(\text{Vol}_{d-1}(\partial W_n))$ , where  $\partial W_n$  denotes the boundary of  $W_n$  and  $\text{Vol}_{d-1}(\cdot)$  stands for the  $(d-1)$ th intrinsic volume (Theorem 1.15). In other words, if the fluctuations of  $\hat{H}_n^\xi$  are not of volume order, then they are at most of surface order.

Coming back to our set-up, when a functional  $H_n^\xi(\mathcal{P})$  is expressible as a sum of local  $U$ -statistics or, more generally, as a sum of exponentially stabilizing score functions  $\xi$ , then a key step towards proving the central limit theorem is to show that the correlation functions of the  $\xi$ -weighted measures defined via Palm expectations  $\mathbb{E}_{x_1, \dots, x_k}$  (cf Section 1.1) and given by

$$(1.6) \quad m^{(k_1, \dots, k_{p+q})}(x_1, \dots, x_{p+q}; n) := \mathbb{E}_{x_1, \dots, x_{p+q}} (\xi(x_1, \mathcal{P}_n)^{k_1} \dots \xi(x_{p+q}, \mathcal{P}_n)^{k_{p+q}}) \rho^{(p+q)}(x_1, \dots, x_{p+q}),$$

similar to those of the input process  $\mathcal{P}$ , approximately factorize into

$$m^{(k_1, \dots, k_p)}(x_1, \dots, x_p; n) m^{(k_{p+1}, \dots, k_{p+q})}(x_{p+1}, \dots, x_{p+q}; n),$$

uniformly in  $n \leq \infty$ , up to an additive error decaying faster than any power of the separation distance  $s$ , defined at (1.3). Here  $x_1, \dots, x_{p+q}$  are distinct points in  $W_n$  and  $k_1, \dots, k_{p+q} \in \mathbb{N}$ . This result, spelled out in Theorem 1.11, is at the heart of our approach. We then give two proofs of the central limit theorem (Theorem 1.13) for the purely atomic random measures (1.4) via the cumulant method and as a corollary derive the asymptotic normality of  $H_n^\xi(\mathcal{P})$  and  $\int f d\mu_n^\xi$ ,  $f$  a test function, as  $n \rightarrow \infty$ . The proof of expectation and variance asymptotics (Theorem 1.12) mainly relies upon the refined Campbell theorem.

In contrast to the afore-mentioned works, our proof of the fast decay of correlations of the  $\xi$ -weighted measures depends heavily on a factorial moment expansion for expected values of functionals of a general point process  $\mathcal{P}$ . This expansion, which originates in [12, 13], is expressed in terms of iterated difference operators of the considered functional on the null configuration of points and integrated against factorial moment measures of the point process. It is valid for general point processes, in contrast to the Fock space representation of Poisson functionals, which involves the

same difference operators but is deeply related to chaos expansions [40]. Further connections with the literature are discussed in the remarks following Theorems 1.14 and 1.15.

Our interest in these issues was stimulated by similarities in the methods of [43], [6, 7, 69] and [53]. The articles [7, 69] prove central limit theorems for stabilizing functionals of Poisson and rarified Gibbsian point processes, respectively, while [53] proves central limit theorems for linear statistics  $\sum_{x \in \mathcal{P}_n} \xi(x)$  of point processes having fast decay of correlations. These papers all establish the fast decay of correlations of the  $\xi$ -weighted measures as at (1.21) below, and then use the resulting volume order cumulant bounds to show asymptotic normality. This paper unifies and extends the results of [6, 7, 53, 69, 71] to input having fast decay of correlations. The idea of using correlation functions to show asymptotic normality via cumulants goes back to [43]. The earlier work of [45] has stimulated our investigation of variance asymptotics.

Having described the goals and context of this paper, we now describe more precisely the assumptions on allowable score and input pairs  $(\xi, \mathcal{P})$  as well as our main results. The generality of allowable pairs  $(\xi, \mathcal{P})$  considered here necessitates several definitions which go as follows.

**1.1. Admissible point processes having fast decay of correlations.** Throughout  $\mathcal{P} \subset \mathbb{R}^d$  denotes a simple point process. By a simple point process we mean a random element taking values in  $\mathcal{N}$ , the space of locally finite simple point sets in  $\mathbb{R}^d$  (or equivalently Radon counting measures  $\mu$  such that  $\mu(\{x\}) \in \{0, 1\}$  for all  $x \in \mathbb{R}^d$ ) and equipped with the canonical  $\sigma$ -algebra  $\mathcal{B}$ . Given a simple point process  $\mathcal{P}$  we interchangeably use the following representations of  $\mathcal{P}$  :

$$\mathcal{P}(\cdot) := \sum_i \delta_{X_i}(\cdot) \text{ (random measure); } \mathcal{P} := \{X_i\}_{i \geq 1} \text{ (random set),}$$

where  $X_i, i \geq 1$ , are  $\mathbb{R}^d$ -valued random variables (given a measurable numbering of points, which is irrelevant for the results presented in this paper). Points of  $\mathbb{R}^d$  are denoted by  $x$  or  $y$  whereas points of  $(\mathbb{R}^d)^{k-1}$  are denoted by  $\mathbf{x}$  or  $\mathbf{y}$ . We let  $\mathbf{0}$  denote a point at the origin of  $\mathbb{R}^d$ .

For a bounded function  $f$  on  $\mathbb{R}^d$  and a simple counting measure  $\mu$ , let  $\mu(f) := \langle f, \mu \rangle$  denote the integral of  $f$  with respect to  $\mu$ . For a bounded set  $B \subset \mathbb{R}^d$  we let  $\mu(B) = \mu(\mathbf{1}_B) = \text{card}(\mu \cap B)$ , with  $\mu$  in the last expression interpreted as the set of its atoms.

For a simple Radon counting measure  $\mu$  and  $k \in \mathbb{N}$ , its  $k$ th factorial power is

$$\mu^{(k)} := \begin{cases} \sum_{\text{distinct } x_1, \dots, x_k \in \mu} \delta_{(x_1, \dots, x_k)} & \text{when } \mu(\mathbb{R}^d) \geq k, \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $\mu^{(k)}$  is a Radon counting measure on  $(\mathbb{R}^d)^k$ . Consistently, for a set  $\mathcal{X} \subset \mathbb{R}^d$ , we denote  $\mathcal{X}^{(k)} := \{(x_1, \dots, x_k) \in (\mathbb{R}^d)^k : x_i \in \mathcal{X}, x_i \neq x_j \text{ for } i \neq j\}$ . The  $k$ th order factorial moment measure of the (simple) point process  $\mathcal{P}$  is defined as  $\alpha^{(k)}(\cdot) := \mathbb{E}(\mathcal{P}^{(k)}(\cdot))$  on  $(\mathbb{R}^d)^k$  i.e.,  $\alpha^{(k)}(\cdot)$  is the intensity measure of the point process  $\mathcal{P}^{(k)}(\cdot)$ . Its Radon-Nikodyn density  $\rho^{(k)}(x_1, \dots, x_k)$  (provided it exists) is the  $k$ -point correlation function (or  $k$ th joint intensity) and is characterized by the relation

$$(1.7) \quad \alpha^{(k)}(B_1 \times \dots \times B_k) = \mathbb{E} \left( \prod_{1 \leq i \leq k} \mathcal{P}(B_i) \right) = \int_{B_1 \times \dots \times B_k} \rho^{(k)}(x_1, \dots, x_k) dx_1 \dots dx_k,$$





$\omega$ -mixing as in Definition 1.1 with  $\omega(n, x) = C_n \phi(c_n x)$  for some correlation decay constants  $c_n > 0$  and  $C_n < \infty$  and a fast decreasing function  $\phi : \mathbb{R}^+ \rightarrow [0, 1]$ , called a correlation decay function.

More explicitly, an admissible point process has *fast decay of correlations*, if for all  $p, q \in \mathbb{N}$  and all  $(x_1, \dots, x_{p+q}) \in (\mathbb{R}^d)^{p+q}$

$$(1.10) \quad |\rho^{(p+q)}(x_1, \dots, x_{p+q}) - \rho^{(p)}(x_1, \dots, x_p) \rho^{(q)}(x_{p+1}, \dots, x_{p+q})| \leq C_{p+q} \phi(c_{p+q} s),$$

where  $s := d(\{x_1, \dots, x_p\}, \{x_{p+1}, \dots, x_{p+q}\})$  is as at (1.3) and  $C_k, c_k, \phi$  are as in Definition 1.2. Without loss of generality, we assume that  $c_k$  is non-increasing in  $k$ , and that  $C_k \in [1, \infty)$  is non-decreasing in  $k$ . As a by-product of our proof of the asymptotic normality of  $\mu_n^\xi$  in (1.4), we establish that the fast decay of correlations of  $\mathcal{P}$  implies that it is Brillinger mixing; cf Remark (vi) in Section 1.4 and Remarks at the end of Section 4.4.2.

Admissible point processes having fast decay of correlations are ubiquitous and include certain determinantal, permanental, and Gibbs point processes, as explained in Section 2.2. The  $k$ -point correlation functions of an admissible point process having fast decay of correlations are bounded i.e.,

$$(1.11) \quad \sup_{(x_1, \dots, x_k) \in (\mathbb{R}^d)^k} \rho^{(k)}(x_1, \dots, x_k) \leq \kappa_k < \infty,$$

for some constants  $\kappa_k$ , which without loss of generality are assumed non-decreasing in  $k$ . Also without loss of generality, assume  $\kappa_0 := \max\{\rho^{(1)}(\mathbf{0}), 1\}$ . For stationary  $\mathcal{P}$  with intensity  $\rho^{(1)}(\mathbf{0}) \in (0, \infty)$  we have that (1.10) implies (1.11) with

$$(1.12) \quad \kappa_k \leq (\rho^{(1)}(\mathbf{0}))^k + \sum_{i=2}^k C_i (\rho^{(1)}(\mathbf{0}))^{k-i} \leq k C_k \kappa_0^k.$$

The bound (1.12) helps to determine when point processes having fast decay of correlations also have exponential moments, as in Section 2.1.

**1.2. Admissible score functions.** Throughout we restrict to translation-invariant score functions  $\xi : \mathbb{R}^d \times \mathcal{N} \rightarrow \mathbb{R}$ , i.e., those which are measurable in each coordinate,  $\xi(x, \mathcal{X}) = 0$  if  $x \notin \mathcal{X} \in \mathcal{N}$ , and for all  $y \in \mathbb{R}^d$ , satisfy  $\xi(\cdot + y, \cdot + y) = \xi(\cdot, \cdot)$ .

We introduce classes (A1) and (A2) of *admissible* score and input pairs  $(\xi, \mathcal{P})$ . Specific examples of admissible input pairs of both classes are provided in Sections 2.2 and 2.3. The first class allows for admissible input  $\mathcal{P}$  as in Definition 1.2 whereas the second considers admissible input  $\mathcal{P}$  having fast decay of correlations (1.10), subject to  $c_k \equiv 1$  and growth conditions on the decay constants  $C_k$  and the decay function  $\phi$ .

**DEFINITION 1.3** (Class (A1) of admissible score and input pairs  $(\xi, \mathcal{P})$ ). *Admissible input  $\mathcal{P}$  consists of admissible point processes having fast decay of correlations as in Definition 1.2. Admissible score functions are of the form*

$$(1.13) \quad \xi(x, \mathcal{X}) := \frac{1}{k!} \sum_{\mathbf{x} \in \mathcal{X}^{(k-1)}} h(x, \mathbf{x}),$$

for some  $k \in \mathbb{N}$  and a symmetric, translation-invariant function  $h : \mathbb{R}^d \times (\mathbb{R}^d)^{k-1} \rightarrow \mathbb{R}$  such that



$$(1.17) \quad \liminf_{t \rightarrow \infty} \frac{\log \phi(t)}{t^b} \in (-\infty, 0)$$

for some constant  $b \in (0, \infty)$ . Admissible score functions  $\xi$  for this class are exponentially stabilizing on the input  $\mathcal{P}$  and satisfy a power growth condition, namely there exists  $\hat{c} \in [1, \infty)$  such that for all  $r \in (0, \infty)$

$$(1.18) \quad |\xi(x, \mathcal{X} \cap B_r(x))| \mathbf{1}[\text{card}(\mathcal{X} \cap B_r(x)) = n] \leq (\hat{c} \max(r, 1))^n.$$

The condition  $c_k \equiv 1$  is equivalent to  $c_* := \inf c_k > 0$ . This follows since we may replace the fast decreasing function  $\phi(\cdot)$  by  $\phi(c_* \times \cdot)$ , with  $c_k \equiv 1$  for this new fast decreasing function. Score functions of class (A1) also satisfy the power growth condition (1.18) since in this case the left hand side of (1.18) is at most  $\|h\|_\infty n^{(k-1)}/k$ . Thus the generalization from (A1) to (A2) consists in replacing local U-statistics by exponentially stabilizing score functions satisfying the power growth condition. This is done at the price of imposing stronger conditions on the input process, requiring in particular that it has finite exponential moments, as explained in Section 2.1.

**1.3. Fast decay of correlations of the  $\xi$ -weighted measures.** The following  $p$ -moment condition involves the score function  $\xi$  and the input  $\mathcal{P}$ . We shall describe in Section 2.1 ways to control the  $p$ -moments of input pairs of class (A1) and (A2).

**DEFINITION 1.8 (Moment condition).** Given  $p \in [1, \infty)$ , say that the pair  $(\xi, \mathcal{P})$  satisfies the  $p$ -moment condition if

$$(1.19) \quad \sup_{1 \leq n \leq \infty} \sup_{1 \leq p' \leq [p]} \sup_{x_1, \dots, x_{p'} \in W_n} \mathbb{E}_{x_1, \dots, x_{p'}} \max\{|\xi(x_1, \mathcal{P}_n)|, 1\}^p \leq \tilde{M}_p < \infty$$

for some constant  $\tilde{M}_p := \tilde{M}_p^\xi$ , where  $\sup$  signifies *ess sup* with respect to  $\alpha^{(p)}$ . Without loss of generality we assume that  $\tilde{M}_p$  is increasing in  $p$  for all  $p$  such that (1.19) holds.

We next consider the decay of the functions at (1.6), the so-called correlation functions of the  $\xi$ -weighted measures at (1.5). These functions indeed play the same role as the  $k$ -point correlation functions of the simple point process  $\mathcal{P}$ . When  $\xi \equiv 1$  they obviously reduce to the correlation functions of  $\mathcal{P}$ . For general  $\xi$  and  $k_i \equiv 1$  they are densities (‘mixed moment densities’ in the language of [7]) of the higher-order moment measures of the  $\xi$ -weighted measures with all distinct arguments. In the case of repeated arguments, the moment measures of a simple point process ‘collapse’ to appropriate lower dimensional ones. This is neither the case for non-simple point processes nor for our  $\xi$ -weighted measures, where general exponents  $k_i$  are required to properly take into account repeated arguments.

When  $k_i \equiv 1$  for all  $1 \leq i \leq p$ , we write  $m_{(p)}(x_1, \dots, x_p; n)$  instead of  $m^{(1, \dots, 1)}(x_1, \dots, x_p; n)$ . Abbreviate  $m^{(k_1, \dots, k_p)}(x_1, \dots, x_p; \infty)$  by  $m^{(k_1, \dots, k_p)}(x_1, \dots, x_p)$ . These functions exist whenever (1.19) is satisfied for  $p$  set to  $k_1 + \dots + k_p$  and provided the  $p$ -point correlation function  $\rho^{(p)}$  exists. As for the input process  $\mathcal{P}$  we consider mixing properties and fast decay of correlations for the  $\xi$ -weighted measures at (1.5).

DEFINITION 1.9 ( $\tilde{\omega}$ -mixing correlation functions of  $\xi$ -weighted measures). *The correlation functions (1.6) are said to be  $\omega$ -mixing if there exists a decreasing function  $\tilde{\omega} : \mathbb{N} \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that for all  $p \in \mathbb{N}$ ,  $\lim_{x \rightarrow \infty} \tilde{\omega}(p, x) = 0$  and for all  $p, q \in \mathbb{N}$ , distinct  $x_1, \dots, x_{p+q} \in \mathbb{R}^d$  and  $n \in \mathbb{N} \cup \{\infty\}$*

$$(1.20) \quad \left| m^{(k_1, \dots, k_{p+q})}(x_1, \dots, x_{p+q}; n) - m^{(k_1, \dots, k_p)}(x_1, \dots, x_p; n) m^{(k_{p+1}, \dots, k_{p+q})}(x_{p+1}, \dots, x_{p+q}; n) \right| \leq \tilde{\omega}(K, s),$$

where  $K := \sum_{i=1}^{p+q} k_i$  and  $s := d(\{x_1, \dots, x_p\}, \{x_{p+1}, \dots, x_{p+q}\})$  is as at (1.3).

DEFINITION 1.10 (Fast decay of correlations of the  $\xi$ -weighted measures). *The  $\xi$ -weighted measures are said to have fast decay of correlations if their correlations functions are  $\tilde{\omega}$ -mixing as in Definition 1.9 with  $\tilde{\omega}(n, x) = \tilde{C}_n \tilde{\phi}(\tilde{c}_n x)$  for some fast decreasing function  $\tilde{\phi} : \mathbb{R}^+ \rightarrow [0, 1]$  and some constants  $\tilde{c}_n > 0$  and  $\tilde{C}_n < \infty$ .*

More explicitly, the  $\xi$ -weighted measures (1.5) have fast decay of correlations if there exists a fast-decreasing function  $\tilde{\phi}$  and constants  $\tilde{C}_k < \infty$ ,  $\tilde{c}_k > 0$ ,  $k \in \mathbb{N}$  such that for all  $n \in \mathbb{N} \cup \{\infty\}$ ,  $p, q \in \mathbb{N}$  and any collection of positive integers  $k_1, \dots, k_{p+q}$ , we have

$$(1.21) \quad \left| m^{(k_1, \dots, k_{p+q})}(x_1, \dots, x_{p+q}; n) - m^{(k_1, \dots, k_p)}(x_1, \dots, x_p; n) m^{(k_{p+1}, \dots, k_{p+q})}(x_{p+1}, \dots, x_{p+q}; n) \right| \leq \tilde{C}_K \tilde{\phi}(\tilde{c}_K s),$$

where  $x_1, \dots, x_{p+q}$ ,  $K$  and  $s$  are as in Definition 1.9.

Our first theorem shows that the fast decay of correlations is inherited from the input process  $\mathcal{P}$  by the  $\xi$ -weighted measures for a wide class of score functions and input. This key result forms the starting point of our approach.

THEOREM 1.11. *Let  $(\xi, \mathcal{P})$  be an admissible score and input pair of class (A1) or (A2) such that the  $p$ -moment condition (1.19) holds for all  $p \in (1, \infty)$ . Then the correlations of the  $\xi$ -weighted measures decay fast as at (1.21).*

We prove this theorem in Section 3, where it is also shown that it subsumes more specialized results of [7, 69].

**1.4. Main results.** We give the limit theory for the measures  $\mu_n^\xi$ ,  $n \geq 1$ , defined at (1.4). Given a score function  $\xi$  on admissible input  $\mathcal{P}$  we set <sup>2</sup>

$$(1.22) \quad \sigma^2(\xi) := \mathbb{E}_0 \xi^2(\mathbf{0}, \mathcal{P}) \rho^{(1)}(\mathbf{0}) + \int_{\mathbb{R}^d} (m_{(2)}(\mathbf{0}, x) - m_{(1)}(\mathbf{0})^2) dx.$$

The following result provides expectation and variance asymptotics for  $\mu_n^\xi(f)$ , with  $f$  belonging to the space  $\mathcal{B}(W_1)$  of bounded measurable functions on  $W_1$ .

<sup>2</sup>For a stationary point process  $\mathcal{P}$ , its Palm expectation  $\mathbb{E}_0$  (and consequently  $m_{(1)}(\mathbf{0})$ ,  $m_{(2)}(\mathbf{0}, x) dx$ ) is meaningfully defined e.g. via the Palm-Matthes approach.

**THEOREM 1.12.** *Let  $\mathcal{P}$  be an admissible point process on  $\mathbb{R}^d$ .*

(i) *If  $\xi$  satisfies exponential stabilization (1.15) and the  $p$ -moment condition (1.19) for some  $p \in (1, \infty)$  then for all  $f \in \mathcal{B}(W_1)$*

$$(1.23) \quad \left| n^{-1} \mathbb{E} \mu_n^\xi(f) - \mathbb{E}_0 \xi(\mathbf{0}, \mathcal{P}) \rho^{(1)}(\mathbf{0}) \int_{W_1} f(x) dx \right| = O(n^{-1/d}).$$

*If  $\xi$  only satisfies stabilization (1.14) and the  $p$ -moment condition (1.19) for some  $p \in (1, \infty)$ , then the right hand side of (1.23) is  $o(1)$ .*

(ii) *Assume that the second correlation function  $\rho^{(2)}$  of  $\mathcal{P}$  exists and is bounded as in (1.11), that  $\xi$  satisfies (1.14), and that  $(\xi, \mathcal{P})$  satisfies the  $p$ -moment condition (1.19) for some  $p \in (2, \infty)$ . If the second-order correlations of the  $\xi$ -weighted measures decay fast, i.e. satisfy (1.21) with  $p = q = k_1 = k_2 = 1$  and all  $n \in \mathbb{N} \cup \{\infty\}$ , then for all  $f \in \mathcal{B}(W_1)$*

$$(1.24) \quad \lim_{n \rightarrow \infty} n^{-1} \text{Var} \mu_n^\xi(f) = \sigma^2(\xi) \int_{W_1} f(x)^2 dx \in [0, \infty),$$

*whereas for all  $f, g \in \mathcal{B}(W_1)$*

$$(1.25) \quad \lim_{n \rightarrow \infty} n^{-1} \text{Cov}(\mu_n^\xi(f), \mu_n^\xi(g)) = \sigma^2(\xi) \int_{W_1} f(x)g(x) dx.$$

We remark that (1.23) and (1.24) together show convergence in probability

$$n^{-1} \mu_n^\xi(f) \xrightarrow{P} \mathbb{E}_0 \xi(\mathbf{0}, \mathcal{P}) \rho^{(1)}(\mathbf{0}) \int_{W_1} f(x) dx \text{ as } n \rightarrow \infty.$$

The proof of variance asymptotics (1.24) requires fast decay of the second-order correlations of the  $\xi$ -weighted measures. Fast decay of *all* higher-order correlations as in Definition 1.10 yields Gaussian fluctuations of  $\mu_n^\xi, n \geq 1$ , under moment conditions on the atom sizes (i.e., under moment conditions on  $\xi$ ) and a variance lower bound. Let  $N$  denote a mean zero normal random variable with variance 1. We write  $f(n) = \Omega(g(n))$  when  $g(n) = O(f(n))$ , i.e., when  $\liminf_{n \rightarrow \infty} |f(n)/g(n)| > 0$ .

**THEOREM 1.13.** *Let  $\mathcal{P}$  be an admissible point process on  $\mathbb{R}^d$  and let the pair  $(\xi, \mathcal{P})$  satisfy the  $p$ -moment condition (1.19) for all  $p \in (1, \infty)$ . If the correlations of the  $\xi$ -weighted measures at (1.5) decay fast as in Definition 1.10 and if  $f \in \mathcal{B}(W_1)$  satisfies*

$$(1.26) \quad \text{Var} \mu_n^\xi(f) = \Omega(n^\nu)$$

*for some  $\nu \in (0, \infty)$ , then as  $n \rightarrow \infty$*

$$(1.27) \quad (\text{Var} \mu_n^\xi(f))^{-1/2} (\mu_n^\xi(f) - \mathbb{E} \mu_n^\xi(f)) \xrightarrow{\mathcal{D}} N.$$

Combining Theorem 1.11 and Theorem 1.13 yields the following theorem, which is well-suited for off-the-shelf use in applications, as seen in Section 2.3.

**THEOREM 1.14.** *Let  $(\xi, \mathcal{P})$  be an admissible pair of class (A1) or (A2) such that the  $p$ -moment condition (1.19) holds for all  $p \in (1, \infty)$ . If  $f \in \mathcal{B}(W_1)$  satisfies condition (1.26) for some  $\nu \in (0, \infty)$ , then  $\mu_n^\xi(f)$  is asymptotically normal as in (1.27), as  $n \rightarrow \infty$ .*



satisfies normal convergence (1.27) as established in [69, 73].

(v) *Mixing conditions.* Central limit theorems for geometric functionals of mixing point processes (random fields) are established in [3, 16, 35, 31, 33, 32, 56]. The geometric functionals considered in these papers are different than the ones considered here; furthermore the relation between the mixing conditions in these papers and  $\omega$ -mixing correlation functions as in Definition 1.1 is unclear. Though correlation functions are simpler than mixing coefficients, which depend on  $\sigma$ -algebras generated by the point processes, our decay rates appear more restrictive than those needed in afore-mentioned papers. A careful investigation of the relations between the various notions of mixing and fast decay of correlations lies beyond the scope of our limit results and will be treated in a separate paper. In the case of point processes on discrete spaces, such a study is easier, c.f. [65].

(vi) *Brillinger mixing and fast decay of correlations.* Brillinger mixing [35, Section 3.5] is defined via finiteness of integrals of the *reduced cumulant measures* (see Section 4.3.2). The very definition of Brillinger mixing implies volume-order growth of cumulants; the converse follows using the ideas in the proof of [10, Theorem 3.2]. The key to proving our announced central limit theorems is to show that the fast decay of correlations of the  $\xi$ -weighted measures (1.5) implies volume-order growth of cumulants and hence Brillinger mixing; see the remarks at the beginning of Section 4.3 and also those and at the end of Section 4.4.2.

(vii) *Multivariate central limit theorem.* We may use the Cramér-Wold device to extend Theorems 1.12 and 1.14 to the multivariate setting as follows. Let  $(\xi, \mathcal{P})$  be a pair satisfying the hypotheses of Theorems 1.12 and 1.14. If  $f_i \in \mathcal{B}(W_1)$ ,  $1 \leq i \leq k$ , satisfy the variance limit (1.24) with  $\sigma^2(\xi) > 0$ , then as  $n \rightarrow \infty$  the fidis

$$\left( \frac{\mu_n^\xi(f_1) - \mathbb{E}\mu_n^\xi(f_1)}{\sqrt{n}}, \dots, \frac{\mu_n^\xi(f_k) - \mathbb{E}\mu_n^\xi(f_k)}{\sqrt{n}} \right)$$

converge to that of a centred Gaussian field having covariance kernel  $f, g \mapsto \sigma^2(\xi) \int_{W_1} f(x)g(x)dx$ .

(viii) *Deterministic radius of stabilization.* It may be shown that our main results go through without the condition (1.17) if the radius of stabilization  $R^\xi(x, \mathcal{P})$  is bounded by a non-random (deterministic) constant and if (1.16) and (1.18) are satisfied. However we are unable to find any interesting examples of point processes satisfying (1.10) but not (1.17).

(ix) *Fast decay of the correlation of the  $\xi$ -weighted measures; Theorem 1.11.* Though the cumulant method is common to [7, 69, 53] and this article, a distinguishing and novel feature of our approach is the proof of fast decay of correlations of the  $\xi$ -weighted measures (1.21), and consequently their Brillinger mixing, for a wide class of functionals and point processes. As mentioned in the introduction, the proof of this result is via factorial moment expansions, which differs from the approach of [7, 69, 53] (see the remarks at the beginning of Section 3). Fast decay of correlations of the  $\xi$ -weighted measures (1.21) appears to be of independent interest. It features in the proofs of moderate deviation principles and laws of the iterated logarithms for stabilizing functionals of Poisson point process [5], [22]. Fast decay of correlations (1.21) yields volume order cumulant bounds, useful in establishing concentration inequalities as well as moderate deviations, as explained in [27, Lemma 4.2].

(x) *Normal approximation.* Difference operators (which appear in our factorial moment expansions) are also a key tool in the Malliavin-Stein method [54, 55]. This method yields presumably optimal rates





(1.13), then the disintegration formula (1.9) yields

$$\sigma^2(\xi, \gamma) = \sum_{j=0}^k (j!(k-j-1)!(k-j-1)!)^{-1} \int_{\mathbb{R}^d} dx \gamma(x) \left( \int_{(B_r(\mathbf{0}) \cap B_r(x))^j \times B_r(\mathbf{0})^{k-j-1} \times B_r(x)^{k-j-1}} h(\mathbf{0}, \mathbf{y}, \mathbf{z}) h(x, \mathbf{x}, \mathbf{z}) \left[ \rho^{(k)}(\mathbf{0}, \mathbf{y}, \mathbf{z}) \rho^{(k)}(x, \mathbf{x}, \mathbf{z}) - \rho^{(2k-j)}(\mathbf{0}, \mathbf{y}, \mathbf{z}, x, \mathbf{x}) \right] dz dy dx \right).$$

(ii) Theorem 1.12 and Theorem 1.15 extend [45, Propositions 1 and 2], which are valid only for  $\xi \equiv 1$ , to general functionals. If an admissible pair  $(\xi, \mathcal{P})$  of type (A1) or (A2) is such that  $\widehat{H}_n^\xi(\mathcal{P})$  does not have volume-order variance growth, then Theorems 1.12 and 1.15 show that  $\widehat{H}_n^\xi(\mathcal{P})$  has at most surface-order variance growth.

**2. Examples and applications.** Before providing examples and applications of our general results, we briefly discuss the moment assumptions involved in our main theorems.

**2.1. Moments of point processes having fast decay of correlations.** We say that  $\mathcal{P}$  has exponential moments if for all bounded Borel  $B \subset \mathbb{R}^d$  and all  $t \in \mathbb{R}^+$  we have

$$(2.1) \quad \mathbb{E}[t^{\mathcal{P}(B)}] < \infty.$$

Similarly, say that  $\mathcal{P}$  has all moments if for all bounded Borel  $B \subset \mathbb{R}^d$  and all  $k \in \mathbb{N}$ , we have

$$(2.2) \quad \mathbb{E}[\mathcal{P}(B)^k] < \infty.$$

**Remarks.**

(i) The point process  $\mathcal{P}$  has exponential moments whenever  $\sum_{k=1}^{\infty} \kappa_k t^k / k! < \infty$  for all  $t \in \mathbb{R}^+$  with  $\kappa_k$  as in (1.11) (cf. the expansion of the probability generating function of a random variable in terms of factorial moments [18, Proposition 5.2.III.]). By (1.12) an admissible point process having fast decay of correlations has exponential moments provided

$$(2.3) \quad \sum_{k=1}^{\infty} \frac{C_k t^k}{(k-1)!} < \infty, \quad t \in \mathbb{R}^+.$$

Note that input of type (A2) has exponential moments since by (1.16), we have  $C_k = O(k^{ak})$ ,  $a \in [0, 1)$ , making (2.3) summable. For pairs  $(\xi, \mathcal{P})$  of type (A2) with radius of stabilization bounded by  $r_0 \in [1, \infty)$ , by (1.18) the  $p$ -moment in (1.19) is consequently controlled by a finite exponential moment, i.e., for  $x_1, \dots, x_{p'} \in W_n$ ,

$$(2.4) \quad \mathbb{E}_{x_1, \dots, x_{p'}} \max\{|\xi(x_1, \mathcal{P}_n)|, 1\}^p \leq \mathbb{E}_{x_1, \dots, x_{p'}} (\max\{\hat{c}r_0, 1\})^{p\mathcal{P}(B_{r_0}(x_1))}.$$

Finally, if  $\mathcal{P}$  has exponential moments under its stationary probability  $\mathbb{P}$ , the same is true under  $\mathbb{P}_{x_1, \dots, x_k}$  for  $\alpha^{(k)}$  almost all  $x_1, \dots, x_k$ <sup>3</sup>.

<sup>3</sup> Indeed, if  $\mathbb{E}_{x_1, \dots, x_k} [\rho^{\mathcal{P}(B_r(x_1))}] = \infty$  for  $x_1, \dots, x_k \in B'$  for some bounded  $B' \in \mathbb{R}^d$  such that  $\alpha^{(k)}(B'^k) > 0$  then  $\mathbb{E}_{x_1, \dots, x_k} [\rho^{\mathcal{P}(B_r(x_1))}] \leq \mathbb{E}_{x_1, \dots, x_k} [\rho^{\mathcal{P}(B'_r)}] = \infty$  with  $B'_r = B' \oplus B_r(0) = \{y' + y : y' \in B', y \in B_r(0)\}$  the  $r$ -parallel set of  $B'$ . Integrating with respect to  $\alpha^{(k)}$  in  $B'^k$ , by the Campbell formula  $\mathbb{E}[(\mathcal{P}(B'_r))^k \rho^{\mathcal{P}(B'_r)}] = \infty$ , which contradicts the existence of exponential moments under  $\mathbb{P}$ .

(ii) For pairs  $(\xi, \mathcal{P})$  of type (A1), the  $p$ -moment (1.19) satisfies for  $x_1, \dots, x_{p'} \in W_n$

$$(2.5) \quad \mathbb{E}_{x_1, \dots, x_{p'}} \max\{|\xi(x_1, \mathcal{P}_n)|, 1\}^p \leq \left(\frac{\|h\|_\infty}{k}\right)^p \mathbb{E}_{x_1, \dots, x_{p'}} [(\mathcal{P}(B_r(x_1)))^{(k-1)p}].$$

We next show that (2.5) may be controlled by moments of Poisson random variables. For any Borel set  $B \subset (\mathbb{R}^d)^k$ , the definition of factorial moment measures gives  $\alpha^{(k)}(B) \leq \kappa_k \text{Vol}_{dk}(B)$ . Since moments may be expressed as a linear combination of factorial moments, for  $k \in \mathbb{N}$  and a bounded Borel subset  $B \subset \mathbb{R}^d$ , using (1.8) we have

$$(2.6) \quad \mathbb{E}[(\mathcal{P}(B))^k] = \sum_{j=0}^k \left\{ \begin{matrix} k \\ j \end{matrix} \right\} \alpha^{(j)}(B^j) \leq \kappa_k \sum_{j=0}^k \left\{ \begin{matrix} k \\ j \end{matrix} \right\} \text{Vol}_{jd}(B)^j = \kappa_k \mathbb{E}(\text{Po}(\text{Vol}_d(B))^k),$$

where  $\left\{ \begin{matrix} k \\ j \end{matrix} \right\}$  stand for the *Stirling numbers of the second kind*,  $\text{Po}(\lambda)$  denotes a Poisson random variable with mean  $\lambda$  and where  $\kappa_j$ 's are non-decreasing in  $j$ . Thus by (1.12), an admissible point process having fast decay of correlations has all moments, as in (2.2). If  $\mathcal{P}$  has all moments under its stationary probability  $\mathbb{P}$ , the same is true under  $\mathbb{P}_{x_1, \dots, x_k}$  for  $\alpha^{(k)}$  almost all  $x_1, \dots, x_k$  (by the same arguments as in Footnote 3).

**2.2. Examples of point processes having fast decay of correlations.** The notion of a stabilizing functional is well established in the stochastic geometry literature but since the notion of fast decay of correlations for point processes (1.10) is less well studied, we first establish that some well-known point processes enjoy this property. For more details on the first five examples, we refer to [9].

### 2.2.1. Class A1 input.

**Permanental input.** The point process  $\mathcal{P}$  is permanental if its correlation functions are defined by  $\rho^{(k)}(x_1, \dots, x_k) := \text{per}(K(x_i, x_j))_{1 \leq i, j \leq k}$ , where the permanent of an  $n \times n$  matrix  $M$  is  $\text{per}(M) := \sum_{\pi \in S_n} \prod_{i=1}^n M_{i, \pi(i)}$ , with  $S_n$  denoting the permutation group of the first  $n$  integers and  $K(\cdot, \cdot)$  is the Hermitian kernel of a locally trace class integral operator  $\mathcal{K} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  [9, Assumption 4.2.3]. A kernel  $K$  is *fast-decreasing* if

$$(2.7) \quad |K(x, y)| \leq \omega(|x - y|), \quad x, y \in \mathbb{R}^d,$$

for some fast-decreasing  $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ . Lemma 5.5 in Section 5 shows that if a stationary permanental point process has a fast-decreasing kernel as at (2.7), then it is an admissible point process having fast decay of correlations with decay function  $\phi = \omega$  and with correlation decay constants satisfying

$$(2.8) \quad C_k := k k! \|K\|^{k-1}, \quad c_k \equiv 1,$$

where  $\|K\| := \sup_{x, y} |K(x, y)|$  and we can choose  $\kappa_k = k! \|K\|^k$ . However, a trace class permanental point process in general does not have exponential moments, i.e., the right-hand side of (2.1) might be infinite for some bounded  $B$  and  $\rho$  large enough. <sup>4</sup>

<sup>4</sup>This is because, the number of points of a (trace-class) permanental p.p. in a compact set  $B$  is a sum of independent geometric random variables  $\text{Geo}(1/(1 + \lambda))$  where  $\lambda$  runs over all eigenvalues of the integral operator defining the process truncated to  $B$ .

The permanental point process with kernel  $K$  may be represented as a Cox point process (see Section 2.2.3) directed by the random measure  $\eta(B) := \int_B (Z_1(x)^2 + Z_2(x)^2) dx$ ,  $B \subset \mathbb{R}^d$ , where the intensity  $Z_1(x)^2 + Z_2(x)^2$  is a sum of i.i.d. Gaussian random fields with zero mean and covariance function  $K/2$  [71, Thm 6.13]. Thus mean zero Gaussian random fields with a fast decaying covariance function  $K/2$  yield a permanental (Cox) point process with kernel  $K$  and having fast decay of correlations.

**$\alpha$ -Permanental point processes.** See [9, Section 4.10], [46], and [71] for more details on this class of point processes which generalize permanental point processes. Given  $\alpha \geq 0$  and a kernel  $K$  which is Hermitian, non-negative definite and locally trace class, a point process  $\mathcal{P}$  is said to be  $\alpha$ -permanental<sup>5</sup> if its correlation functions satisfy

$$(2.9) \quad \rho^{(k)}(x_1, \dots, x_k) = \sum_{\pi \in S_k} \alpha^{k-\nu(\pi)} \prod_{i=1}^k K(x_i, x_{\pi(i)})$$

where  $S_k$  stands for the usual symmetric group and  $\nu(\cdot)$  denotes the number of cycles in a permutation. The right hand side is the  $\alpha$ -permanent of the matrix  $((K(x_i, x_j))_{i,j \leq k}$ . The special cases  $\alpha = 0$  and  $\alpha = 1$  respectively give the Poisson point process with intensity  $K(\mathbf{0}, \mathbf{0})$  and the permanental point process with kernel  $K$ . In what follows, we assume  $\alpha = 1/m$  for  $m \in \mathbb{N}$ , i.e.  $1/\alpha$  is a positive integer. Existence of such  $\alpha$ -permanental point processes is guaranteed by [71, Theorem 1.2]. The property of these point processes most important to us is that an  $\alpha$ -permanental point process with kernel  $K$  is a superposition of  $1/\alpha$  i.i.d. copies of a permanental point process with kernel  $\alpha K$  (see [9, Section 4.10]). Also from definition (2.9), we obtain

$$\rho^{(k)}(x_1, \dots, x_k) \leq \|K\|^k \alpha^k \sum_{\pi \in S_k} (\alpha^{-1})^{\nu(\pi)},$$

and so we can take  $\kappa_k = \prod_{i=0}^{k-1} (j\alpha + 1) \|K\|^k$  for an  $\alpha$ -permanental point process. The following result is a consequence of the upcoming Proposition 2.3 and the identity (2.8) for decay constants of a permanental point process with kernel  $\alpha K$ .

**PROPOSITION 2.1.** *Let  $\alpha = 1/m$  for some  $m \in \mathbb{N}$  and let  $\mathcal{P}_\alpha$  be the stationary  $\alpha$ -permanental point process with a kernel  $K$  which is Hermitian, non-negative definite and locally trace class. Assume also that  $|K(x, y)| \leq \omega(|x - y|)$  for some fast-decreasing  $\omega$ . Then  $\mathcal{P}_\alpha$  is an admissible point process having fast decay of correlations with decay function  $\phi = \omega$  and correlation decay constants  $C_k = km^{1-k(m-1)} m! (k!)^m \|K\|^{km-1}$ ,  $c_k = 1$ .*

**Zero set of Gaussian entire function (GEF).** A Gaussian entire function  $f(z)$  is the sum  $\sum_{j \geq 0} X_j \frac{z^j}{\sqrt{j!}}$  where  $X_j$  are i.i.d. with the standard normal density on the complex plane. The zero set  $f^{-1}(\{0\})$  gives rise to the point process  $\mathcal{P}_{GEF} := \sum_{x \in f^{-1}(\{0\})} \delta_x$  on  $\mathbb{R}^2$ . The point process  $\mathcal{P}_{GEF}$  is an admissible point process having fast decay of correlations [53, Theorem 1.4], exhibiting local repulsion of

<sup>5</sup>In contrast to terminology in [9, 71], here we distinguish the two cases (i)  $\alpha \geq 0$  ( $\alpha$ -permanental) and (ii)  $\alpha \leq 0$  ( $\alpha$ -determinantal)



Consequently,  $\phi$  satisfies the requisite exponential decay (1.17) whenever  $\omega$  itself satisfies (1.17).

The Ginibre ensemble of eigenvalues of  $N \times N$  matrices with independent standard complex Gaussian entries is a leading example of a determinantal point process. The limit of the Ginibre ensemble as  $N \rightarrow \infty$  is the Ginibre point process (or the infinite Ginibre ensemble), here denoted  $\mathcal{P}_{GIN}$ . It is the prototype of a stationary determinantal point process and has kernel

$$K(z_1, z_2) := \exp(z_1 \bar{z}_2) \exp\left(-\frac{|z_1|^2 + |z_2|^2}{2}\right) = \exp\left(i\text{Im}(z_1 \bar{z}_2) - \frac{|z_1 - z_2|^2}{2}\right), \quad z_1, z_2 \in \mathbb{C}.$$

More generally, for  $0 < \beta \leq 1$ , the  $\beta$ -Ginibre (determinantal) point process (see [28]) has kernel

$$K_\beta(z_1, z_2) := \exp\left(\frac{1}{\beta} z_1 \bar{z}_2\right) \exp\left(-\frac{|z_1|^2 + |z_2|^2}{2\beta}\right), \quad z_1, z_2 \in \mathbb{C}.$$

When  $\beta = 1$ , we obtain  $\mathcal{P}_{GIN}$  and as  $\beta \rightarrow 0$  we obtain the Poisson point process. Thus the  $\beta$ -Ginibre point process interpolates between the Ginibre and Poisson point processes. Identifying the complex plane with  $\mathbb{R}^2$  we see that all  $\beta$ -Ginibre point processes are admissible point processes having fast decay of correlations satisfying (1.16) and (1.17).

*Moment Conditions.* Let  $p \in [1, \infty)$  and let  $\mathcal{P}$  be a stationary determinantal point process with a continuous and fast-decreasing kernel. We now show that the  $p$ -moment condition (1.19) holds for pairs  $(\xi, \mathcal{P})$  of class (A1) or (A2), provided  $\xi$  has a deterministic radius of stabilization, say  $r_0 \in [1, \infty)$ . First, for all  $(x_1, \dots, x_p) \in (\mathbb{R}^d)^p$ , all increasing  $F : \mathbb{N} \rightarrow \mathbb{R}^+$  and all bounded Borel sets  $B$  we have [28, Theorem 2]

$$\mathbb{E}_{x_1, \dots, x_p}^! (F(\mathcal{P}(B))) \leq \mathbb{E}(F(\mathcal{P}(B))).$$

Thus using (2.4), the above inequality and stationarity of  $\mathcal{P}$ , we get that for any bounded stabilizing score function  $\xi$  of class (A2),

(2.16)

$$\sup_{1 \leq n \leq \infty} \sup_{1 \leq p' \leq [p]} \sup_{x_1, \dots, x_{p'} \in W_n} \mathbb{E}_{x_1, \dots, x_{p'}} \max\{|\xi(x_1, \mathcal{P}_n)|, 1\}^p \leq \mathbb{E}(\max\{\hat{c}r_0, 1\}^{p\mathcal{P}(B_{r_0}(\mathbf{0})) + p^2}) < \infty.$$

The finiteness of the last term follows from the fact that determinantal input considered here is of class (A2) and, by Remark (i) at the beginning of Section 2.1, such input has finite exponential moments.

**$\alpha$ -Determinantal point processes.** Similar to permanental point processes, we generalize determinantal point processes to include their  $\alpha$ -determinantal versions, by requiring that the correlation functions satisfy (2.9) for some  $\alpha \leq 0$ . In what follows, we shall assume that  $\alpha = -1/m$ ,  $m \in \mathbb{N}$ . Existence of such  $\alpha$ -determinantal point processes again follows from [71, Theorem 1.2]. Likewise, an  $\alpha$ -determinantal point process with kernel  $K$  is a superposition of  $-1/\alpha$  i.i.d. copies of a determinantal point process with kernel  $-\alpha K$  ([9, Section 4.10]). By [71, Proposition 4.3], we can take  $\kappa_k = K(\mathbf{0}, \mathbf{0})^k$  for an  $\alpha$ -determinantal point process. Analogously to Proposition 2.1, the next result follows from Proposition 2.3 below and the identity (2.15) for correlation decay constants of a determinantal point process with kernel  $-\alpha K$ .

**PROPOSITION 2.2.** *Let  $\alpha = -1/m$  for some  $m \in \mathbb{N}$  and  $\mathcal{P}_\alpha$  be the stationary  $\alpha$ -determinantal point process with a kernel  $K$  which is Hermitian, non-negative definite and locally trace class. As-*



cluster point processes or Matérn hard-core point processes with finite dependence radius.

**Thinned Poisson point processes.** Suppose  $\mathcal{P}$  is a Poisson point process,  $\xi(\cdot, \mathcal{P}) \in \{0, 1\}$ , and consider the thinned point process  $\tilde{\mathcal{P}} := \sum_{x \in \mathcal{P}} \xi(x, \mathcal{P}) \delta_x$ . If  $\xi$  stabilizes exponentially fast then Lemma 5.2 of [7] shows that  $\tilde{\mathcal{P}}$  is an admissible point process having fast decay of correlations. This set-up includes finite-range dependent point processes  $\mathcal{P}$  as well as Matérn cluster and Matérn hard-core point process with exponentially decaying dependence radii. Tractable procedures generating thinnings of Poisson point processes are in [2].

**Thinned general point processes.** Suppose  $(\xi, \mathcal{P})$  is an admissible pair of class A2 and suppose further that  $\xi(\cdot, \mathcal{P}) \in \{0, 1\}$ . Then  $\mu_n^\xi$  is a thinned point process and the correlation functions of  $\mu_n^\xi$  coincide with  $m^{(1, \dots, 1)}(x_1, \dots, x_k; \infty)$  in (1.6). Theorem 1.11 implies these functions have fast decay of correlations and hence  $\mu_n^\xi$  is an admissible point process having fast decay of correlations. For similar examples and generalizations, termed *generalized shot-noise Cox point process*, see [51].

**Superpositions of i.i.d. point processes.** Apart from thinning another natural operation on point processes generating new point processes consists of independent superposition. We show that this operation preserves fast decay of correlations.

Let  $\mathcal{P}_1, \dots, \mathcal{P}_m$ ,  $m \in \mathbb{N}$ , be i.i.d. copies of an admissible point process  $\mathcal{P}$  with correlation functions  $\rho$  and having fast decay of correlations. Let  $\rho_0$  denote the correlation functions of the point process  $\mathcal{P}_0 := \cup_{i=1}^m \mathcal{P}_i$ . For any  $k \geq 1$  and distinct  $x_1, \dots, x_k \in \mathbb{R}^d$  the following relation holds

$$(2.18) \quad \rho_0^{(k)}(x_1, \dots, x_k) = \sum_{\sqcup_{i=1}^m S_i = [k]} \prod_{i=1}^m \rho(S_i),$$

where  $\sqcup$  stands for disjoint union and where we abbreviate  $\rho^{(|S_i|)}(x_j : j \in S_i)$  by  $\rho(S_i)$ . Here  $S_i$  may be empty, in which case we set  $\rho(\emptyset) = 1$ . It follows from (2.18) that  $\mathcal{P}_0$  is an admissible point process with intensity  $m\rho^{(1)}(\mathbf{0})$ . Further, we can take  $\kappa_k(\mathcal{P}_0) = (\kappa_k)^m m^k$ . The proof of the next proposition, which shows that  $\mathcal{P}_0$  has fast decay of correlations, is in the Appendix.

**PROPOSITION 2.3.** *Let  $m \in \mathbb{N}$  and  $\mathcal{P}_1, \dots, \mathcal{P}_m$  be i.i.d. copies of an admissible point process  $\mathcal{P}$  having fast decay of correlations with decay function  $\phi$  and correlation decay constants  $C_k$  and  $c_k$ . Then  $\mathcal{P}_0 := \cup_{i=1}^m \mathcal{P}_i$  is an admissible point process having fast decay of correlations with decay function  $\phi$  and correlation decay constants  $m^k m! (\kappa_k)^{m-1} C_k$  and  $c_k$ . Further, if  $\mathcal{P}$  is admissible input of type (A2) with  $\kappa_k \leq \lambda^k$  for some  $\lambda \in (0, \infty)$ , then  $\mathcal{P}_0$  is also admissible input of type (A2).*

We have already used this proposition in the context of fast decay of correlations of  $\alpha$ -permanental and determinantal point processes.

**2.3. Applications.** Having provided examples of admissible point processes, we now use Theorems 1.12 and 1.14 to deduce the limit theory for geometric and topological statistics of these point processes. Examples include statistics arising in combinatorial and differential topology, integral geometry, and computational geometry. When the input is one of the Gibbs point processes described in Section 2.2, then these results can be deduced from [69, 73]. The examples are not exhaustive and

indeed include functionals in stochastic geometry already discussed in e.g. [7, 64]. There are further applications to (i) random packing models on input having fast decay of correlations (extending [62]), (ii) statistics of percolation models (extending e.g. [41, 61]), and (iii) statistics of extreme points of input having fast decay of correlations (extending [4, 73]). Details are left to the reader.

We shall need to assume moment bounds and variance lower bounds in the upcoming applications in Sections 2.3.1- 2.3.3. In view of (2.14) and (2.17), the moment conditions are valid for input pairs  $(\xi, \mathcal{P})$  of class (A1) when  $\mathcal{P}$  is either  $\alpha$ -determinantal with  $-1/\alpha \in \mathbb{N}$  or  $\mathcal{P}_{GEF}$ . Similarly, in view of (2.17), the moment conditions are valid for input pairs  $(\xi, \mathcal{P})$  of class (A2) when  $\xi$  has a bounded radius of stabilization and when  $\mathcal{P}$  is  $\alpha$ -determinantal with  $-1/\alpha \in \mathbb{N}$ .

*2.3.1. Statistics of simplicial complexes.* A nonempty family  $\Delta$  of finite subsets of a set  $V$  is an *abstract simplicial complex* if  $\mathcal{Y} \in \Delta$  and  $\mathcal{Y}_0 \subset \mathcal{Y}$  implies that  $\mathcal{Y}_0 \in \Delta$ . Elements of  $\Delta$  are called *faces/simplices* and the *dimension of a face* is one less than its cardinality. The 0-dimensional faces are *vertices*. The collection of all faces of  $\Delta$  with dimension less than  $k$  is a sub-complex called the *k-skeleton* of  $\Delta$  and denoted by  $\Delta^{\leq k}$ . The 1-skeleton of a simplicial complex is a graph whose vertices are 0-dimensional faces and whose edges are 1-dimensional faces. The simplicial complex, or ‘complex’ for short, represents a combinatorial generalization of a graph and is a fundamental object in combinatorial as well as computational topology [21]

Given a finite point set  $\mathcal{X}$  in  $\mathbb{R}^d$  (or generally, in a metric space) there are various ways to define a complex that captures some of the geometry/topology of  $\mathcal{X}$ . One such complex is the Čech complex. Recall that if  $\mathcal{X} \subset \mathbb{R}^d$  is a finite set of points and  $r \in (0, \infty)$ , then the Čech complex of radius  $r$  is the abstract complex

$$\mathcal{C}(\mathcal{X}, r) := \left\{ \sigma \subset \mathcal{X} : \bigcap_{x \in \sigma} B_r(x) \neq \emptyset \right\}.$$

By the nerve theorem [11, Theorem 10.7], the Čech complex is homotopy equivalent (in particular, same topological invariants) to the classical germ-grain model

$$(2.19) \quad \mathcal{C}_B(\mathcal{X}, r) := \bigcup_{x \in \mathcal{X}} B_x(r).$$

The 1-skeleton of the Čech complex,  $\mathcal{C}(\mathcal{X}, r)^{\leq 1}$ ,  $\mathcal{X}$  random, is the *random geometric graph* [58], denoted by  $G(\mathcal{X}, r)$ . One can study many geometric or topological statistics similar to those described below for the Čech complex for other geometric complexes (for example, see [77, Section 3.2]) or geometric graphs. Indeed, motivated by problems in topological data analysis, random geometric complexes on Poisson or binomial point processes were studied in [36] and later were extended to stationary point processes in [74].

We next establish the limit theory for statistics of random Čech complexes. *The central limit theorems are applicable whenever the input  $\mathcal{P}$  is either  $\alpha$ -determinantal ( $|\alpha| = \frac{1}{m}$ ,  $m \in \mathbb{N}$ ) with kernel as at (2.7),  $\mathcal{P}_{GEF}$ , or rarified Gibbsian input.* In all that follows we fix  $r \in (0, \infty)$ .

**Simplex counts or clique counts.** Let  $\Gamma$  be a complex on  $k$ -vertices such that  $\Gamma^{\leq 1}$  is a connected



graph. For  $x \in \mathbb{R}^d$  and  $\mathbf{x} := (x_1, \dots, x_{k-1}) \in (\mathbb{R}^d)^{k-1}$ , let

$$h^\Gamma(x, \mathbf{x}) := \mathbf{1}[\mathcal{C}(\{x, x_1, \dots, x_{k-1}\}, r) \cong \Gamma],$$

where  $\cong$  denotes simplicial isomorphism. For an admissible point process  $\mathcal{P}$  as in Definition 1.2, we put

$$\gamma^{(k)}(x, \mathcal{P}) := \frac{1}{k!} \sum_{\mathbf{x} \in (\mathcal{P} \cap B_r(x))^{k-1}} h^\Gamma(x, \mathbf{x}),$$

that is  $(\gamma^{(k)}, \mathcal{P})$  is an admissible pair of type (A1). If  $\Gamma$  denotes the  $(k-1)$ -simplex, then  $H_n^{\gamma^{(k)}}(\mathcal{P})$  is the number of  $(k-1)$ -simplices in  $\mathcal{C}(\mathcal{P}_n, r)$  and for  $k=2$ ,  $H_n^{\gamma^{(k)}}(\mathcal{P})$  is the *edge count* in the random geometric graph  $G(\mathcal{P}_n, r)$ . Theorem 3.4 of [74] establishes expectation asymptotics for  $n^{-1}H_n^{\gamma^{(k)}}(\mathcal{P})$  for stationary input. The next result establishes variance asymptotics and asymptotic normality of  $n^{-1}H_n^{\gamma^{(k)}}(\mathcal{P})$ . It is an immediate consequence of Theorem 1.12(ii) and Theorem 1.14. Let  $\sigma^2(\gamma^{(k)})$  be as at (1.22), with  $\xi$  put to be  $\gamma^{(k)}$ .

**THEOREM 2.4.** *Let  $k \in \mathbb{N}$ . If  $\mathcal{P}$  is an admissible point process having fast decay of correlations as at (1.10) and the pair  $(\gamma^{(k)}, \mathcal{P})$  satisfies the moment condition (1.19) for all  $p \in (1, \infty)$ , then*

$$\lim_{n \rightarrow \infty} n^{-1} \text{Var} H_n^{\gamma^{(k)}}(\mathcal{P}) = \sigma^2(\gamma^{(k)}).$$

*Additionally, if  $\text{Var} H_n^{\gamma^{(k)}}(\mathcal{P}) = \Omega(n^\nu)$  for some  $\nu \in (0, \infty)$ , then as  $n \rightarrow \infty$*

$$(\text{Var} H_n^{\gamma^{(k)}}(\mathcal{P}))^{-1/2} \left( H_n^{\gamma^{(k)}}(\mathcal{P}) - \mathbb{E} H_n^{\gamma^{(k)}}(\mathcal{P}) \right) \xrightarrow{\mathcal{D}} N.$$

Up to now, the central limit theorem for clique counts has been restricted to binomial or Poisson input, cf. [20, 23, 39, 58, 66]. Theorem 2.4 shows that asymptotic normality holds for more general input.

**Edge lengths.** For  $x, y \in \mathbb{R}^d$ , let  $h(x, y) := |x - y| \mathbf{1}[|x - y| \leq r]$ . The  $U$ -statistic

$$\xi^L(x, \mathcal{X}) := \frac{1}{2} \sum_{y \in \mathcal{X} \cap B_r(x)} h(x, y),$$

is of generic type (1.13) and  $H_n^{\xi^L}(\mathcal{P})$  is the *total edge length* of the geometric graph  $G(\mathcal{P}_n, r)$ . The following is an immediate consequence of Theorems 1.12 and 1.14.

**THEOREM 2.5.** *For any admissible point process  $\mathcal{P}$  having fast decay of correlations as at (1.10) with the pair  $(\xi^L, \mathcal{P})$  satisfying the moment condition (1.19) for all  $p \in (1, \infty)$ , we have*

$$|n^{-1} \mathbb{E} H_n^{\xi^L}(\mathcal{P}) - \mathbb{E}_0 \xi^L(\mathbf{0}, \mathcal{P}) \rho^{(1)}(\mathbf{0})| = O(n^{-1/d}), \text{ and } \lim_{n \rightarrow \infty} n^{-1} \text{Var} H_n^{\xi^L}(\mathcal{P}) = \sigma^2(\xi^L).$$

*Moreover, if  $\text{Var} H_n^{\xi^L}(\mathcal{P}) = \Omega(n^\nu)$  for some  $\nu \in (0, \infty)$  then as  $n \rightarrow \infty$*

$$(\text{Var} H_n^{\xi^L}(\mathcal{P}))^{-1/2} \left( \widehat{H}_n^{\xi^L}(\mathcal{P}) - \mathbb{E} H_n^{\xi^L}(\mathcal{P}) \right) \xrightarrow{\mathcal{D}} N.$$



with (1.13), define the Morse score function

$$(2.20) \quad \xi^M(x, \mathcal{X}) := \frac{1}{k!} \sum_{\mathbf{x} \in (\mathcal{X} \cap B_{2r}(x))^k} h(x, \mathbf{x}) \mathbf{1}[\mathcal{X}(Q(x, \mathbf{x}) \setminus \{x, \mathbf{x}\}) = 0].$$

Then  $\xi^M$  satisfies the power growth condition (1.18) and is of type (A2) (and nearly of type (A1)), with the understanding that  $h(x, x_1, \dots, x_k) = 0$  whenever  $\max_{1 \leq i \leq k} |x_i - x| \in (2r, \infty)$ . The statistic  $H^{\xi^M}(\mathcal{P})$  is simply the number  $N_k(\mathcal{P}_n, r)$  of index  $k$  Morse critical points generated by  $\mathcal{P}_n$  which are within a distance  $r$  of  $\mathcal{P}_n$ , whereas  $\mu_n^{\xi^M}$  is the random measure generated by index  $k$  critical points. Index 0 Morse critical points are trivially the points of  $\mathcal{X}$  and so in this case  $N_0(\mathcal{X}) = \text{card}(\mathcal{X})$ . Thus we shall be interested in asymptotics for Morse critical points only of index  $k \in \mathbb{N}$ .

The next result establishes variance asymptotics and asymptotic normality of  $N_k(\mathcal{P}_n, r)$ , valid when  $\mathcal{P}$  is of class (A2). It is an immediate consequence of Theorem 1.12(ii), Theorem 1.14, and the fact that for input  $\mathcal{P}$  of class (A2),  $(\xi^M, \mathcal{P})$  is an input pair of class (A2). Let  $\sigma^2(\xi^M)$  be as at (1.22), with  $\xi$  put to be  $\xi^M$ .

**THEOREM 2.7.** *For all  $k \in \{1, \dots, d\}$  and class (A2) input  $\mathcal{P}$  with the pair  $(\xi^M, \mathcal{P})$  satisfying the moment condition (1.19) for all  $p \in (1, \infty)$ , we have*

$$\lim_{n \rightarrow \infty} n^{-1} \text{Var} N_k(\mathcal{P}_n, r) = \sigma^2(\xi^M).$$

Moreover if  $\text{Var} N_k(\mathcal{P}_n, r) = \Omega(n^\nu)$  for some  $\nu \in (0, \infty)$ , then as  $n \rightarrow \infty$

$$(\text{Var} N_k(\mathcal{P}_n, r))^{-1/2} (N_k(\mathcal{P}_n, r) - \mathbb{E} N_k(\mathcal{P}_n, r)) \xrightarrow{\mathcal{D}} N.$$

### Remarks.

(i) Theorem 5.2 of [74] establishes expectation asymptotics for  $n^{-1} N_k(\mathcal{P}_n, r)$  for stationary input, though without a rate of convergence and [15] establishes a central limit theorem but only for the case of Poisson or binomial point processes.

(ii) The Morse inequalities relate the Morse critical points (local functionals) to the Betti numbers (global functionals) of the Boolean model and in particular, imply that the changes in the homology of the Boolean model  $\mathcal{C}_B(\mathcal{P}, r)$  occurs at radii  $r = R(\mathbf{x})$  whenever  $C(\mathbf{x})$  is a Morse critical point. A trivial consequence is that the  $k$ th Betti number of  $\mathcal{C}(\mathcal{P}_n, r)$  is upper bounded by  $N_k(\mathcal{P}_n, r)$ .

(iii) Examples of similar score functions satisfying a modified version of (A1) (similar to (2.20)) include component counts of random geometric graphs [58, Chapter 3], simplex counts of degree  $k$  in the Čech complex, and simplex counts in either an alpha complex [77, Sec 3.2] or an appropriate discrete Morse complex on  $\mathcal{P}_n$  [24].

**2.3.3. Statistics of germ-grain models.** We furnish two more applications of Theorem 1.14 when  $\xi$  has a deterministic radius of stabilization. The applications concern the germ-grain model, a classic model in stochastic geometry [68].

**$k$ -covered region of the germ-grain model.** The following is a statistic of interest in coverage processes [29]. For locally-finite  $\mathcal{X} \subset \mathbb{R}^d$  and  $x \in \mathcal{X}$ , define the score function

$$\beta^{(k)}(x, \mathcal{X}) := \int_{y \in B_r(x)} \frac{\mathbf{1}[\mathcal{X}(B_r(y)) \geq k]}{\mathcal{X}(B_r(y))} dy.$$

Clearly,  $\beta^{(k)}$  is an exponentially stabilizing score function as in Definition 1.1 with stabilization radius  $2r$ . Define the  $k$ -covered region of the germ-grain model  $\mathcal{C}_B(\mathcal{P}_n, r)$  at (2.19) by  $\mathcal{C}_B^k(\mathcal{P}_n, r) = \{y : \mathcal{P}_n(B_r(y)) \geq k\}$ . Thus  $H_n^{\beta^{(k)}}(\mathcal{P})$  is the volume of  $\mathcal{C}_B^k(\mathcal{P}_n, r)$ . When  $k = 1$ ,  $H_n^{\beta^{(k)}}(\mathcal{P})$  is the volume of the germ-grain model having germs in  $\mathcal{P}_n$ . Clearly  $\beta^{(k)}$  is bounded by the volume of a radius  $r$  ball and so  $\xi$  satisfies the power growth condition (1.18). The following is an immediate consequence of Theorems 1.12 and 1.14 and the fact that if  $\mathcal{P}$  is of class (A2) then the input pair  $(\beta^{(k)}, \mathcal{P})$  is also of class (A2).

**THEOREM 2.8.** *For all  $k \in \mathbb{N}$  and any point process  $\mathcal{P}$  of class (A2) with the pair  $(\beta^{(k)}, \mathcal{P})$  satisfying the moment condition (1.19) for all  $p \in (1, \infty)$ , we have*

$$|n^{-1} \mathbb{E} \text{Vol}_d(\mathcal{C}_B^k(\mathcal{P}_n, r)) - \mathbb{E}_0 \beta^{(k)}(\mathbf{0}, \mathcal{P}) \rho^{(1)}(\mathbf{0})| = O(n^{-1/d}),$$

and

$$\lim_{n \rightarrow \infty} n^{-1} \text{Var} \text{Vol}_d(\mathcal{C}_B^k(\mathcal{P}_n, r)) = \sigma^2(\beta^{(k)}).$$

Moreover, if  $\text{Var} \text{Vol}_d(\mathcal{C}_B^k(\mathcal{P}_n, r)) = \Omega(n^\nu)$  for some  $\nu \in (0, \infty)$ , then as  $n \rightarrow \infty$

$$(2.21) \quad \frac{\text{Vol}_d(\mathcal{C}_B^k(\mathcal{P}_n, r)) - \mathbb{E} \text{Vol}_d(\mathcal{C}_B^k(\mathcal{P}_n, r))}{\sqrt{\text{Var} \text{Vol}_d(\mathcal{C}_B^k(\mathcal{P}_n, r))}} \xrightarrow{\mathcal{D}} N.$$

In the case of Poisson input and  $k = 1$ , [29] establishes a central limit theorem for  $\mathcal{C}_B^1(\mathcal{P}_n, r)$ . For general  $k$ , the central limit theorem for Poisson input can be deduced from the general results in [61, 7] with presumably optimal bounds following from [42, Proposition 1.4].

**Intrinsic volumes of the germ-grain model.** Let  $\mathcal{K}$  denote the set of *convex bodies*, i.e., compact, convex subsets of  $\mathbb{R}^d$ . The *intrinsic volumes*  $V_0, \dots, V_d$  are non-negative functionals on  $\mathcal{K}$  which satisfy *Steiner's formula*

$$V_d(K \oplus B_r(\mathbf{0})) = \sum_{j=0}^d r^{d-j} \theta_{d-j} V_j(K), \quad K \in \mathcal{K},$$

where  $r > 0$ ,  $\oplus$  denotes Minkowski addition,  $V_j$  denotes the  $j$ -dimensional Lebesgue measure, and  $\theta_j := \pi^{j/2} / \Gamma(j/2 + 1)$  is the volume of the unit ball in  $\mathbb{R}^j$ . Intrinsic volumes satisfy translation invariance and additivity, i.e., for  $K_1, \dots, K_m \in \mathcal{K}$ ,

$$V_j(\cup_{i=1}^m K_i) = \sum_{k=1}^m (-1)^{k+1} \sum_{1 \leq i_1 < \dots < i_k \leq m} V_j(\cap_{l=1}^k K_{i_l}), \quad j \in \{0, \dots, d\}.$$

This identity allows an extension of intrinsic volumes to real-valued functionals on the family of finite unions of convex bodies [68, Ch. 14]. Intrinsic volumes coincide with *quer-mass integrals* or *Minkowski functional* up to a normalization. The  $V_j$ 's define certain  $j$ -dimensional volumes of  $K$ ,

independently of the ambient space.  $V_d$  is the  $d$ -dimensional volume,  $2V_{d-1}$  is the surface measure, and  $V_0$  is the *Euler-Poincaré characteristic* which may also be expressed as an alternating sum of simplex counts, Morse critical points, or Betti numbers [21, Sections IV.2, VI.2]. Save for  $V_d$  and  $V_{d-1}$ , the remaining  $V_j$ 's may assume negative values on unions of convex bodies.

For finite  $\mathcal{X}$  and  $r > 0$ , we express  $V_j(\mathcal{C}_B(\mathcal{X}, r)), j \in \{0, \dots, d\}$ , as a sum of bounded stabilizing scores, which goes as follows. For  $x_1 \in \mathcal{X}$ , define the score

$$\xi_j(x_1, \mathcal{X}) := \sum_{k=1}^{\infty} (-1)^{k+1} \sum_{(x_2, \dots, x_k) \in (\mathcal{X} \cap B_{2r}(x_1))^{(k)}} \frac{V_j(B_r(x_1) \cap \dots \cap B_r(x_k))}{k!}.$$

The score  $\xi_j$  is translation invariant with radius of stabilization  $R^{\xi_j}(x_1, \mathcal{P}) \leq \lceil 3r \rceil$ . By additivity, we have  $V_j(\mathcal{C}_B(\mathcal{P}_n, r)) = H_n^{\xi_j}(\mathcal{P})$  for  $j \in \{0, \dots, d\}$ . The homogeneity relation  $V_j(B_r(\mathbf{0})) = r^j V_j(B_1(\mathbf{0})) = r^j \binom{d}{j} \frac{\theta_d}{\theta_{d-j}}$  and the monotonicity of  $V_j$  on  $\mathcal{K}$  yield

$$|\xi_j(x_1, \mathcal{X})| \mathbf{1}[\mathcal{X}(B_{2r}(x_1)) = l] \leq r^j \binom{d}{j} \frac{\theta_d}{\theta_{d-j}} \sum_{k=1}^l \binom{l}{k} \leq 2^l r^j \binom{d}{j} \frac{\theta_d}{\theta_{d-j}}.$$

In other words,  $\xi_j, 0 \leq j \leq d$ , satisfy the power growth condition (1.18) and are exponentially stabilizing. Theorems 1.12 and 1.14 yield the following limit theorems, where we note that  $(\xi_j, \mathcal{P})$  is of class (A2) whenever  $\mathcal{P}$  is of class (A2).

**THEOREM 2.9.** *Fix  $r \in (0, \infty)$  as above. For all  $j \in \{0, \dots, d\}$  and any point process  $\mathcal{P}$  of class (A2) with the pair  $(\xi_j, \mathcal{P})$  satisfying the  $p$ -moment condition (1.19) for all  $p \in (1, \infty)$ , we have*

$$|n^{-1} \mathbb{E} V_j(\mathcal{C}_B(\mathcal{P}_n, r)) - \mathbb{E}_0 \xi_j(\mathbf{0}, \mathcal{P}) \rho^{(1)}(\mathbf{0})| = O(n^{-1/d}), \text{ and } \lim_{n \rightarrow \infty} n^{-1} \text{Var} V_j(\mathcal{C}_B(\mathcal{P}_n, r)) = \sigma^2(\xi_j).$$

Moreover, if  $\text{Var} V_j(\mathcal{C}_B(\mathcal{P}_n, r)) = \Omega(n^\nu)$  for some  $\nu \in (0, \infty)$  then as  $n \rightarrow \infty$

$$\frac{V_j(\mathcal{C}_B(\mathcal{P}_n, r)) - \mathbb{E} V_j(\mathcal{C}_B(\mathcal{P}_n, r))}{\sqrt{\text{Var} V_j(\mathcal{C}_B(\mathcal{P}_n, r))}} \xrightarrow{\mathcal{D}} N.$$

### Remarks.

(i) Theorem 2.9 extends the analogous central limit theorems of [34], which are confined to Poisson input, to any point process of class (A2).

(ii) We may likewise prove central limit theorems for other functionals of germ-grain models, including mixed volumes, integrals of surface area measures [67, Chapters 4 and 5], and total measures of translative integral geometry [68, Section 6.4]. These functionals, like intrinsic volumes, are expressed as sums of bounded stabilizing scores and thus, under suitable assumptions, the limit theory for these functionals follows from Theorems 1.12 and 1.14.

**2.3.4. Edge-lengths in  $k$ -nearest neighbors graph.** We now use the full force of Theorems 1.12 and 1.14, applying them to sums of score functions whose radius of stabilization has an exponentially decaying tail.

Statistics of the Voronoi tessellation as well as of graphs in computational geometry such as the  $k$ -nearest neighbors graph and sphere of influence graph may be expressed as sums of exponentially

stabilizing score functionals [61] and hence via Theorems 1.12 and 1.14, we may deduce the limit theory for these statistics. To illustrate, we establish a weak law of large numbers, variance asymptotics, and a central limit theorem for the total edge-length of the  $k$ -nearest neighbors graph on a  $\alpha$ -determinantal point process  $\mathcal{P}_\alpha$  with  $-1/\alpha \in \mathbb{N}$  and a fast-decreasing kernel as in (2.7). As noted in Proposition 2.2, such an  $\alpha$ -determinantal point process is of class (A2) as in Definition 1.7.

As shown in Corollary 5.9, we may explicitly upper bound void probabilities for  $\mathcal{P}$ , allowing us to deduce exponential stabilization for score functions on  $\mathcal{P}$ . This is a recurring phenomena, and it is often the case that to show exponential stabilization of statistics, it suffices to control the Palm probability content of large Euclidean balls. This opens the way towards showing that other relevant statistics of random graphs exhibit exponential stabilization on  $\mathcal{P}$ . This includes intrinsic volumes of faces of Voronoi tessellations [68, Section 10.2], edge-lengths in a radial spanning tree [70, Lemma 3.2], proximity graphs including the Gabriel graph, and global Morse critical points i.e., critical points as defined in Section 2.3.2 but without the restriction  $\mathbf{1}[R(x, \mathbf{x}) \leq r]$ .

Given locally finite  $\mathcal{X} \subset \mathbb{R}^d$  and  $k \in \mathbb{N}$ , the (undirected)  $k$ -nearest neighbors graph  $NG(\mathcal{X})$  is the graph with vertex set  $\mathcal{X}$  obtained by including an edge  $\{x, y\}$  if  $y$  is one of the  $k$  nearest neighbors of  $x$  and/or  $x$  is one of the  $k$  nearest neighbors of  $y$ . In the case of a tie we may break the tie via some pre-defined total order (say lexicographic order) on  $\mathbb{R}^d$ . For any finite  $\mathcal{X} \subset \mathbb{R}^d$  and  $x \in \mathcal{X}$ , we let  $\mathcal{E}(x)$  be the edges  $e$  in  $NG(\mathcal{X})$  which are incident to  $x$ . Defining

$$\xi_L(x, \mathcal{X}) := \frac{1}{2} \sum_{e \in \mathcal{E}(x)} |e|,$$

we write the total edge length of  $NG(\mathcal{X})$  as  $L(NG(\mathcal{X})) = \sum_{x \in \mathcal{X}} \xi_L(x, \mathcal{X})$ . Let  $\sigma^2(\xi_L)$  be as at (1.22), with  $\xi$  put to be  $\xi_L$ .

**THEOREM 2.10.** *Let  $\mathcal{P} := \mathcal{P}_\alpha$  be a stationary  $\alpha$ -determinantal point process on  $\mathbb{R}^d$  with  $-1/\alpha \in \mathbb{N}$  and a fast-decreasing kernel  $K$  as at (2.7). We have*

$$|n^{-1} \mathbb{E}L(NG(\mathcal{P}_n)) - \mathbb{E}_0 \xi_L(\mathbf{0}, \mathcal{P}) K(\mathbf{0}, \mathbf{0})| = O(n^{-1/d}), \text{ and } \lim_{n \rightarrow \infty} n^{-1} \text{Var}L(NG(\mathcal{P}_n)) = \sigma^2(\xi_L).$$

If  $\text{Var}L(NG(\mathcal{P}_n)) = \Omega(n^\nu)$  for some  $\nu \in (0, \infty)$  then as  $n \rightarrow \infty$

$$\frac{L(NG(\mathcal{P}_n)) - \mathbb{E}L(NG(\mathcal{P}_n))}{\sqrt{\text{Var}L(NG(\mathcal{P}_n))}} \xrightarrow{\mathcal{D}} N.$$

**Remark.** Theorem 2.10 extends Theorem 6.4 of [59] which is confined to Poisson input. In this context, the work [42] provides a rate of normal approximation.

*Proof.* We want to show that  $(\xi_L, \mathcal{P})$  is an admissible score and input pair of type (A2) and then apply Theorem 1.14. Note that  $\mathcal{P}$  is an admissible point process which has fast decay of correlations satisfying (1.16) and (1.17). Thus we only need to show that  $\xi_L$  is exponentially stabilizing, that  $\xi_L$  satisfies the power growth condition (1.18), and the  $p$ -moment condition (1.19). When  $d = 2$ , we show exponential stabilization of  $\xi_L$  by closely following the proof of Lemma 6.1 of [63]. This goes as follows. For each  $t > 0$ , construct six disjoint equal triangles  $T_j(t)$ ,  $1 \leq j \leq 6$ , such that  $x$  is a vertex of each triangle and each edge has length  $t$ . Let the random variable  $R$  be the minimum  $t$  such

that  $\mathcal{P}_n(T_j) \geq k + 1$  for all  $1 \leq j \leq 6$ . Notice that  $R \in [r, \infty)$  implies that there is a ball inscribed in some  $T_j(t)$  with center  $c_j$  of radius  $\gamma r$  which does not contain  $k + 1$  points. Combining Corollary 5.9 in the Appendix and the fact that  $\mathcal{P}$  has kernel  $K$ , the probability of this event satisfies

$$\begin{aligned} \mathbb{P}_{x_1, \dots, x_p}[R > r] &\leq 6\mathbb{P}_{x_1, \dots, x_p}[\mathcal{P}(B_{\gamma r}(c_1)) \leq k - 1] \leq 6\mathbb{P}_{x_1, \dots, x_p}^{\dagger}[\mathcal{P}(B_{\gamma r}(c_1)) \leq k - 1] \\ &\leq 6e^{m(2k+p-2)/8} e^{-K(\mathbf{0}, \mathbf{0})\pi\gamma^2 r^2/8}, \end{aligned}$$

that is to say that  $R$  has exponentially decaying tails. As in Lemma 6.1 of [63], we find that  $R^\xi(x, \mathcal{P}_n) := 4R$  is a radius of stabilization for  $\xi_L$ , showing that (1.15) holds with  $c = 2$ . For  $d > 2$ , we may extend these geometric arguments (cf. the proof of Theorem 6.4 of [59]) to define a random variable  $R$  serving as a radius of stabilization. Mimicking the above arguments we may likewise show that  $R$  has exponentially decaying tails.

For all  $r \in (0, \infty)$  and  $l \in \mathbb{N}$  we notice that (1.18) holds because

$$|\xi_L(x, \mathcal{X} \cap B_r(x))| \mathbf{1}[\mathcal{X}(B_r(x)) = l] \leq r \cdot \min(l, 6) \leq (cr)^l.$$

Since vertices in the  $k$ -nearest neighbors graph have degree bounded by  $kC(d)$  as in Lemma 8.4 of [75], and since each edge incident to  $x$  has length at most  $4R$ , it follows that  $|\xi_L(x, \mathcal{P}_n)| \leq k \cdot C(d) \cdot 4R$ . Since  $R$  has moments of all orders,  $(\xi_L, \mathcal{P})$  satisfies the  $p$ -moment condition (1.19) for all  $p \geq 1$ . Thus  $\xi_L$  satisfies all conditions of Theorem 1.14 and we deduce Theorem 2.10 as desired.  $\square$

**3. Proof of the fast decay (1.21) for correlations of the  $\xi$ -weighted measures.** We show the decay bound (1.21) via a factorial moment expansion for the expectation of functionals of point processes. Notice that (1.21) holds for any exponentially stabilizing score function  $\xi$  satisfying the  $p$ -moment condition (1.19) for all  $p \in [1, \infty)$  on a Poisson point process  $\mathcal{P}$ . Indeed if  $x, y \in \mathbb{R}^d$  and  $r_1, r_2 > 0$  satisfy  $r_1 + r_2 < |x - y|$  then  $\xi(x, \mathcal{P})\mathbf{1}[R^\xi(x, \mathcal{P}) \leq r_1]$  and  $\xi(y, \mathcal{P})\mathbf{1}[R^\xi(y, \mathcal{P}) \leq r_2]$  are independent random variables. This yields the fast decay (1.21) with  $k_1 = \dots = k_{p+q} = 1$  and  $\tilde{C}_n \leq c_1^n$  with  $c_1$  a constant, as in [7, Lemma 5.2]. On the other hand, if  $\mathcal{P}$  is rarified Gibbsian input and  $\xi$  is exponentially stabilizing, then [69, Lemma 3.4] shows the fast decay bound (1.21) with  $k_1 = \dots = k_{p+q} = 1$ . These methods depend on quantifying the region of spatial dependencies of Gibbsian points via exponentially decaying diameters of their ancestor clans. Such methods apparently neither extend to determinantal input nor to the zero set  $\mathcal{P}_{GEF}$  of a Gaussian entire function. On the other hand, for  $\mathcal{P}_{GEF}$  and for  $\xi \equiv 1$ , the paper [53] uses the Kac-Rice-Hammersley formula and complex analysis tools to show (1.21) with  $k_1 = \dots = k_{p+q} = 1$ . All three proofs are specific to either the underlying point process or to the score function  $\xi$ . The following more general and considerably different approach includes these results as special cases.

**3.1. Difference operators and factorial moment expansions.** We introduce some notation and collect auxiliary results required for an application of the much-needed factorial moment expansions [12, 13] for general point processes. Equip  $\mathbb{R}^d$  with a total order  $\prec$  defined using the lexicographical ordering of the polar co-ordinates. For  $\mu \in \mathcal{N}$  and  $x \in \mathbb{R}^d$ , define the measure  $\mu|_x(\cdot) := \mu(\cdot \cap \{y : y \prec x\})$ . Note that since  $\mu$  is a locally finite measure and the ordering is defined via polar co-ordinates  $\mu|_x$  is a finite measure for all  $x \in \mathbb{R}^d$ . Let  $o$  denote the null-measure i.e.,  $o(B) = 0$  for all Borel subsets

$B$  of  $\mathbb{R}^d$ . For a measurable function  $\psi : \mathcal{N} \rightarrow \mathbb{R}$ ,  $l \in \mathbb{N} \cup \{0\}$ , and  $x_1, \dots, x_l \in \mathbb{R}^d$ , we define the factorial moment expansion (FME) kernels [12, 13] as follows. For  $l \geq 1$ ,

$$(3.1) \quad D_{x_1, \dots, x_l}^l \psi(\mu) = \sum_{i=0}^l (-1)^{l-i} \sum_{J \subset \binom{[l]}{i}} \psi(\mu|_{x_*} + \sum_{j \in J} \delta_{x_j}) = \sum_{J \subset [l]} (-1)^{l-|J|} \psi(\mu|_{x_*} + \sum_{j \in J} \delta_{x_j}),$$

where  $\binom{[l]}{j}$  denotes the collection of all subsets of  $[l] := \{1, \dots, l\}$  with cardinality  $j$  and  $x_* := \min\{x_1, \dots, x_l\}$ , with the minimum taken with respect to the order  $\prec$ . For  $l = 0$ , put  $D^0 \psi(\mu) := \psi(o)$ . Note that  $D_{x_1, \dots, x_l}^l \psi(\mu)$  is a symmetric function of  $x_1, \dots, x_l$ .<sup>7</sup>

We say that  $\psi$  is  $\prec$ -continuous at  $\infty$  if for all  $\mu \in \mathcal{N}$  we have

$$\lim_{x \uparrow \infty} \psi(\mu|_x) = \psi(\mu).$$

We first recall the FME expansion proved in [12, cf. Theorem 3.2] for dimension one and then extended to higher-dimensions [13, cf. Theorem 3.1]. Recall that  $\mathbb{E}_{y_1, \dots, y_l}^!$  denote expectations with respect to reduced Palm probabilities.

**THEOREM 3.1.** *Let  $\mathcal{P}$  be a simple point process and let  $\psi : \mathcal{N} \rightarrow \mathbb{R}$  be  $\prec$ -continuous at  $\infty$ . Assume that for all  $l \geq 1$*

$$(3.2) \quad \int_{\mathbb{R}^{dl}} \mathbb{E}_{y_1, \dots, y_l}^! [|D_{y_1, \dots, y_l}^l \psi(\mathcal{P})|] \rho^{(l)}(y_1, \dots, y_l) dy_1 \dots dy_l < \infty$$

and

$$(3.3) \quad \frac{1}{l!} \int_{\mathbb{R}^{dl}} \mathbb{E}_{y_1, \dots, y_l}^! [D_{y_1, \dots, y_l}^l \psi(\mathcal{P})] \rho^{(l)}(y_1, \dots, y_l) dy_1 \dots dy_l \rightarrow 0 \text{ as } l \rightarrow \infty.$$

Then  $\mathbb{E}[\psi(\mathcal{P})]$  has the following factorial moment expansion

$$(3.4) \quad \mathbb{E}[\psi(\mathcal{P})] = \psi(o) + \sum_{l=1}^{\infty} \frac{1}{l!} \int_{\mathbb{R}^{dl}} D_{y_1, \dots, y_l}^l \psi(o) \rho^{(l)}(y_1, \dots, y_l) dy_1 \dots dy_l.$$

Consider now admissible pairs  $(\xi, \mathcal{P})$  of type (A1) or (A2) and  $x_1, \dots, x_p \in \mathbb{R}^d$ . The proof of (1.21) given in the next sub-section is based on the FME expansion for  $\mathbb{E}_{x_1, \dots, x_p}[\psi(\mathcal{P}_n)]$ , where  $\psi(\mu)$  is the following product of the score functions

$$(3.5) \quad \psi(\mu) := \psi_{k_1, \dots, k_p}(x_1, \dots, x_p; \mu) := \prod_{i=1}^p \xi(x_i, \mu)^{k_i}$$

with  $k_1, \dots, k_p \geq 1$ . However, under  $\mathbb{P}_{x_1, \dots, x_p}$  the point process  $\mathcal{P}_n$  has fixed atoms at  $x_1, \dots, x_p$ , which complicates the form of its factorial moment measures. It is more handy to consider these points as parameters of the following modified functional

$$(3.6) \quad \psi^! (\mu) := \psi_{k_1, \dots, k_p}^! (x_1, \dots, x_p; \mu) := \prod_{i=1}^p \xi \left( x_i, \mu + \sum_{j=1}^p \delta_{x_j} \right)^{k_i}$$

<sup>7</sup>For  $x_l \prec x_{l-1} \prec \dots \prec x_1$  the functional  $D_{x_1, \dots, x_l}^l \psi(\mu)$  is equal to the iterated difference operator:  $D_{x_1}^1 \psi(\mu) = \psi(\mu|_{x_1} + \delta_{x_1}) - \psi(\mu|_{x_1})$ ,  $D_{x_1, \dots, x_l}^l \psi(\mu) = D_{x_l}^1 (D_{x_1, \dots, x_{l-1}}^{l-1} \psi(\mu))$ .



and to not count points  $x_1, \dots, x_p$  in  $\mathcal{P}$ , i.e., consider  $\mathcal{P}$  under the reduced Palm probabilities  $\mathbb{P}_{x_1, \dots, x_p}^\dagger$ . Obviously  $\mathbb{E}_{x_1, \dots, x_p}[\psi(\mathcal{P}_n)] = \mathbb{E}_{x_1, \dots, x_p}^\dagger[\psi^\dagger(\mathcal{P}_n)]$  and the latter expectation is more suitable for FME expansion with respect to the correlation functions  $\rho_{x_1, \dots, x_p}^{(l)}(y_1, \dots, y_l)$  of  $\mathcal{P}$  with respect to the Palm probabilities  $\mathbb{P}_{x_1, \dots, x_p}^\dagger$ . The following consequence of Theorem 3.1 allows us to use FME expansions to prove (1.21).

LEMMA 3.2. *Assume that either (i)  $(\xi, \mathcal{P})$  is an admissible score and input pair of type (A1) or (ii)  $(\xi, \mathcal{P})$  satisfies the power growth condition (1.18), with  $\xi$  having a radius of stabilization satisfying  $\sup_{x \in \mathcal{P}} R^\xi(x, \mathcal{P}) \leq r$  a.s. for some  $r \in (1, \infty)$  and with  $\mathcal{P}$  having exponential moments. Then for distinct  $x_1, \dots, x_p \in \mathbb{R}^d$ , non-negative integers  $k_1, \dots, k_p$  and  $n \leq \infty$  the functional  $\psi^\dagger$  at (3.6) admits the FME*

$$(3.7) \quad \begin{aligned} \mathbb{E}_{x_1, \dots, x_p}[\psi_{k_1, \dots, k_p}(x_1, \dots, x_p; \mathcal{P}_n)] &= \mathbb{E}_{x_1, \dots, x_p}^\dagger[\psi_{k_1, \dots, k_p}^\dagger(x_1, \dots, x_p; \mathcal{P}_n)] \\ &= \psi_{k_1, \dots, k_p}^\dagger(x_1, \dots, x_p; o) \\ &\quad + \sum_{l=1}^{\infty} \frac{1}{l!} \int_{\mathbb{R}^{dl}} D_{y_1, \dots, y_l}^l \psi_{k_1, \dots, k_p}^\dagger(x_1, \dots, x_p; o) \rho_{x_1, \dots, x_p}^{(l)}(y_1, \dots, y_l) dy_1 \dots dy_l. \end{aligned}$$

When  $(\xi, \mathcal{P})$  is of type (A1), the series (3.7) has at most  $(k-1) \sum_{i=1}^p k_i$  non-zero terms, where  $k$  is as in (1.13).

PROOF. Throughout we fix non-negative integers  $k_1, \dots, k_p$  and suppress them when writing  $\psi^\dagger$ ; i.e.,  $\psi^\dagger(x_1, \dots, x_p; \mathcal{P}_n) := \psi_{k_1, \dots, k_p}^\dagger(x_1, \dots, x_p; \mathcal{P}_n)$ . The bounded radius of stabilization for  $\xi$  implies  $\psi^\dagger$  is  $\prec$ -continuous at  $\infty$ .

Consider first  $\psi^\dagger$  at (3.6) with  $\xi$  as in case (ii); later we consider the simpler case (i). We show the validity of the expansion (3.7) as follows. Let  $y_1, \dots, y_l \in \mathbb{R}^d$ . The difference operator  $D_{y_1, \dots, y_l}^l$  vanishes as soon as  $y_k \notin \cup_{i=1}^p B_r(x_i)$  for some  $k \in \{1, \dots, l\}$ , that is to say

$$(3.8) \quad D_{y_1, \dots, y_l}^l \psi^\dagger(x_1, \dots, x_p; \mu) = 0.$$

To prove this, set  $\mu_J := \mu|_{y_*} + \sum_{j \in J} \delta_{y_j}$  for  $J \subset [l]$  and  $y_* := \min\{y_1, \dots, y_l\}$ , with the minimum taken with respect to  $\prec$  order. From (3.1) we obtain

$$\begin{aligned} &D_{y_1, \dots, y_l}^l \psi^\dagger(x_1, \dots, x_p; \mu) \\ &= \sum_{J \subset [l], k \notin J} (-1)^{l-|J|} \psi^\dagger(x_1, \dots, x_p; \mu_J) + \sum_{J \subset [l], k \in J} (-1)^{l-|J|-1} \psi^\dagger(x_1, \dots, x_p; \mu_{J \cup \{k\}}) = 0, \end{aligned}$$

where the last equality follows by noting that for  $J \subset [l]$  with  $k \notin J$ ,  $\psi^\dagger(x_1, \dots, x_p; \mu_J) = \psi^\dagger(x_1, \dots, x_p; \mu_{J \cup \{k\}})$  because  $R^\xi(x, \mathcal{P}) \in [1, r]$  by assumption.

Henceforth we put

$$(3.9) \quad K_p := \sum_{i=1}^p k_i, \quad K_q := \sum_{i=1}^q k_{p+i}, \quad K := \sum_{i=1}^{p+q} k_i.$$

Consider now  $y_1, \dots, y_l \in \cup_{i=1}^p B_r(x_i)$ . For  $J \subset [l]$ , from  $1 \leq R^\xi(x, \mathcal{P}) \leq r$  and (1.18) we have



$$\begin{aligned}
\psi^!(x_1, \dots, x_p; \mu_J) &\leq \prod_{i=1}^p \|h\|_\infty^{k_i} (\mu(\cup_{i=1}^p B_r(x_i)) + |J| + p)^{k_i(k-1)} \\
(3.14) \qquad \qquad \qquad &\leq \|h\|_\infty^{K_p} (\mu(\cup_{i=1}^p B_r(x_i)) + |J| + p)^{K_p(k-1)}.
\end{aligned}$$

The number of subsets of  $[l]$  is  $2^l$  and so by (3.1), we obtain

$$\begin{aligned}
|D_{y_1, \dots, y_l}^l \psi^!(x_1, \dots, x_p; \mu)| &\leq \|h\|_\infty^{K_p} \sum_{J \subset [l]} (\mu(\cup_{i=1}^p B_r(x_i)) + |J| + p)^{K_p(k-1)} \\
(3.15) \qquad \qquad \qquad &\leq \|h\|_\infty^{K_p} 2^l (\mu(\cup_{i=1}^p B_r(x_i)) + l + p)^{K_p(k-1)}.
\end{aligned}$$

Consider  $\psi^!(x_1, \dots, x_p; \mathcal{P}_n)$  with  $\psi^!$  defined as above. Using the refined Campbell theorem (1.9), the bound (3.15), and following the calculations as in (3.12), we obtain

$$\begin{aligned}
&\frac{1}{l!} \int_{\mathbb{R}^{dl}} (\mathbb{E}_{x_1, \dots, x_p}^! \rho_{y_1, \dots, y_l}^l [D_{y_1, \dots, y_l}^l \psi^!(x_1, \dots, x_p; \mathcal{P}_n)] \rho_{x_1, \dots, x_p}^{(l)}(y_1, \dots, y_l) dy_1 \dots dy_l \\
&\leq \|h\|_\infty^{K_p} 2^l \mathbb{E}_{x_1, \dots, x_p} [\mathcal{P}(\cup_{i=1}^p B_r(x_i))^l (\mathcal{P}(\cup_{i=1}^p B_r(x_i)) + l + p)^{K_p(k-1)}].
\end{aligned}$$

Since  $\mathcal{P}$  has all moments under the Palm measure (see Remark (ii) at the beginning of Section 2.1), the finiteness of the last term and hence the validity of the condition (3.2) for  $l \in [1, K_p(k-1)]$  follows. This justifies the FME expansion (3.7), with finitely many non-zero terms, when  $\psi^!$  is the product of score functions of class (A1).  $\square$

**3.2. Proof of Theorem 1.11.** First assume that  $(\xi, \mathcal{P})$  is of class (A2). Later we consider the simpler case that  $(\xi, \mathcal{P})$  is of class (A1). For fixed  $p, q, k_1, \dots, k_{p+q} \in \mathbb{N}$ , consider correlation functions  $m^{(k_1, \dots, k_{p+q})}(x_1, \dots, x_{p+q}; n)$ ,  $m^{(k_1, \dots, k_p)}(x_1, \dots, x_p; n)$ ,  $m^{(k_{p+1}, \dots, k_{p+q})}(x_{p+1}, \dots, x_{p+q}; n)$  of the  $\xi$ -weighted measures at (1.6). We abbreviate  $\psi_{k_1, \dots, k_p}(x_1, \dots, x_p; \mu)$  by  $\psi(x_1, \dots, x_p; \mu)$  as at (3.5), and similarly for  $\psi(x_{p+1}, \dots, x_{p+q}; \mu)$  and  $\psi(x_1, \dots, x_{p+q}; \mu)$ .

Given  $x_1, \dots, x_{p+q} \in W_n$  we recall  $s := d(\{x_1, \dots, x_p\}, \{x_{p+1}, \dots, x_{p+q}\})$ . Without loss of generality we assume  $s \in (4, \infty)$ . Recalling the definition of  $b$  at (1.17) and that of  $K$  at (3.9), we may assume without loss of generality that  $b \in (0, d)$ . Put

$$(3.16) \qquad t := t(s) := \left(\frac{s}{4}\right)^{b(1-a)/(2(K+d))},$$

where  $a \in [0, 1)$  is at (1.16). Since  $s \in (4, \infty)$  and  $K \geq 2$ , we easily have  $t \in (1, s/4)$ . Given stabilization radii  $R^\xi(x_i, \mathcal{P}_n)$ ,  $1 \leq i \leq p+q$ , we put

$$\tilde{\xi}(x_i, \mathcal{P}_n) := \xi(x_i, \mathcal{P}_n \cap B_{R^\xi(x_i, \mathcal{P}_n)}(x)) \mathbf{1}[R^\xi(x_i, \mathcal{P}_n) \leq t]$$

considered under  $\mathbb{E}_{x_1, \dots, x_p}$ . We denote by  $\tilde{m}^{(k_1, \dots, k_p)}$  the correlation functions of the  $\tilde{\xi}$ -weighted atomic measure, that is

$$\tilde{m}^{(k_1, \dots, k_p)}(x_1, \dots, x_p; n) := \mathbb{E}_{x_1, \dots, x_p} [\tilde{\xi}(x_1, \mathcal{P}_n)^{k_1} \dots \tilde{\xi}(x_p, \mathcal{P}_n)^{k_p}] \rho^{(p)}(x_1, \dots, x_p).$$



and also satisfy the power-growth condition (1.18) since  $|\tilde{\xi}| \leq |\xi|$ . Thus the pair  $(\tilde{\xi}, \mathcal{P})$  satisfies the assumptions of Lemma 3.2. The corresponding version of  $\tilde{\psi}$ , accounting for the fixed atoms of  $\mathcal{P}_n$  is

$$\tilde{\psi}^l(x_1, \dots, x_p; \mu) := \prod_{i=1}^p \tilde{\xi}(x_i, \mu + \sum_{i=1}^p \delta_{x_i})^{k_i}$$

and similarly for  $\tilde{\psi}^l(x_{p+1}, \dots, x_q; \mathcal{P}_n)$  and  $\tilde{\psi}^l(x_1, \dots, x_{p+q}; \mathcal{P}_n)$ .

Put  $B_{t,n}(x_i) := B_t(x_i) \cap W_n$ . Applying (3.7), the multiplicative identity (2.11) and (3.8), we obtain

$$\begin{aligned} \tilde{m}^{(k_1, \dots, k_{p+q})}(x_1, \dots, x_{p+q}) &= \mathbb{E}_{x_1, \dots, x_{p+q}}^l [\tilde{\psi}^l(x_1, \dots, x_{p+q}; \mathcal{P}_n)] \rho^{(p+q)}(x_1, \dots, x_{p+q}) \\ &= \sum_{l=0}^{\infty} \frac{1}{l!} \int_{(W_n)^l} D_{y_1, \dots, y_l}^l \tilde{\psi}^l(o) \rho^{(l+p+q)}(x_1, \dots, x_{p+q}, y_1, \dots, y_l) dy_1 \dots dy_l \\ &= \sum_{l=0}^{\infty} \frac{1}{l!} \int_{(\cup_{i=1}^{p+q} B_{t,n}(x_i))^l} D_{y_1, \dots, y_l}^l \tilde{\psi}^l(o) \rho^{(l+p+q)}(x_1, \dots, x_{p+q}, y_1, \dots, y_l) dy_1 \dots dy_l. \end{aligned}$$

Applying (3.1) when  $\mu$  is the null measure, this gives for  $\alpha^{(p+q)}$  almost all  $x_1, \dots, x_{p+q}$

$$\begin{aligned} \tilde{m}^{(k_1, \dots, k_{p+q})}(x_1, \dots, x_{p+q}) &= \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{j=0}^l \frac{l!}{j!(l-j)!} \int_{(\cup_{i=1}^p B_{t,n}(x_i))^j \times (\cup_{i=1}^q B_{t,n}(x_{p+i}))^{l-j}} D_{y_1, \dots, y_l}^l \tilde{\psi}^l(x_1, \dots, x_{p+q}; o) \\ &\quad \times \rho^{(l+p+q)}(x_1, \dots, x_{p+q}, y_1, \dots, y_l) dy_1 \dots dy_l \\ &= \sum_{l=0}^{\infty} \sum_{j=0}^l \frac{1}{j!(l-j)!} \int_{(\cup_{i=1}^p B_{t,n}(x_i))^j \times (\cup_{i=1}^q B_{t,n}(x_{p+i}))^{l-j}} \end{aligned}$$

(3.20)

$$\times \sum_{J \subset [l]} (-1)^{l-|J|} \tilde{\psi}^l(x_1, \dots, x_{p+q}; \sum_{j \in J} \delta_{y_j}) \times \rho^{(l+p+q)}(x_1, \dots, x_{p+q}, y_1, \dots, y_l) dy_1 \dots dy_l.$$

To compare the  $(p+q)$ th correlation functions of the  $\xi$ -weighted measures with the product of their  $p$ th and  $q$ th correlation functions, we shall use the fact that  $R^{\tilde{\xi}}(x_i, \mathcal{P}_n) \in (0, t]$  (cf. Lemma 5.2) implies the following factorization, which holds for  $y_1, \dots, y_j \in \cup_{i=1}^p B_t(x_i)$  and  $y_{j+1}, \dots, y_l \in \cup_{i=1}^q B_t(x_{p+i})$ , with  $t \in (1, s/4)$  (making  $\cup_{i=1}^p B_t(x_i)$  and  $\cup_{i=1}^q B_t(x_{p+i})$  disjoint):

$$(3.21) \quad \tilde{\psi}^l(x_1, \dots, x_{p+q}; \sum_{i=1}^l \delta_{y_i}) = \tilde{\psi}^l(x_1, \dots, x_p; \sum_{i=1}^j \delta_{y_i}) \tilde{\psi}^l(x_{p+1}, \dots, x_{p+q}; \sum_{i=j+1}^l \delta_{y_i}).$$

Using the expansion (3.7) along with (3.21), we next derive an expansion for the product of  $p$ th and  $q$ th correlation functions of the  $\xi$ -weighted measures. Recalling the multiplicative identity (2.11) as well as the identity  $\mathbb{E}_{x_1, \dots, x_p} [\psi(\mathcal{P}_n)] = \mathbb{E}_{x_1, \dots, x_p}^l [\psi^l(\mathcal{P}_n)]$  (cf. (3.6)), we obtain

$$\begin{aligned} \tilde{m}^{(k_1, \dots, k_p)}(x_1, \dots, x_p) \tilde{m}^{(k_{p+1}, \dots, k_q)}(x_{p+1}, \dots, x_{p+q}) \\ = \mathbb{E}_{x_1, \dots, x_p}^l [\tilde{\psi}^l(x_1, \dots, x_p; \mathcal{P}_n)] \mathbb{E}_{x_{p+1}, \dots, x_{p+q}}^l [\tilde{\psi}^l(x_{p+1}, \dots, x_{p+q}; \mathcal{P}_n)] \\ \quad \times \rho^{(p)}(x_1, \dots, x_p) \rho^{(q)}(x_{p+1}, \dots, x_{p+q}) \end{aligned}$$



$$(3.25) \leq \phi\left(\frac{s}{2}\right) \sum_{l=0}^{\infty} \sum_{j=0}^l \frac{C_{l+p+q}}{j!(l-j)!} \int_{(\cup_{i=1}^p B_t(x_i))^j \times (\cup_{i=1}^q B_t(x_{p+i}))^{l-j}} 2^l (\hat{c}t)^{jK_p + (l-j)K_q + K} dy_1 \dots dy_l.$$

Consequently, bounding  $K_p$  and  $K_q$  by  $K$  we obtain

$$(3.26) \begin{aligned} & |\tilde{m}^{(k_1, \dots, k_{p+q})}(x_1, \dots, x_{p+q}) - \tilde{m}^{(k_1, \dots, k_p)}(x_1, \dots, x_p) \tilde{m}^{(k_{p+1}, \dots, k_q)}(x_{p+1}, \dots, x_{p+q})| \\ & \leq \phi\left(\frac{s}{2}\right) \sum_{l=0}^{\infty} C_{l+p+q} 2^l (\hat{c}t)^{(l+1)K} ((p+q)\theta_d t^d)^l \sum_{j=0}^l \frac{1}{j!(l-j)!} \\ & \leq \phi\left(\frac{s}{2}\right) \sum_{l=0}^{\infty} \frac{C_{l+p+q}}{l!} 4^l (\hat{c}t)^{(l+1)K} ((p+q)\theta_d t^d)^l \leq \phi\left(\frac{s}{2}\right) \sum_{l=0}^{\infty} \frac{C_{l+K}}{l!} 4^l (\hat{c}t)^{(l+1)K} (K\theta_d t^d)^l, \end{aligned}$$

where  $\theta_d := \pi^{d/2}/\Gamma(d/2 + 1)$  is the volume of the unit ball in  $\mathbb{R}^d$  and where the last inequality uses  $p+q \leq K$ . The bound (1.16) yields  $C_{l+K} = O((l+K)^{a(l+K)})$ . Thus there are constants  $c_1, c_2$  and  $c_3$  depending only on  $a, d$  and  $K$  such that

$$\sum_{l=0}^{\infty} \frac{C_{l+K}}{l!} 4^l (\hat{c}t)^{(l+1)K} (K\theta_d t^d)^l \leq t^K \sum_{l=0}^{\infty} \frac{c_1 c_2^l l^{c_3} (t^{K+d})^l \cdot l^{al}}{l!}.$$

By Stirling's formula, there are constants  $c_4, c_5$  and  $c_6$  depending only on  $a, d$  and  $K$  such that

$$t^K \sum_{l=0}^{\infty} \frac{C_{l+K}}{l!} 4^l (\hat{c}t)^{lK} (K\theta_d t^d)^l \leq t^K \sum_{l=0}^{\infty} \frac{c_4 c_5^l l^{c_6} (t^{K+d})^l}{([l(1-a)])!},$$

where for  $r \in \mathbb{R}$ ,  $[r]$  is the greatest integer less than  $r$ . We compute

$$(3.27) \begin{aligned} & t^K \sum_{l=0}^{\infty} \frac{C_{l+K}}{l!} 4^l (\hat{c}t)^{lK} (K\theta_d t^d)^l \leq t^K \sum_{n=0}^{\infty} \sum_{\{l: [l(1-a)] = n\}} \frac{c_4 c_5^l l^{c_6} (t^{K+d})^l}{n!} \\ & \leq t^K \sum_{n=0}^{\infty} \frac{c_4 c_5^n n^{c_6} (t^{K+d})^{(n+1)/(1-a)}}{(1-a)n!} \leq c_7 \exp(c_8 t^{(K+d)/(1-a)}) \end{aligned}$$

where  $c_7$  and  $c_8$  depend only on  $a, d$  and  $K$ .

Recalling from (3.16) that  $t := (s/4)^{b(1-a)/(2(K+d))}$  we obtain

$$\sum_{l=0}^{\infty} \frac{C_{l+K}}{l!} 4^l (\hat{c}t)^{(l+1)K} (K\theta_d t^d)^l \leq c_7 \exp\left(c_8 \left(\frac{s}{4}\right)^{\frac{b}{2}}\right).$$

By (1.17), there is a constant  $c_9$  depending only on  $a$  such that for all  $s$  we have  $\phi(s) \leq c_9 \exp(-s^b/c_9)$ . Combining this with (3.26) and (3.27) gives

$$|\tilde{m}^{(k_1, \dots, k_{p+q})} - \tilde{m}^{(k_1, \dots, k_p)} \tilde{m}^{(k_{p+1}, \dots, k_q)}| \leq c_7 c_9 \exp\left(\frac{-(s/2)^b}{c_9} + c_8 \left(\frac{s}{4}\right)^{\frac{b}{2}}\right).$$

This along with (3.19) establishes (1.21) when  $(\xi, \mathcal{P})$  is an admissible pair of class (A2).

Now we establish (1.21) when  $(\xi, \mathcal{P})$  is of class (A1). Let  $k$  be as in (1.13). Follow the arguments for case (A2) word for word using that  $\sup_{x \in \mathcal{P}} R^\xi(x, \mathcal{P}) \leq r$ . Notice that for  $l \in ((k-1)K, \infty)$





$$\begin{aligned}
\text{Var}\mu_n^\xi(f) &= \mathbb{E} \sum_{x \in \mathcal{P}_n} f(n^{-1/d}x)^2 \xi^2(x, \mathcal{P}_n) \\
&\quad + \mathbb{E} \sum_{x, y \in \mathcal{P}_n, x \neq y} f(n^{-1/d}x) f(n^{-1/d}y) \xi(x, \mathcal{P}_n) \xi(y, \mathcal{P}_n) - \left( \mathbb{E} \sum_{x \in \mathcal{P}_n} f(n^{-1/d}x) \xi(x, \mathcal{P}_n) \right)^2 \\
(4.1) \quad &= \int_{W_n} f(n^{-1/d}u)^2 \mathbb{E}_u(\xi^2(u, \mathcal{P}_n)) \rho^{(1)}(u) \, du \\
(4.2) \quad &+ \int_{W_n \times W_n} f(n^{-1/d}u) f(n^{-1/d}v) (m_{(2)}(u, v; n) - m_{(1)}(u; n) m_{(1)}(v; n)) \, dudv.
\end{aligned}$$

Since  $\xi$  satisfies the  $p$ -moment condition (1.19) for  $p > 2$ , we have that  $\xi^2$  satisfies the  $p$ -moment condition for  $p > 1$ . Also,  $\xi$  and  $\xi^2$  have the same radius of stabilization. Thus, the proof of expectation asymptotics, with  $\xi$  replaced by  $\xi^2$ , shows that the first term in (4.1), multiplied by  $n^{-1}$ , converges to

$$\mathbb{E}_0 \xi^2(\mathbf{0}, \mathcal{P}) \rho^{(1)}(\mathbf{0}) \int_{W_1} f(x)^2 \, dx;$$

cf. expectation asymptotics (1.23). Setting  $x = n^{-1/d}u$  and  $z = v - u = v - n^{1/d}x$ , the second term in (4.2), multiplied by  $n^{-1}$ , may be rewritten as

$$\begin{aligned}
(4.3) \quad &\int_{W_1} \int_{W_n - n^{1/d}x} f(x + n^{-1/d}z) f(x) \\
&\quad \times [m_{(2)}(n^{1/d}x, n^{1/d}x + z; n) - m_{(1)}(n^{1/d}x; n) m_{(1)}(n^{1/d}x + z; n)] \, dz dx.
\end{aligned}$$

Setting  $\mathcal{P}_n^x := \mathcal{P} \cap (W_n - n^{1/d}x)$ , the translation invariance of  $\xi$  and stationarity of  $\mathcal{P}$  yields

$$\begin{aligned}
m_{(2)}(n^{1/d}x, n^{1/d}x + z; n) &= m_{(2)}(\mathbf{0}, z; \mathcal{P}_n^x) \\
m_{(1)}(n^{1/d}x; n) &= m_{(1)}(\mathbf{0}; \mathcal{P}_n^x) \\
m_{(1)}(n^{1/d}x + z; n) &= m_{(1)}(z; \mathcal{P}_n^x).
\end{aligned}$$

Putting aside for the moment technical details, one expects that the above moments converge to  $m_{(2)}(\mathbf{0}, z)$ ,  $m_{(1)}(\mathbf{0})$  and  $m_{(1)}(z) = m_{(1)}(\mathbf{0})$ , respectively, when  $n \rightarrow \infty$ . Moreover, splitting the inner integral in (4.3) into two terms

$$(4.4) \quad \int_{W_n - n^{1/d}x} (\dots) \, dz = \int_{W_n - n^{1/d}x} \mathbf{1}[|z| \leq M] (\dots) \, dz + \int_{W_n - n^{1/d}x} \mathbf{1}[|z| > M] (\dots) \, dz$$

for any  $M > 0$ , we see (at least when  $f$  is continuous) that the first term in the right-hand side of (4.4) converges to the desired value

$$\int_{\mathbb{R}^d} f(x)^2 [m_{(2)}(\mathbf{0}, z) - m_{(1)}(\mathbf{0})^2] \, dz$$

when first  $n \rightarrow \infty$  and then  $M \rightarrow \infty$ . By the fast decay of the second-order correlations of the  $\xi$ -weighted measures, i.e., by (1.21) with  $p = q = k_1 = k_2 = 1$  and all  $n \in \mathbb{N} \cup \{\infty\}$ , the absolute value of the second term in (4.4) can be bounded uniformly in  $n$  by



The  $p$ th moment of  $X_n$  and  $X$  under  $\mathbb{E}_{\mathbf{0},z}$  can be bounded by  $\tilde{M}_p$  using the  $p$ -moment condition (1.19) with  $p > 2$  as in (4.5). Stabilization (1.14) with  $l = 2$  gives

(4.8)

$$\mathbb{P}_{\mathbf{0},z}(X_n \neq X) \leq \mathbb{P}_{\mathbf{0},z}((\max(R^\xi(u, \mathcal{P}), R^\xi(u, \mathcal{P}_n)) > n^{1/d}d(x, \partial W_1)) \leq 2\varphi(a_2 n^{1/d}d(x, \partial W_1))$$

with the right-hand side converging to 0 for all  $x \notin \partial W_1$ . This proves that  $\mathbb{E}_{\mathbf{0},z}(X_n - X)^2$  and (by the very same arguments)  $\mathbb{E}_{\mathbf{0},z}(Y_n - Y)^2$  converge to 0 as  $n \rightarrow \infty$  for all  $x \notin \partial W_1$ . Concluding this part of the proof, we have shown that the expression in (4.7) converges to 0 and thus  $m_{(2)}(\mathbf{0}, z; \mathcal{P}_n^x)$  converges to  $m_{(2)}(\mathbf{0}, z)$ . Using similar arguments, we derive

$$\begin{aligned} |m_{(1)}(\mathbf{0}, \mathcal{P}_n^x) - m_{(1)}(\mathbf{0})| &= |\mathbb{E}_{\mathbf{0}}(X_n) - \mathbb{E}_{\mathbf{0}}(X)|\rho^{(1)}(\mathbf{0}) \\ &\leq \kappa_1((\mathbb{E}_{\mathbf{0}}(X_n)^2)^{1/2} + (\mathbb{E}_{\mathbf{0}}(X^2))^{1/2})(\mathbb{P}_{\mathbf{0}}(X_n \neq X))^{1/2}, \end{aligned}$$

by the  $p$ -moment condition (1.19) and the stabilization property (1.14) for  $p = 1$  one can show that  $m_{(1)}(\mathbf{0}, \mathcal{P}_n^x)$  converges to  $m_{(1)}(\mathbf{0})$  uniformly in  $x$  for all  $x \in W_1 \setminus \partial W_1$ . Exactly the same arguments assure convergence of  $m_{(1)}(z, \mathcal{P}_n^x)$  to  $m_{(1)}(z) = m_{(1)}(\mathbf{0})$ . This concludes the proof of Lemma 4.1.  $\square$

In order to complete the proof of variance asymptotics for general  $f \in \mathcal{B}(W_1)$  (not necessarily continuous) we use arguments borrowed from the proof of [59, Theorem 2.1]. Recall that  $x \in W_1$  is a Lebesgue point for  $f$  if  $(\text{Vol}_d B_\epsilon(x))^{-1} \int_{B_\epsilon(x)} |f(z) - f(x)| dz \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Denote by  $\mathcal{C}_f$  all Lebesgue points of  $f$  in  $W_1$ . By the Lebesgue density theorem almost every  $x \in W_1$  is a Lebesgue point of  $f$  and thus for any  $M > 0$  and  $n$  large enough the double integral in (4.3) is equal to

$$\begin{aligned} &\int_{W_1} \mathbf{1}[x \in \mathcal{C}_f] f(x) \int_{W_n - n^{1/d}x} f(x + n^{-1/d}z) h_n^\xi(x, z) dz dx \\ &= \int_{W_1} \mathbf{1}[x \in \mathcal{C}_f] f(x) \int_{|z| \leq M} f(x + n^{-1/d}z) h_n^\xi(x, z) dz dx \\ &\quad + \int_{W_1} \mathbf{1}[x \in \mathcal{C}_f] f(x) \int_{W_n - n^{1/d}x} \mathbf{1}(|z| > M) f(x + n^{-1/d}z) h_n^\xi(x, z) dz dx. \end{aligned}$$

As already explained, by the fast decay of the second-order correlations of the  $\xi$ -weighted measures, the second term converges to 0 as first  $n \rightarrow \infty$  and then  $M \rightarrow \infty$ . Considering the first term, by the uniform boundedness of  $h_n^\xi(x, z)$ , using the dominated convergence theorem, it is enough to prove for any Lebesgue point  $x$  of  $f$  and fixed  $M$  that

$$\lim_{n \rightarrow \infty} \int_{|z| < M} h_n^\xi(x, z) f(x + n^{-1/d}z) dz = f(x) \int_{|z| < M} h_\infty^\xi(x, z) dz.$$

In this regard notice that

$$\begin{aligned} &\int_{|z| < M} |h_n^\xi(x, z) f(x + n^{-1/d}z) - h_\infty^\xi(x, z) f(x)| dz \\ &\leq \int_{|z| < M} C_h \times |f(x + n^{-1/d}z) - f(x)| + |h_n^\xi(x, z) - h_\infty^\xi(x, z)| \times \|f\|_\infty dz \\ &\leq C_h n \int_{|z| < n^{-1/d}M} |f(x + z) - f(x)| dz + \|f\|_\infty \times \int_{|z| < M} |h_n^\xi(x, z) - h_\infty^\xi(x, z)| dz. \end{aligned}$$



Applying this identity to the right hand side of (4.10), then multiplying (4.10) by  $n^{1/d}$  and taking limits we obtain

$$(4.12) \quad \lim_{n \rightarrow \infty} n^{-(d-1)/d} \text{Var} \widehat{H}_n^\xi(\mathcal{P}) = - \lim_{n \rightarrow \infty} n^{-(d-1)/d} \int_{\mathbb{R}^d} c(\mathbf{0}, z) \gamma_{W_n}(z) dz.$$

As in [45], we have  $n^{-(d-1)/d} \gamma_{W_n}(z) \leq C|z|$ , and therefore again, by the fast-decreasing property of  $c(\mathbf{0}, z)$  we conclude that  $n^{-(d-1)/d} c(\mathbf{0}, z) \gamma_{W_n}(z)$  is dominated by an integrable function of  $z$ . Also, as in [45, Lemma 1], for all  $z \in \mathbb{R}^d$  we have  $\lim_{n \rightarrow \infty} n^{-(d-1)/d} \gamma_{W_n}(z) = \gamma(z)$ . The dominated convergence theorem yields (1.31) as desired,

$$\lim_{n \rightarrow \infty} n^{-(d-1)/d} \text{Var} \widehat{H}_n^\xi(\mathcal{P}) = - \int_{\mathbb{R}^d} c(\mathbf{0}, z) \gamma(z) dz. \quad \square$$

### 4.3. First proof of the central limit theorem.

4.3.1. *The method of cumulants.* We use the method of cumulants to prove Theorem 1.13. We shall define cumulants precisely in Section 4.3.2. Write  $\bar{\mu}_n^\xi$  for the centered measure  $\mu_n^\xi - \mathbb{E}\mu_n^\xi$  and recall that we write  $\langle f, \mu \rangle$  for  $\int f d\mu$ . The guiding principle is that as soon as the  $k$ th order cumulants  $C_n^k$  for  $(\text{Var} \langle f, \mu_n^\xi \rangle)^{-1/2} \langle f, \bar{\mu}_n^\xi \rangle$  vanish as  $n \rightarrow \infty$  for  $k$  large, then

$$(4.13) \quad (\text{Var} \langle f, \mu_n^\xi \rangle)^{-1/2} \langle f, \bar{\mu}_n^\xi \rangle \xrightarrow{\mathcal{D}} N.$$

We establish the vanishing of  $C_n^k$  for  $k$  large by showing that the fast decay of correlation functions for the  $\xi$ -weighted measures at (1.5) implies volume order growth (i.e., growth of order  $O(n)$ ) for the  $k$ th order cumulant for  $\langle f, \bar{\mu}_n^\xi \rangle$ ,  $k \geq 2$ , and then use the assumption  $\text{Var} \langle f, \mu_n^\xi \rangle = \Omega(n^\nu)$ .

*Our approach.* The  $O(n)$  growth of the  $k$ th order cumulant for  $\langle f, \bar{\mu}_n^\xi \rangle$  is established by controlling the growth of  $k$ th order cumulant measures for  $\mu_n^\xi$ , here denoted by  $c_n^k$ , and which are defined analogously to moment measures. We first prove a general result (see (4.18) and (4.19) below) showing that integrals of the cumulant measures  $c_n^k$  may be controlled by a finite sum of integrals of so-called  $(S, T)$  semi-cluster measures, where  $(S, T)$  is a generic partition of  $\{1, \dots, k\}$ . This result holds for any  $\mu_n^\xi$  of the form (1.4) and depends neither on choice of input  $\mathcal{P}$  nor on the localization properties of  $\xi$ . Semi-cluster measures for  $\mu_n^\xi$  have the appealing property that they involve differences of measures on product spaces with product measures, and thus their Radon-Nikodym derivatives involve differences of correlation functions of the  $\xi$ -weighted measures.

In general, bounds on cumulant measures in terms of semi-cluster measures are not terribly informative. However, when  $\xi$ , together with  $\mathcal{P}$ , satisfy moment bounds and fast decay of correlations (1.21), then the situation changes. First, integrals of  $(S, T)$  semi-cluster measures on properly chosen subsets  $W(S, T)$  of  $W_n^k$ , with  $(S, T)$  ranging over partitions of  $\{1, \dots, k\}$ , exhibit  $O(n)$  growth. This is because the subsets  $W(S, T)$  are chosen so that the Radon-Nikodym derivative of the  $(S, T)$  semi-cluster measure, being a difference of the correlation functions of the  $\xi$ -weighted measures, may be controlled by (1.21) for points  $(v_1, \dots, v_k) \in W(S, T)$ . Second, it conveniently happens that  $W_n^k$  is precisely the union of  $W(S, T)$ , as  $(S, T)$  ranges over partitions of  $\{1, \dots, k\}$ . Therefore, combining these observations, we see that every cumulant measure on  $W_n^k$  is a sum ranging over partitions  $(S, T)$  of  $\{1, \dots, k\}$



*Moment measures.* Given a random measure  $\mu$  on  $\mathbb{R}^d$ , the  $k$ -th moment measure  $M^k = M^k(\mu)$  is the one (Sect 5.4 and Sect 9.5 of [18]) satisfying

$$\langle f_1 \otimes \dots \otimes f_k, M^k(\mu) \rangle = \mathbb{E}[\langle f_1, \mu \rangle \dots \langle f_k, \mu \rangle] = \mathbb{E} \left[ \sum_{x \in \mathcal{P}_n} f_1\left(\frac{x}{n^{1/d}}\right) \xi(x, \mathcal{P}_n) \cdots \sum_{x \in \mathcal{P}_n} f_k\left(\frac{x}{n^{1/d}}\right) \xi(x, \mathcal{P}_n) \right]$$

for all  $f_1, \dots, f_k \in \mathbb{B}(\mathbb{R}^d)$ , where  $f_1 \otimes \dots \otimes f_k : (\mathbb{R}^d)^k \rightarrow \mathbb{R}$  is given by  $f_1 \otimes \dots \otimes f_k(x_1, \dots, x_k) = f_1(x_1) \dots f_k(x_k)$ .

As on p. 143 of [18], when  $\mu$  is a counting measure,  $M^k$  may be expressed as a sum of factorial moment measures  $M_{[j]}$ ,  $1 \leq j \leq k$ , (as defined on p. 133 of [18]):

$$M^k(d(x_1 \times \dots \times x_k)) = \sum_{j=1}^k \sum_{\mathcal{V}} M_{[j]}(\Pi_{i=1}^j dy_i(\mathcal{V})) \delta(\mathcal{V}),$$

where, to quote from [18], the inner sum is taken over all partitions  $\mathcal{V}$  of the  $k$  coordinates into  $j$  non empty disjoint subsets, the  $y_i(\mathcal{V})$ ,  $1 \leq i \leq j$ , constitute an arbitrary selection of one coordinate from each subset, and  $\delta(\mathcal{V})$  is a  $\delta$  function which equals zero unless equality holds among the coordinates in each non-empty subset of  $\mathcal{V}$ .

When  $\mu$  is the atomic measure  $\mu_n^\xi$ , we write  $M_n^k$  for  $M^k(\mu_n^\xi)$ . By the Campbell formula, considering repetitions in the  $k$ -fold product of  $\mathbb{R}^d$ , and putting  $\tilde{y}_i := y_i(\mathcal{V})$  and  $\mathcal{V} := (\mathcal{V}_1, \dots, \mathcal{V}_j)$  we have that

$$\begin{aligned} \langle f \otimes \dots \otimes f, M_n^k \rangle &= \mathbb{E}[\langle f, \mu_n^\xi \rangle \dots \langle f, \mu_n^\xi \rangle] \\ &= \sum_{j=1}^k \sum_{\mathcal{V}} \int_{(W_n)^j} \Pi_{i=1}^k f\left(\frac{y_i}{n^{1/d}}\right) \mathbb{E}_{\tilde{y}_1, \dots, \tilde{y}_j} [\Pi_{i=1}^j \xi^{|\mathcal{V}_i|}(\tilde{y}_i, \mathcal{P}_n)] \rho^{(j)}(\tilde{y}_1, \dots, \tilde{y}_j) \Pi_{i=1}^j dy_i(\mathcal{V}) \delta(\mathcal{V}). \end{aligned}$$

In other words, recalling Lemma 9.5.IV of [18] we get

$$(4.17) \quad dM_n^k(y_1, \dots, y_k) = \sum_{j=1}^k \sum_{\mathcal{V}} m^{(|\mathcal{V}_1|, \dots, |\mathcal{V}_j|)}(\tilde{y}_1, \dots, \tilde{y}_j; n) \Pi_{i=1}^j dy_i(\mathcal{V}) \delta(\mathcal{V}).$$

*Cumulant measures.* The  $k$ th cumulant measure  $c_n^k := c^k(\mu_n)$  is defined analogously to the  $k$ th moment measure via

$$\langle f_1 \otimes \dots \otimes f_k, c^k(\mu_n) \rangle = c(\langle f_1, \mu_n \rangle \dots \langle f_k, \mu_n \rangle)$$

where  $c(X_1, \dots, X_k)$  denotes the joint cumulant of the random variables  $X_1, \dots, X_k$ .

The existence of the cumulant measures  $c_n^l$ ,  $l = 1, 2, \dots$  follows from the existence of moment measures in view of the representation (4.16). Thus, we have the following representation for cumulant measures :

$$c_n^l = \sum_{T_1, \dots, T_p} (-1)^{p-1} (p-1)! M_n^{T_1} \cdots M_n^{T_p},$$

where  $T_1, \dots, T_p$  ranges over all unordered partitions of the set  $1, \dots, l$  (see p. 30 of [44]). Henceforth for  $T_i \subset \{1, \dots, l\}$ , let  $M_n^{T_i}$  denote a copy of the moment measure  $M^{[T_i]}$  on the product space  $W^{T_i}$ . Multiplication denotes the usual product of measures: For  $T_1, T_2$  disjoint sets of integers and for measurable  $B_1 \subset (\mathbb{R}^d)^{T_1}, B_2 \subset (\mathbb{R}^d)^{T_2}$  we have  $M_n^{T_1} M_n^{T_2}(B_1 \times B_2) = M_n^{T_1}(B_1) M_n^{T_2}(B_2)$ . The first cumulant measure coincides with the expectation measure and the second cumulant measure





that  $d(v^S, v^T) \geq D_k(v)/k^2$ . If this were not the case then given  $v := (v_1, \dots, v_k)$ , the distance between any two components of  $v$  must be strictly less than  $D_k(v)/k^2$  and we would get  $\max_{i \leq k} \sum_{j=1}^k |v_i - v_j| \leq (k-1)kD_k/k^2 < D_k$ , a contradiction. Thus  $W_n^k$  is the union of sets  $W(S, T)$ ,  $(S, T) \in \Xi(k)$ , as asserted. We next describe the behavior of the differential  $d(U_n^{S_1, T_1} M_n^{|S_2|} M_n^{|T_2|})$  on  $W(S, T)$ .

*Semi-cluster measures on  $W(S, T)$ .* Next, given  $S_1 \subset S$  and  $T_1 \subset T$ , notice that  $d(v^{S_1}, v^{T_1}) \geq d(v^S, v^T)$  where  $v^{S_1}$  denotes the projection of  $v^S$  onto  $W_n^{S_1}$  and  $v^{T_1}$  denotes the projection of  $v^T$  onto  $W_n^{T_1}$ . Let  $\Pi(S_1, T_1)$  be the partitions of  $S_1$  into  $j_1$  sets  $\mathcal{V}_1, \dots, \mathcal{V}_{j_1}$ , with  $1 \leq j_1 \leq |S_1|$ , and the partitions of  $T_1$  into  $j_2$  sets  $\mathcal{V}_{j_1+1}, \dots, \mathcal{V}_{j_1+j_2}$ , with  $1 \leq j_2 \leq |T_1|$ . Thus an element of  $\Pi(S_1, T_1)$  is a partition of  $S_1 \cup T_1$ .

If a partition  $\mathcal{V}$  of  $S_1 \cup T_1$  does not belong to  $\Pi(S_1, T_1)$ , then there is a partition element of  $\mathcal{V}$  containing points in  $S_1$  and  $T_1$  and thus, recalling (4.17), we have  $\delta(\mathcal{V}) = 0$  on the set  $W(S, T)$ . Thus we make the crucial observation that, *on the set  $W(S, T)$  the differential  $d(M_n^{S_1 \cup T_1})$  collapses into a sum over partitions in  $\Pi(S_1, T_1)$ .* Thus  $d(M_n^{S_1 \cup T_1})$  and  $d(M_n^{S_1} M_n^{T_1})$  both involve sums of measures on common diagonal subspaces, as does their difference, made precise as follows.

LEMMA 4.2. *On the set  $W(S, T)$  we have*

$$(4.20) \quad d(U_n^{S_1, T_1}) = \sum_{j_1=1}^{|S_1|} \sum_{j_2=1}^{|T_1|} \sum_{\mathcal{V} \in \Pi(S_1, T_1)} [\dots] \Pi_{i=1}^{j_1+j_2} dy_i(\mathcal{V}) \delta(\mathcal{V})$$

where

$$[\dots] := m^{(|\mathcal{V}_1|, \dots, |\mathcal{V}_{j_1}|, |\mathcal{V}_{j_1+1}|, \dots, |\mathcal{V}_{j_1+j_2}|)}(\tilde{y}_1, \dots, \tilde{y}_{j_1}, \tilde{y}_{j_1+1}, \dots, \tilde{y}_{j_1+j_2}; n) \\ - m^{(|\mathcal{V}_1|, \dots, |\mathcal{V}_{j_1}|)}(\tilde{y}_1, \dots, \tilde{y}_{j_1}; n) m^{(|\mathcal{V}_{j_1+1}|, \dots, |\mathcal{V}_{j_1+j_2}|)}(\tilde{y}_{j_1+1}, \dots, \tilde{y}_{j_1+j_2}; n).$$

The representations of  $dM_n^{|S_2|}$  and  $dM_n^{|T_2|}$  follow from (4.17), that is to say

$$(4.21) \quad dM_n^{|S_2|} = \sum_{j_3=1}^{|S_2|} \sum_{\mathcal{V} \in \Pi(S_2)} m^{(|\mathcal{V}_1|, \dots, |\mathcal{V}_{j_3}|)}(\tilde{y}_1, \dots, \tilde{y}_{j_3}; n) \Pi_{i=1}^{j_3} dy_i(\mathcal{V}) \delta(\mathcal{V}),$$

where  $\Pi(S_2)$  runs over partitions of  $S_2$  into  $j_3$  sets,  $1 \leq j_3 \leq |S_2|$ . Similarly

$$(4.22) \quad dM_n^{|T_2|} = \sum_{j_4=1}^{|T_2|} \sum_{\mathcal{V} \in \Pi(T_2)} m^{(|\mathcal{V}_1|, \dots, |\mathcal{V}_{j_4}|)}(\tilde{y}_1, \dots, \tilde{y}_{j_4}; n) \Pi_{i=1}^{j_4} dy_i(\mathcal{V}) \delta(\mathcal{V}),$$

where  $\Pi(T_2)$  runs over partitions of  $T_2$  into  $j_4$  sets,  $1 \leq j_4 \leq |T_2|$ .

**4.3.3. Fast decay of correlations and semi-cluster measures.** The previous section established properties of semi-cluster and cumulant measures valid for any  $\mu_n^\xi$  of the form (1.4). If  $\xi$  with  $\mathcal{P}$  exhibit fast decay of correlations (1.21) of the  $\xi$ -weighted measures and satisfies moment bounds, we now assert that each integral in (4.19) is  $O(n)$ .

LEMMA 4.3. *Assume  $\xi$  satisfies moment bounds (1.19) for all  $p \geq 1$  and exhibits fast decay of correlations (1.21) in its  $\xi$ -weighted measure. For each partition element  $(S, T)$  of  $\Xi(k)$  we have*

$$(4.23) \quad \int_{W(S,T) \subset W_n^S \times W_n^T} |d(U_n^{S_1, T_1} M_n^{|S_2|} M_n^{|T_2|})| = O(n).$$

PROOF. The differential  $d(U_n^{S_1, T_1} M_n^{|S_2|} M_n^{|T_2|})$  is a sum

$$\sum_{j_1=1}^{|S_1|} \sum_{j_2=1}^{|T_1|} \sum_{j_3=1}^{|S_2|} \sum_{j_4=1}^{|T_2|} [\dots][\dots][\dots]$$

of products of three factors, one factor coming from each of the summands in (4.20)- (4.22). By Theorem 1.11, on the set  $W(S, T)$  the factor arising from (4.20) is bounded in absolute value by

$$\tilde{C}_k \tilde{\phi}\left(\frac{\tilde{c}_k D_k(y)}{k^2}\right).$$

By the moment bound (1.19) the two remaining factors arising from summands in (4.21)- (4.22) are bounded by a constant  $M'(k)$  depending only on  $k$ .

Thus we have

$$\begin{aligned} \int_{W(S,T)} |d(U_n^{S_1, T_1} M_n^{|S_2|} M_n^{|T_2|})| &\leq \tilde{C}_k (M'(k))^2 \sum_{j=1}^k \sum_{\mathcal{V}} \int_{W(S,T)} \tilde{\phi}\left(\frac{\tilde{c}_k D_k(y)}{k^2}\right) \Pi_{i=1}^j dy_i(\mathcal{V}) \delta(\mathcal{V}) \\ &\leq \tilde{C}_k (M'(k))^2 \sum_{j=1}^k \sum_{\mathcal{V}} \int_{(W_n)^j} \tilde{\phi}\left(\frac{\tilde{c}_k D_k(y)}{k^2}\right) \Pi_{i=1}^j dy_i(\mathcal{V}) \delta(\mathcal{V}). \end{aligned}$$

Here  $\mathcal{V}$  runs over all partitions of the  $k$  coordinates into  $j$  non-empty disjoint subsets. We assert that all summands are  $O(n)$ . We show this when  $j = k$ , as the proof for the remaining indices  $j \in \{1, \dots, k-1\}$  is similar. Write

$$\begin{aligned} &\int_{y_1 \in W_n} \dots \int_{y_k \in W_n} \tilde{\phi}\left(\frac{\tilde{c}_k D_k(y)}{k^2}\right) dy_1 \dots dy_k \\ &= \int_{y_1 \in W_n} \int_{w_2 \in W_{n-y_1}} \dots \int_{w_k \in W_{n-y_1}} \tilde{\phi}\left(\frac{\tilde{c}_k D_k(\mathbf{0}, w_2, \dots, w_k)}{k^2}\right) dy_1 dw_2 \dots dw_k. \end{aligned}$$

Now  $D_k(\mathbf{0}, w_2, \dots, w_k) \geq \sum_{i=2}^k |w_i|$ . Letting  $e_k := \tilde{c}_k(k-1)/k^2$  gives

$$\begin{aligned} &\int_{y_1 \in W_n} \dots \int_{y_k \in W_n} \tilde{\phi}\left(\frac{\tilde{c}_k D_k(y)}{k^2}\right) dy_1 \dots dy_k \leq n \int_{w_2 \in \mathbb{R}^d} \dots \int_{w_k \in \mathbb{R}^d} \tilde{\phi}\left(\frac{e_k}{k-1} \sum_{i=2}^k |w_i|\right) dw_2 \dots dw_k \\ &\leq n \int_{w_2 \in \mathbb{R}^d} \dots \int_{w_k \in \mathbb{R}^d} \tilde{\phi}(\Pi_{i=2}^k |w_i|^{1/(k-1)}) dw_2 \dots dw_k = O(n), \end{aligned}$$

where the first inequality follows from the decreasing behavior of  $\tilde{\phi}$ , the second inequality follows from the arithmetic geometric mean inequality, and the last equality follows since  $\tilde{\phi}$  is decreasing faster than any polynomial. We similarly bound the other summands for  $j \in \{1, \dots, k-1\}$ , completing the proof of Lemma 4.3.  $\square$

4.3.4. *Proof of Theorem 1.13.* By the bound (4.19) and Lemma 4.3 we obtain (4.14). Letting  $C_n^k$  be the  $k$ th cumulant for  $(\text{Var}\langle f, \mu_n^\xi \rangle)^{-1/2} \langle f, \mu_n^\xi \rangle$ , we obtain  $C_n^1 = 0$ ,  $C_n^2 = 1$ , and for all  $k = 3, 4, \dots$

$$C_n^k = O(n(\text{Var}\langle f, \mu_n^\xi \rangle)^{-k/2}).$$

Since  $\text{Var}\langle f, \mu_n^\xi \rangle = \Omega(n^\nu)$  by assumption, it follows that if  $k \in (2/\nu, \infty)$ , then the  $k$ th cumulant tends  $C_n^k$  to zero as  $n \rightarrow \infty$ . By a classical result of Marcinkiewicz (see e.g. [72, Lemma 3]), we get that all cumulants  $C_n^k$ ,  $k \geq 3$ , converge to zero as  $n \rightarrow \infty$ . This gives (4.13) as desired and completes the proof of Theorem 1.13.  $\square$

4.4. **Second proof of the central limit theorem.** We now give a second proof of the central limit theorem which we believe is of independent interest. Even though this proof is also based on the cumulant method used in Section 4.3.1, we shall bound the cumulants using a different approach, using Ursell functions of the  $\xi$ -weighted measure and establishing a property equivalent to Brillinger mixing; see Remarks at the end of Section 4.4.2. Though much of this proof can be read independently of the proof in Section 4.3, we repeatedly use the definition of moments and cumulants from Section 4.3.2.

*Our approach.* We shall adapt the approach in [53, Sec. 4] replacing  $\mathcal{P}_{GEF}$  by our  $\xi$ -weighted measures, which are purely atomic measures. As noted in Section 1.3, the correlation functions of the  $\xi$ -weighted measure are generalizations of the correlations functions of the simple point process, but the extension of the approach used in [53, Sec. 4] requires some care regarding the repeated arguments captured by general exponents  $k_i$  in (1.6).

4.4.1. *Ursell functions of the  $\xi$ -weighted measures.* Recall the definition of the correlation functions (1.6) of the  $\xi$ -weighted measures

$$m^{(k_1, \dots, k_p)}(x_1, \dots, x_p; n) := \mathbb{E}_{x_1, \dots, x_p} ((\xi(x_1, \mathcal{P}_n))^{k_1} \dots (\xi(x_p, \mathcal{P}_n))^{k_p}) \rho^{(p)}(x_1, \dots, x_p).$$

We will drop dependence on  $n$ , i.e.,  $m^{(k_1, \dots, k_p)}(x_1, \dots, x_p; n) = m^{(k_1, \dots, k_p)}(x_1, \dots, x_p)$  unless asymptotics in  $n$  is considered.

Inspired by the approach in [8, Section 2] we now introduce *Ursell functions*  $m_{\top}^{(k_1, \dots, k_p)}$  (sometimes called *truncated correlation function*) of the  $\xi$ -weighted measures. Define  $m_{\top}^{(k_1, \dots, k_p)}$  by taking  $m_{\top}^{(k)}(x) := m^{(k)}(x)$  for all  $k \in \mathbb{N}$  and inductively

$$(4.24) \quad m_{\top}^{(k_1, \dots, k_p)}(x_1, \dots, x_p) := m^{(k_1, \dots, k_p)}(x_1, \dots, x_p) - \sum_{\substack{\gamma \in \Pi[p] \\ |\gamma| > 1}} \prod_{i=1}^{|\gamma|} m_{\top}^{(k_j: j \in \gamma(i))}(x_j : j \in \gamma(i)).$$

for distinct  $x_1, \dots, x_p \in W_n$  and all integers  $k_1, \dots, k_p$ ,  $p \geq 1$ , and (implicitly)  $n \leq \infty$ . It is straightforward to prove that these functions satisfy the following relations. They extend the known relations for point processes, where  $m^{(k_1, \dots, k_p)}(x_1, \dots, x_p) = \rho^{(p)}(x_1, \dots, x_p)$  depend only on  $p$ , but we were unable to find them in the literature for (signed) purely atomic random measures, as our  $\xi$ -weighted measures. Assuming  $1 \in \gamma(1)$  in (4.24) and summing over partitions of  $\{1, \dots, p\} \setminus \gamma(1)$ , we get the

following relation :

$$(4.25) \quad m^{(k_1, \dots, k_p)}(x_1, \dots, x_p) = m_{\top}^{(k_1, \dots, k_p)}(x_1, \dots, x_p) + \sum_{\substack{I \subsetneq \{1, \dots, p\} \\ 1 \in I}} m_{\top}^{(k_j: j \in I)}(x_j : j \in I) m^{(k_j: j \in I^c)}(x_j : j \in I^c),$$

where  $I^c := \{1, \dots, p\} \setminus I$ . Using (4.25), by induction with respect to  $p$ , one obtains the direct relation to the correlation functions

$$(4.26) \quad m_{\top}^{(k_1, \dots, k_p)}(x_1, \dots, x_p) = \sum_{\gamma \in \Pi[p]} (-1)^{|\gamma|-1} (|\gamma| - 1)! \prod_{i=1}^{|\gamma|} m^{(k_j: j \in \gamma(i))}(x_j : j \in \gamma(i)).$$

This extends the relation [53, (27)], valid for point processes. We say that a partition  $\gamma = \{\gamma(1), \dots, \gamma(l)\} \in \Pi(p)$  *refines* partition  $\sigma = \{\sigma(1), \dots, \sigma(l_1)\} \in \Pi(p)$  if for all  $i \in \{1, \dots, l\}$ ,  $\gamma(i) \subset \sigma(j)$  for some  $j \in \{1, \dots, l_1\}$ . Otherwise, the partition  $\gamma$  is said to *mix* partition  $\sigma$ . Now using (4.24), we get for any  $I \subsetneq \{1, \dots, p\}$

$$(4.27) \quad m^{(k_j: j \in I)}(x_j : j \in I) m^{(k_j: j \in I^c)}(x_j : j \in I^c) = \sum_{\substack{\gamma \in \Pi[p] \\ \gamma \text{ refines } \{I, I^c\}}} \prod_{i=1}^{|\gamma|} m_{\top}^{(k_j: j \in \gamma(i))}(x_j : j \in \gamma(i)),$$

and therefore, again in view of (4.24)

$$(4.28) \quad m_{\top}^{(k_1, \dots, k_p)}(x_1, \dots, x_p) = m^{(k_1, \dots, k_p)}(x_1, \dots, x_p) - m^{(k_j: j \in I)}(x_j : j \in I) m^{(k_j: j \in I^c)}(x_j : j \in I^c) + \sum_{\substack{\gamma \in \Pi[p], |\gamma| > 1 \\ \gamma \text{ mixes } \{I, I^c\}}} \prod_{i=1}^{|\gamma|} m_{\top}^{(k_j: j \in \gamma(i))}(x_j : j \in \gamma(i)).$$

This extends the relation [53, last displayed formula in the proof of Claim 4.1] valid for point processes.

**4.4.2. Fast decay of correlations and bounds for Ursell functions.** We show now that fast decay of correlations (1.21) of the  $\xi$ -weighted measures implies some bounds on the Ursell functions of these measures. Since  $m^{(k_1, \dots, k_p)}(x_1, \dots, x_p; n)$  is invariant with respect to any joint permutation of its arguments  $(k_1, \dots, k_p)$  and  $(x_1, \dots, x_p)$ , fast decay of correlations (1.21) of the  $\xi$ -weighted measures may be rephrased as follows : There exists a fast-decreasing function  $\tilde{\phi}$  and constants  $\tilde{C}_k, \tilde{c}_k$ , such that for any collection of positive integers  $k_1, \dots, k_p, p \geq 2$ , satisfying  $k_1 + \dots + k_p = k$ , for any nonempty, proper subset  $I \subsetneq \{1, \dots, p\}$ , for all  $n \leq \infty$  and all configurations  $x_1, \dots, x_p \in W_n$  of distinct points we have

$$(4.29) \quad \left| m^{(k_1, \dots, k_p)}(x_1, \dots, x_p; n) - m^{(k_j: j \in I)}(x_j : j \in I; n) m^{(k_j: j \in I^c)}(x_j : j \in I^c; n) \right| \leq \tilde{C}_k \tilde{\phi}(\tilde{c}_k s),$$

where  $s := d(\{x_j : j \in I\}, \{x_j : j \in I^c\})$ .

Now we consider the bounds of Ursell functions of the  $\xi$ -weighted measures. Following the idea of [53, Claim 4.1] one proves that fast decay of correlations (1.21) of the  $\xi$ -weighted measures and the  $p$ -moment condition (1.19) imply that there exists a fast-decreasing function  $\tilde{\phi}_{\top}$  and constants  $\tilde{C}_k^{\top}$ ,

$\tilde{c}_k^\top$ , such that for any collection of positive integers  $k_1, \dots, k_p$ ,  $p \geq 2$ , satisfying  $k_1 + \dots + k_p = k$ , for all  $n \leq \infty$  and all configurations  $x_1, \dots, x_p \in W_n$  of distinct points we have

$$(4.30) \quad |m_{\top}^{(k_1, \dots, k_p)}(x_1, \dots, x_p; n)| \leq \tilde{C}_k^\top \tilde{\phi}_\top(c_k^\top \text{diam}(x_1, \dots, x_p)),$$

where  $\text{diam}(x_1, \dots, x_p) := \max_{i,j=1\dots p}(|x_i - x_j|)$ . The proof uses the representation (4.28), fast decay of correlations (1.21) of the  $\xi$ -weighted measures, together with the fact that there exist constants  $c_p^\top$  (depending on the dimension  $d$ ) such that for each configuration  $x_1, \dots, x_p \in W_n$ , there exists a partition  $\{I, I^c\}$  of  $\{1, \dots, p\}$  such that  $d(\{x_j : j \in I\}, \{x_j : j \in I^c\}) \geq \tilde{c}_p^\top \text{diam}(x_1, \dots, x_p)$ .

Next, inequality (4.30) allows one to bound integrals

$$(4.31) \quad \sup_{n \leq \infty} \sup_{x_1 \in W_n} \sup_{\substack{k_1 + \dots + k_p = k \\ k_i > 0}} \int_{(W_n)^{p-1}} |m_{\top}^{(k_1, \dots, k_p)}(x_1, \dots, x_p; n)| dx_2 \cdots dx_p < \infty.$$

Indeed, for a fixed point  $x_1 \in W_n$ , we split  $(W_n)^{p-1}$  into disjoint sets:

$$\begin{aligned} G_0 &:= \{(x_2, \dots, x_p) \in (W_n)^{p-1} : \text{diam}(x_1, \dots, x_p) \leq 1\} \\ G_l &:= \{(x_2, \dots, x_p) \in (W_n)^{p-1} : 2^{l-1} < \text{diam}(x_1, \dots, x_p) \leq 2^l\}, \quad l \geq 1 \end{aligned}$$

and use estimate (4.30) to bound the integral on the left-hand side of (4.31) by

$$\tilde{C}_k^\top + \tilde{C}_k^\top \sum_{l=1}^{\infty} 2^{dl(k-1)} \tilde{\phi}_\top(\tilde{c}_k^\top 2^{l-1}) < \infty$$

since  $\tilde{\phi}_\top$  is fast-decreasing; cf. [53, Claim 4.2].

### Remarks.

(i) A careful inspection of the relation (4.28) shows that in fact the fast decay of correlations (1.21) of the  $\xi$ -weighted measures is equivalent to the bound (4.30) on Ursell functions of these measures.

(ii) Condition (4.31), implied by (4.30), can be interpreted as the Brillinger mixing condition of the  $\xi$ -weighted measures. In fact it is slightly stronger in the sense that the bound on the Ursell functions integrated over  $dx_2 \cdots dx_p$  in the entire space (corresponding to the total reduced cumulant measures) is uniform for the whole family of the  $\xi$ -weighted measures considered on  $W_n$ , parametrized by  $n \leq \infty$  and, for  $n < \infty$  the bound is also uniform over  $x_1 \in W_n$  (which is immediate for reduced cumulant measures in the stationary case  $n = \infty$ ).

4.4.3. *Proof of Theorem 1.13.* The cumulant of order one is equal to the expectation and hence disappears for the considered (centered) random variable  $\bar{\mu}_n^\xi(f)$ . The cumulant of order 2 is equal to the variance and hence equal to 1 in our case. For  $k \geq 2$ , note the following relation between the normalized and the unnormalized cumulants :

$$(4.32) \quad S_k((\text{Var } \bar{\mu}_n^\xi(f))^{-1/2} \mu_n^\xi(f)) = (\text{Var } \mu_n^\xi(f))^{-k/2} \times S_k(\mu_n^\xi(f)).$$

We establish the vanishing of (4.32) for  $k$  large by showing that the  $k$ th order cumulant  $S_k(\mu_n^\xi(f))$

is of order  $O(n)$ ,  $k \geq 2$ , and then use assumption (1.26), i.e.,  $\text{Var}\langle f, \mu_n^\xi \rangle = \Omega(n^\nu)$ . We have

$$M_n^k := \mathbb{E}(\langle f, \mu_n^\xi \rangle)^k = \mathbb{E}\left(\sum_{x_i \in \mathcal{P}_n} f_n(x_i) \xi(x_i, \mathcal{P}_n)\right)^k,$$

where  $f_n(\cdot) = f(\cdot/n^{1/d})$ . Considering appropriately the repetitions of points  $x_i$  in the  $k$ th product of the sum and using the Campbell theorem at (1.9), one obtains

$$(4.33) \quad M_n^k = \sum_{\sigma \in \Pi[k]} \langle \bigotimes_{i=1}^{|\sigma|} f_n^{|\sigma(i)|} m^{(\sigma)}, \lambda_n^{|\sigma|} \rangle,$$

where  $\lambda_n^l$  denotes the Lebesgue measure on  $(W_n)^l$  and  $\bigotimes$  denotes the tensor product of functions

$$\left(\bigotimes_{i=1}^p f_n^{k_j}\right)(x_1, \dots, x_p) = \prod_{i=1}^p (f_n)^{k_j}(x_j), \quad m^{(\sigma)}(x_1, \dots, x_{|\sigma|}; n) := m^{(|\sigma(1)|, \dots, |\sigma(|\sigma|)|)}(x_1, \dots, x_{|\sigma|}; n)$$

Using the above representation and (4.16) the  $k$ th cumulant  $S_k(\mu_n^\xi(f))$  can be expressed as follows

$$(4.34) \quad \begin{aligned} S_k(\mu_n^\xi(f)) &= \sum_{\gamma \in \Pi[k]} (-1)^{|\gamma|-1} (|\gamma| - 1)! \sum_{\substack{\sigma \in \Pi[k] \\ \sigma \text{ refines } \gamma}} \prod_{i=1}^{|\gamma|} \langle \bigotimes_{j=1}^{|\gamma(i)/\sigma|} f_n^{(\gamma(i)/\sigma)(j)} m^{(\gamma(i)/\sigma)}, \lambda_n^{|\gamma(i)/\sigma|} \rangle \\ &= \sum_{\sigma \in \Pi[k]} \sum_{\substack{\gamma \in \Pi[k] \\ \sigma \text{ refines } \gamma}} (-1)^{|\gamma|-1} (|\gamma| - 1)! \prod_{i=1}^{|\gamma|} \langle \bigotimes_{j=1}^{|\gamma(i)/\sigma|} f_n^{(\gamma(i)/\sigma)(j)} m^{(\gamma(i)/\sigma)}, \lambda_n^{|\gamma(i)/\sigma|} \rangle, \end{aligned}$$

where  $\gamma(i)/\sigma$  is the partition of  $\gamma(i)$  induced by  $\sigma$ . Note that for any partition  $\sigma \in \Pi[k]$ , with  $|\sigma(j)| = k_j$ ,  $j = 1, \dots, |\sigma| = p$ , the inner sum in (4.34) can be rewritten as follows:

$$(4.35) \quad \sum_{\gamma \in \Pi[p]} (-1)^{|\gamma|-1} (|\gamma| - 1)! \prod_{i=1}^{|\gamma|} \langle \bigotimes_{j \in \gamma(i)} f_n^{k_j} m^{(k_j: j \in \gamma(i))}, \lambda_n^{|\gamma(i)|} \rangle = \langle \bigotimes_{j=1}^p f_n^{k_j} m_{\top}^{(k_1, \dots, k_p)}, \lambda_n^p \rangle,$$

where the equality is due to (4.26). Consequently

$$(4.36) \quad S_k(\mu_n^\xi(f)) = \sum_{\sigma \in \Pi[k]} \langle \bigotimes_{j=1}^{|\sigma|} f_n^{|\sigma(j)|} m_{\top}^{(|\sigma(1)|, \dots, |\sigma(|\sigma|)|)}, \lambda_n^{|\sigma|} \rangle,$$

which extends the relation [53, Claim 4.3] valid for point processes. The formula (4.36), which expresses the  $k$ th cumulant in terms of the Ursell functions, is the counterpart to the standard formula (4.33) expressing  $k$ th moments in terms of correlation functions. Now, using (4.31) and denoting the supremum therein by  $\hat{C}_k$ , we have that

$$\begin{aligned} & \left| \langle \bigotimes_{j=1}^p f_n^{k_j} m_{\top}^{(k_1, \dots, k_p)}, \lambda_n^p \rangle \right| \leq \int_{W_n^p} \left| \bigotimes_{j=1}^p f_n^{k_j} \right| |m_{\top}^{(k_1, \dots, k_p)}(x_1, \dots, x_p)| dx_1 \dots dx_p \\ & \leq \|f\|_\infty^k \int_{W_n} dx_1 \int_{W_n^{p-1}} |m_{\top}^{(k_1, \dots, k_p)}(x_1, \dots, x_p)| dx_2 \dots dx_p \leq \|f\|_\infty^k \hat{C}_k \text{Vol}_d(W_n). \end{aligned}$$

So, the above bound along with (4.34) and (4.35) gives us that  $S_k(\mu_n^\xi(f)) = O(n)$  for all  $k \geq 2$ . Thus, using the variance lower bound condition (1.26) and the relation (4.32), we get for large enough  $k$ , that  $S_k((\text{Var}\mu_n^\xi(f))^{-1/2}\mu_n^\xi(f)) \rightarrow 0$  as  $n \rightarrow \infty$ . Now, as discussed in (4.13), this suffices to guarantee normal convergence.  $\square$

## 5. Appendix.

**5.1. Facts needed in the proof of fast decay of correlations of the  $\xi$ -weighted measures.** The following facts about U-statistics and the radius of stabilization are used in the proof of Lemma 3.2.

**LEMMA 5.1.** *Let  $f, g$  be two real valued, symmetric functions defined on  $(\mathbb{R}^d)^k$  and  $(\mathbb{R}^d)^l$  respectively. Let  $F := \frac{1}{k!} \sum_{\mathbf{x} \in \mathcal{X}^{(k)}} f(\mathbf{x})$  and  $G := \frac{1}{l!} \sum_{\mathbf{x}' \in \mathcal{X}^{(l)}} g(\mathbf{x}')$  be the corresponding U-statistics of order  $k$  and  $l$  respectively, on the input  $\mathcal{X} \subset \mathbb{R}^d$ . Then we have:*

(i) *The product  $FG$  is a sum of U-statistics of order not greater than  $k + l$ .*

(ii) *Let  $\mathcal{A}$  be a fixed, finite subset of  $\mathbb{R}^d$ . The statistic  $F_{\mathcal{A}} := \frac{1}{k!} \sum_{\mathbf{x} \in (\mathcal{X} \cup \mathcal{A})^{(k)}} f(\mathbf{x})$  is a sum of U-statistics of  $\mathcal{X}$  of order not greater than  $k$ .*

*Proof.* The two statements follow from symmetrizing the inner summands in the below representations

$$FG = \sum_{m=\max(k,l)}^{k+l} \sum_{\mathbf{z} \in \mathcal{X}^{(m)}} \frac{f(z_1, \dots, z_k)g(z_{m-l+1}, \dots, z_m)}{(k+l-m)!(m-k)!(m-l)!},$$

$$F_{\mathcal{A}} = \sum_{m=0}^{\min(|\mathcal{A}|, k)} \sum_{\mathbf{a} \in \mathcal{A}^{(m)}} \sum_{\mathbf{z} \in \mathcal{X}^{(k-m)}} \frac{f(a_1, \dots, a_m, z_1, \dots, z_{k-m})}{m!(k-m)!},$$

For a proof of the first representation note that by symmetry of  $f$ ,  $F(\mathcal{Y}) = f(y_1, \dots, y_k)$  if  $Y = \{y_1, \dots, y_k\}$  and similarly for  $G$ . Thus, we derive that

$$\begin{aligned} FG &= \sum_{\mathcal{Y}_i \subset \mathcal{X}, i=1,2} F(\mathcal{Y}_1)G(\mathcal{Y}_2)\mathbf{1}[|\mathcal{Y}_1| = k, |\mathcal{Y}_2| = l] \\ &= \sum_{m=\max(k,l)}^{k+l} \sum_{\mathcal{Y}_i \subset \mathcal{X}, i=1,2} F(\mathcal{Y}_1)G(\mathcal{Y}_2)\mathbf{1}[|\mathcal{Y}_1 \cup \mathcal{Y}_2| = m, |\mathcal{Y}_1| = k, |\mathcal{Y}_2| = l] \\ &= \sum_{m=\max(k,l)}^{k+l} \sum_{\mathcal{Y} \subset \mathcal{X}} \mathbf{1}[|\mathcal{Y}| = m] \sum_{\mathcal{Y}_i \subset \mathcal{Y}, i=1,2} F(\mathcal{Y}_1)G(\mathcal{Y}_2)\mathbf{1}[\mathcal{Y}_1 \cup \mathcal{Y}_2 = \mathcal{Y}, |\mathcal{Y}_1| = k, |\mathcal{Y}_2| = l] \\ &= \sum_{m=\max(k,l)}^{k+l} \sum_{\mathbf{z} \in \mathcal{X}^{(m)}} \frac{1}{m!} \sum_{\mathcal{Y}_1, \mathcal{Y}_2} F(\mathcal{Y}_1)G(\mathcal{Y}_2)\mathbf{1}[\mathcal{Y}_1 \cup \mathcal{Y}_2 = \{z_1, \dots, z_m\}, |\mathcal{Y}_1| = k, |\mathcal{Y}_2| = l] \\ &= \sum_{m=\max(k,l)}^{k+l} \sum_{\mathbf{z} \in \mathcal{X}^{(m)}} \frac{f(z_1, \dots, z_k)g(z_{m-l+1}, \dots, z_m)}{m!} \sum_{\mathcal{Y}_1, \mathcal{Y}_2} \mathbf{1}[\mathcal{Y}_1 \cup \mathcal{Y}_2 = \{z_1, \dots, z_m\}, |\mathcal{Y}_1| = k, |\mathcal{Y}_2| = l] \\ &= \sum_{m=\max(k,l)}^{k+l} \sum_{\mathbf{z} \in \mathcal{X}^{(m)}} \frac{f(z_1, \dots, z_k)g(z_{m-l+1}, \dots, z_m)}{(k+l-m)!(m-k)!(m-l)!}, \end{aligned}$$





$$|\text{per}([v_1, \dots, v_n])| \leq \frac{n!}{n^{\frac{n}{2}}} \prod_{i=1}^n \sqrt{\bar{v}_i^T v_i} \leq n! \prod_{i=1}^n \|v_i\|,$$

where  $\|v_i\|$  is the  $l_\infty$ -norm of  $v_i$  viewed as an  $n$ -dimensional complex vector.

LEMMA 5.4. *Let  $n \in \mathbb{N}$ . For any two matrices  $K$  and  $L$ , we have*

$$|\text{per}(K) - \text{per}(L)| \leq nn! \|K - L\| \max\{\|K\|, \|L\|\}^{n-1}.$$

Now, in the proof of Lemma 5.3, using the above estimate instead of (5.2), we establish fast decay of correlations (1.10) of permanental point processes with fast-decreasing kernels  $K$ .

LEMMA 5.5. *Let  $\mathcal{P}$  be a stationary permanental point process on  $\mathbb{R}^d$  with a fast-decreasing kernel satisfying  $K(x, y) \leq \omega(|x - y|)$  where  $\omega$  is at (2.7). With  $s$  as at (1.3) and  $n = p + q$ , we have*

$$|\rho^{(n)}(x_1, \dots, x_{p+q}) - \rho^{(p)}(x_1, \dots, x_p) \rho^{(q)}(x_{p+1}, \dots, x_{p+q})| \leq nn! \omega(s) \|K\|^{n-1}.$$

To bound the radius of stabilization of geometric functionals on determinantal point processes, we rely on the following exponential decay of Palm void probabilities. Though the proof is inspired by that of a similar estimate in [49, Lemma 2], we derive a more general and explicit bound.

LEMMA 5.6. *Let  $\mathcal{P}$  be a stationary determinantal point process on  $\mathbb{R}^d$ . Then for  $p, k \in \mathbb{N}$ ,  $\mathbf{x} \in (\mathbb{R}^d)^p$ , and any bounded Borel subset  $B \subset \mathbb{R}^d$ , we have*

$$(5.3) \quad \mathbb{P}_{\mathbf{x}}^!(\mathcal{P}(B) \leq k) \leq e^{(2k+p)/8} e^{-K(\mathbf{0}, \mathbf{0}) \text{Vol}_d(B)/8}.$$

*Proof.* For any determinantal point process  $\mathcal{P}$  (even a non-stationary one), let  $\mathcal{P}_x$  be the reduced Palm point process with respect to  $x \in \mathbb{R}^d$ . From (2.11) (see also [71, Theorem 6.5]), we have that  $\mathcal{P}_x$  is also a determinantal point process and its kernel  $L$  is given by

$$(5.4) \quad L(y_1, y_2) = K(y_1, y_2) - \frac{K(y_1, x)K(x, y_2)}{K(x, x)}.$$

Next we assert that

$$(5.5) \quad \int_{\mathbb{R}^d} |K(x, y)|^2 dy \leq K(x, x), \quad x \in \mathbb{R}^d.$$

To see this, write  $K(x, y) = \sum_j \lambda_j \phi_j(x) \bar{\phi}_j(y)$ ,  $\lambda_j \in [0, 1]$ , where  $\phi_j, j \geq 1$ , is an orthonormal basis for  $L^2(\mathbb{R}^d, dx)$  (cf. Lemma 4.2.2 of [9]). In view of  $\overline{K(x, y)} = K(y, x)$  we get  $\int_{\mathbb{R}^d} |K(x, y)|^2 dy = \int_{\mathbb{R}^d} K(x, y) \overline{K(x, y)} dy = \int_{\mathbb{R}^d} K(x, y) K(y, x) dy = \sum_j \lambda_j^2 \phi_j(x) \bar{\phi}_j(x) \leq \sum_j \lambda_j \phi_j(x) \bar{\phi}_j(x) = K(x, x)$ , whence the assertion (5.5). The bound (5.5) shows for any bounded Borel subset  $B$  and  $x \in \mathbb{R}^d$  that

$$\mathbb{E}_x^!(\mathcal{P}(B)) = \int_B L(y, y) dy = \int_B K(y, y) dy - \frac{1}{K(x, x)} \int_B |K(x, y)|^2 dy \geq \mathbb{E}(\mathcal{P}(B)) - 1.$$

Re-iterating the above inequality, we get that for all  $\mathbf{x} \in \mathbb{R}^{dp}$  and any bounded Borel subset  $B$



random variables. Independence of the Bernoulli random variables gives

$$\begin{aligned}
\mathbb{P}_x(\mathcal{P}(B_r) \leq k) &= \sum_{J \subset \mathbb{N}, |J| \leq k} \prod_{j \in J} \mu_j \prod_{j \notin J} (1 - \mu_j) \leq \sum_{J \subset \mathbb{N}, |J| \leq k} \prod_{j \in J} \lambda_j \prod_{j \notin J} (1 - \lambda_{j+1}) \\
&\leq \sum_{J \subset \mathbb{N}, |J| \leq k} \prod_{j \in J} \lambda_j \prod_{j \notin J} (1 - \lambda_j) \prod_{j-1 \in J \cup \{0\}, j \notin J} (1 - \lambda_j)^{-1} \\
&\leq (1 - \lambda_1)^{-k-1} \sum_{J \subset \mathbb{N}, |J| \leq k} \prod_{j \in J} \lambda_j \prod_{j \notin J} (1 - \lambda_j) = (1 - \lambda_1)^{-k-1} \mathbb{P}[\mathcal{P}(B_r) \leq k].
\end{aligned}$$

The proof of the first inequality in (5.8) for the case  $p = 1$  is complete by noting that  $\lambda_1 = \mathbb{P}(EXP(1) \leq r^2)$  (see [9, Theorems 4.7.1 and 4.7.3]), where  $EXP(1)$  stands for an exponential random variable with mean 1. As said before, iteratively the first inequality in (5.8) can be proven for an arbitrary  $p$ . To complete the proof of the second inequality, we bound  $\mathbb{P}(\mathcal{P}(B_r) \leq k)$  in a manner similar to the proof of [9, Proposition 7.2.1].

Let  $\mathcal{P}^* := \{R_1^2, R_2^2, \dots\} = \{|X|^2 : X \in \mathcal{P}\}$  be the point process of squared moduli of the Ginibre point process. Then, from [9, Theorem 4.7.3], it is known that  $R_i^2 \stackrel{d}{=} \Gamma(i, 1)$  ( $\Gamma(i, 1)$  denotes a gamma random variable with parameters  $i, 1$ ) and are independently distributed. There is a constant  $\beta \in (0, 1)$  such that

$$\mathbb{P}(R_i^2 \geq r^2) \leq e^{-\beta r^2} \mathbb{E}(e^{\beta R_i^2}) \leq e^{-\beta r^2} (1 - \beta)^{-i}, \quad i \geq 1.$$

For  $i < r^2$ , the bound is optimal for  $\beta = 1 - \frac{i}{r^2}$ . For  $r$ , set  $r_* := \lceil r^2 \rceil$ , the ceiling of  $r^2$ . Then,

$$\begin{aligned}
\mathbb{P}(\mathcal{P}(B_r) \leq k) &= \mathbb{P}(\#\{i : R_i^2 \leq r^2\} \leq k) \leq \mathbb{P}(\#\{i \leq r_* : R_i^2 \leq r^2\} \leq k) \\
&\leq \sum_{J \subset [r_*], |J| \leq k} \prod_{i \in J} \mathbb{P}(R_i^2 \leq r^2) \prod_{i \notin J} \mathbb{P}(R_i^2 > r^2) \\
&\leq \sum_{J \subset [r_*], |J| \leq k} \prod_{i \in J} e^{r^2} e^{-\beta r^2} (1 - \beta)^{-i} \prod_{i \notin J} e^{-\beta r^2} (1 - \beta)^{-i} \\
&\leq k r^{2k} e^{k r^2} \prod_{i=1}^{r_*} e^{-a r^2} (1 - \beta)^{-i} = k r^{2k} e^{k r^2} e^{-\frac{1}{4} r^4 (1+o(1))},
\end{aligned}$$

where equality follows by substituting the optimal  $\beta$  for each  $i$ , as in [9, Section 7.2].  $\square$

**5.3. Facts about superposition of independent point processes.** The following facts on superposition of independent point processes were useful in the applications involving  $\alpha$ -determinantal point processes,  $|\alpha| = 1/m, m \in \mathbb{N}$ .

*Proof of Proposition 2.3.* We shall prove the proposition in the case  $m = 2$ ; the general case follows in the same fashion albeit with considerably more notation. Let  $x_1, \dots, x_{p+q}$  be distinct points in  $\mathbb{R}^d$  with  $s$  at (1.3) as usual. For a subset  $S \subset [p+q]$ , we abbreviate  $\rho^{|S|}(x_j : j \in S)$  by  $\rho(S)$ . Using (2.18) we have that

$$\rho_0^{(p+q)}([p+q]) = \sum_{S_1 \sqcup S_2 = [p+q]} \rho(S_1) \rho(S_2) = 2\rho([p+q]) + 2\rho([p])\rho([q])$$



PROPOSITION 5.8. *Let  $\mathcal{P}_1, \dots, \mathcal{P}_m, m \in \mathbb{N}$ , be independent admissible point processes. For  $p, k \in \mathbb{N}$  and a bounded Borel set  $B$ , set*

$$\nu_{p,k}(B) := \sup_{i=1, \dots, m} \sup_{0 \leq p' \leq p} \sup_{x_1, \dots, x_{p'}} \mathbb{P}_{x_1, \dots, x_{p'}}(\mathcal{P}_i(B) \leq k).$$

Let  $\mathcal{P} := \cup_{i=1}^m \mathcal{P}_i$  be the independent superposition. Then,  $\alpha^{(k)}$  a.e.  $x_1, \dots, x_p$ , we have

$$\mathbb{P}_{x_1, \dots, x_p}(\mathcal{P}(B) \leq k) \leq \nu_{p,k}(B)^m.$$

PROOF. We shall show the proposition for  $m = 2$  and the general case follows similarly. Further, we use  $\rho, \rho_1, \rho_2$  to denote the correlation functions of  $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2$  respectively. Let  $A = A_1 \times \dots \times A_p$  where  $A_1, \dots, A_p$  are disjoint bounded Borel subsets. By setting  $f(x_1, \dots, x_p; \mathcal{P}) = \mathbf{1}[\mathcal{P}(B) \leq k]$  in the refined Campbell theorem in (1.9) and using the independence of  $\mathcal{P}_1, \mathcal{P}_2$ , we derive that

$$\begin{aligned} & \int_A \mathbb{P}_{x_1, \dots, x_p}(\mathcal{P}(B) \leq k) \rho^{(p)}(x_1, \dots, x_p) dx_1 \dots dx_p = \mathbb{E}(\mathbf{1}[\mathcal{P}(B) \leq k] \mathcal{P}(A_1) \dots \mathcal{P}(A_p)) \\ & \leq \sum_{S \subset [p]} \mathbb{E} \left( \mathbf{1}[\mathcal{P}_1(B) \leq k] \prod_{i \in S} \mathcal{P}_1(A_i) \right) \mathbb{E} \left( \mathbf{1}[\mathcal{P}_2(B) \leq k] \prod_{i \notin S} \mathcal{P}_2(A_i) \right) \\ & = \int_A \sum_{S \subset [p]} \mathbb{P}_{x_i; i \in S}(\mathcal{P}_1(B) \leq k) \rho_1^{(|S|)}(x_i; i \in S) \mathbb{P}_{x_i; i \notin S}(\mathcal{P}_2(B) \leq k) \rho_2^{(p-|S|)}(x_i; i \notin S) dx_1 \dots dx_p \end{aligned}$$

Thus, by definition of  $\nu_{p,k}(B)$ , we get that for a.e.  $x_1, \dots, x_p$

$$\begin{aligned} & \mathbb{P}_{x_1, \dots, x_p}(\mathcal{P}(B) \leq k) \rho^{(p)}(x_1, \dots, x_p) \\ & \leq \sum_{S \subset [p]} \mathbb{P}_{x_i; i \in S}(\mathcal{P}_1(B) \leq k) \rho_1^{(|S|)}(x_i; i \in S) \mathbb{P}_{x_i; i \notin S}(\mathcal{P}_2(B) \leq k) \rho_2^{(p-|S|)}(x_i; i \notin S) \\ & \leq \nu_{p,k}(B)^2 \sum_{S \subset [p]} \rho_1^{(|S|)}(x_i; i \in S) \rho_2^{(p-|S|)}(x_i; i \notin S) = \nu_{p,k}(B)^2 \rho^{(p)}(x_1, \dots, x_p), \end{aligned}$$

where the last equality follows from (2.18). The proposition now follows from the above inequality.  $\square$

Now, as a trivial corollary of Lemma 5.6 and Proposition 5.8, we obtain the following useful result.

COROLLARY 5.9. *Let  $\mathcal{P}$  be a stationary  $\alpha$ -determinantal point process on  $\mathbb{R}^d$  with  $\alpha = -1/m, m \in \mathbb{N}$ . Then for  $p, k \in \mathbb{N}$ ,  $\mathbf{x} \in (\mathbb{R}^d)^p$ , and any bounded Borel subset  $B \subset \mathbb{R}^d$ , we have*

$$(5.9) \quad \mathbb{P}_{\mathbf{x}}^!(\mathcal{P}(B) \leq k) \leq e^{m(2k+p)/8} e^{-K(\mathbf{0}, \mathbf{0}) \text{Vol}_d(B)/8}.$$

Consider the same assumptions as in Proposition 5.8. For a bounded Borel subset  $B$ , set

$$M_p(B) := \sup_{i=1, \dots, m} \sup_{0 \leq p' \leq p} \sup_{x_1, \dots, x_{p'}} \mathbb{E}_{x_1, \dots, x_{p'}}(t^{\mathcal{P}_i(B)}), \quad t \geq 0, p \geq 1.$$

Now setting  $f(x_1, \dots, x_p; \mathcal{P}) = t^{\mathcal{P}(B)}$  in the proof of the proposition, we may deduce that

$$(5.10) \quad \sup_{0 \leq p' \leq p} \sup_{x_1, \dots, x_{p'}} \mathbb{E}_{x_1, \dots, x_{p'}}(t^{\mathcal{P}(B)}) \leq M_p(B)^m.$$

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## References.

- [1] G. W. Anderson, A. Guionnet and O. Zeitouni (2010), *An Introduction to Random Matrices*, Cambridge University Press, Cambridge, U.K.
- [2] F. Baccelli and P. Bormolen (2012), Extremal versus additive Matérn point processes, *Queueing Syst.* **71**, No. 1-2, 179–197.
- [3] A. Baddeley (1980), A limit theorem for statistics of spatial data, *Adv. in Appl. Probab.*, **12**, 2, 447–461.
- [4] A. D. Barbour and A. Xia (2006), Normal approximation for random sums, *Adv. Appl. Probab.*, **38**, 693–728.
- [5] Yu. Baryshnikov, P. Eichelsbacher, T. Schreiber, and J. E. Yukich (2008), Moderate deviations for some point measures in geometric probability, *Ann. Inst. Henri Poincaré Probab. Stat.*, **44**, 3, 422–446.
- [6] Yu. Baryshnikov and J. E. Yukich (2003), Gaussian fields and random packing, *J. of Stat. Phys.*, **111**, 443–463.
- [7] Yu. Baryshnikov and J. E. Yukich (2005), Gaussian limits for random measures in geometric probability. *Ann. Appl. Probab.*, **15**, no. 1A, 213–253.
- [8] K. Baumann and G. C. Hegerfeldt (1985), A noncommutative Marcinkiewicz theorem, *Publications of the Research Institute for Mathematical Sciences*, **21**, no. 1, 191–204.
- [9] J. Ben Hough, M. Krishnapur, Y. Peres, and B. Virág (2009), *Zeros of Gaussian Analytic Functions and Determinantal Point Processes*, Vol. 51, American Mathematical Society, Providence, RI.
- [10] C. Biscio and F. Lavancier (2016), Brillinger mixing of determinantal point processes and statistical applications, *Electronic J. of Statistics*, **10**, 582–607.
- [11] A. Björner (1995), Topological methods, In *Handbook of Combinatorics*, Vol. 2, 1819–1872.
- [12] B. Błaszczyszyn (1995), Factorial moment expansion for stochastic systems, *Stoch. Proc. Appl.*, **56**, 2, 321–335.
- [13] B. Błaszczyszyn, E. Merzbach and V. Schmidt (1997), A note on expansion for functionals of spatial marked point processes, *Stat. Probab. Letters*, **36**, 3, 299–306.
- [14] B. Błaszczyszyn and D. Yogeshwaran (2014), On comparison of clustering properties of point processes, *Adv. Appl. Probab.*, **46**, 1, 1–20.
- [15] O. Bobrowski and R.J. Adler (2014), Distance functions, critical points, and the topology of random Čech complexes, *Homology Homotopy Appl.*, **16**, 2, 311–344.
- [16] A. Bulinski and E. Spodarev (2012), Central limit theorems for weakly dependent random fields. In *Stochastic Geometry, Spatial Statistics and Random Fields : Asymptotic methods*. Ed. Spodarev, E., Lect. Notes Math., vol. 2068.
- [17] E. Carlen, E. Lieb, M. Loss (2006), An inequality of Hadamard type for permanents, *Meth. Appl. Anal.*, **13**, 1, 1–18.
- [18] D. J. Daley and D. Vere-Jones (2003), *An Introduction to the Theory of Point Processes*, Springer, NY.
- [19] J. Dancis and C. Davis (1987), An interlacing theorem for eigenvalues of self-adjoint operators, *Lin. Alg. Appl.*, **88–89**, 117–122.
- [20] L. Decreasefond, E. Ferraz, H. Randriam, and A. Vergne (2014), Simplicial Homology of Random Configurations, *Adv. in Appl. Probab.* **46**, 2, 1–20.
- [21] H. Edelsbrunner and J. L. Harer (2010), *Computational Topology : An Introduction*, American Mathematical Society, Providence, RI.
- [22] P. Eichelsbacher, M. Raic, and T. Schreiber (2015), Moderate deviations for stabilizing functionals in geometric probability, *Ann. Inst. Henri Poincaré Prob. Stat.*, **51**, No. 1, 89–128.
- [23] P. Eichelsbacher and C. Thäle (2014), New Berry-Esseen bounds for non-linear functionals of Poisson random measures, *Electron. J. Probab.* **102**(19).

- [24] R. Forman (2002), A user's guide to discrete Morse theory, *Sém. Lothar. Combin.*, **48**.
- [25] P. J. Forrester and G. Honner (1999), Exact statistical properties of the zeros of complex random polynomials, *J. of Physics A: Math. and General*, **32**, 16, 2961.
- [26] V. Gershkovich and H. Rubinstein (1997), Morse theory for Min-type functions, *The Asian J. Math.* **1**, 696–715.
- [27] J. Grote and C. Thäle (2016), Gaussian polytopes: a cumulant-based approach, *preprint* arXiv:1602.06148.
- [28] A. Goldman (2010), The Palm measure and the Voronoi tessellation for the Ginibre point process, *Ann. Appl. Probab.*, **20**, No. 1, 90–128.
- [29] P. Hall (1988), *Introduction to the Theory of Coverage Processes*, John Wiley and Sons.
- [30] K-H. Hanisch (1982), On inversion formulae for n-fold Palm distributions of point processes in LCS-spaces, *Math. Nachr.*, **106**, 1, 171–179.
- [31] L. Heinrich (1994), Normal approximation for some mean-value estimates of absolutely regular tessellations, *Math. Meth. Stat.*, **3**, 1, 1–24.
- [32] L. Heinrich (2013), Asymptotic methods in statistics of random point processes, In *Stochastic Geometry, Spatial Statistics and Random Fields : Asymptotic methods*. Ed. Spodarev, E., Lect. Notes Math., Springer, vol. 2068.
- [33] L. Heinrich and I. Molchanov (1999), Central limit theorem for a class of random measures associated with germ-grain models, *Adv. Appl. Probab.*, **31**, 2, 283–314.
- [34] D. Hug, G. Last and M. Schulte (2016) Second order properties and central limit theorems for geometric functionals of Boolean models, *Ann. Appl. Probab.*, **26**, 1, 73–135.
- [35] G. Ivanoff (1982) Central limit theorems for point processes, *Stoch. Proc. Applications.*, **12**, 2, 171–186.
- [36] M. Kahle (2011), Random geometric complexes, *Discrete Comput. Geom.*, **45**, 3, 553–573.
- [37] M. Krishnapur (2006), Overcrowding estimates for zeroes of planar and hyperbolic Gaussian analytic functions, *J. Stat. Phys.*, **124**, No. 6, 1399–1423.
- [38] R. Lachièze-Rey and M. Schulte and J. E. Yukich (2017), Normal approximation for stabilizing functionals, *pre-print* arXiv:1702.00726.
- [39] R. Lachièze-Rey and G. Peccati (2015), New Kolmogorov bounds for functionals of binomial point processes, to appear in *Ann. Appl. Probab.*, arXiv:1505.04640.
- [40] G. Last and M. Penrose (2010), Poisson process Fock space representation, chaos expansion and covariance inequalities, *Probab. Th. Rel. Fields*, **150**, 663–690.
- [41] G. Last and M. Penrose (2012), Percolation and limit theory for the Poisson lilypond model, *Rand. Struct. Alg.*, **42**, 226–249.
- [42] G. Last, G. Peccati and M. Schulte (2016), Normal approximations on the Poisson space: Mehler's formula, second order Poincaré inequalities and stabilization, *Probab. Theory Rel. Fields.*, **165**, 3, 667–723.
- [43] V. A. Malyshev (1975), The central limit theorem for Gibbsian random fields, *Soviet Math. Doklady* **16**, 1141–1145.
- [44] V. A. Malyshev and R. A. Minlos (1991), *Gibbs Random Fields*, Kluwer.
- [45] Ph. A. Martin and T. Yalcin (1980), The charge fluctuations in classical Coulomb systems, *J. Stat. Phys.*, **22**, No. 4, 435–463.
- [46] P. McCullagh and J. Møller (2006), The permanent process, *Adv. Appl. Probab.*, **38**, No. 4, 873–888.
- [47] J. Milnor (1963), *Morse Theory*, Princeton University Press.
- [48] M. Mitzenmacher and E. Upfal (2005), *Probability and Computing Randomized Algorithms and Probabilistic Analysis*, Cambridge University Press, Cambridge.
- [49] N. Miyoshi and T. Shirai. (2016), A sufficient condition for tail asymptotics of SIR distribution in downlink cellular networks. In *14th Int. Symp. on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks (WiOpt)*, 1–7.
- [50] J. Møller (2003), Shot noise Cox processes, *Adv. Appl. Probab.*, **35**, No. 3 614–640.
- [51] J. Møller and G.L. Torrisi (2005), Generalised shot noise Cox processes, *Adv. Appl. Probab.*, **37**, No. 1, 48–74.
- [52] F. Nazarov and M. Sodin (2011), Fluctuations in random complex zeros: asymptotic normality revisited, *Intern. Math. Research Notices*, **24**, 5720- 5759.
- [53] F. Nazarov and M. Sodin (2012), Correlation functions for random complex zeroes: Strong clustering and local universality, *Comm. Math. Phys.*, **310**, No. 1, 75–98.
- [54] I. Nourdin and G. Peccati (2012), *Normal Approximations with Malliavin Calculus : From Stein's Method to Universality*, Cambridge Tracts in Mathematics, **192**, Cambridge University Press, Cambridge.

- [55] G. Peccati and M. Reitzner (Editors) (2016), *Stochastic Analysis for Poisson Point Processes: Malliavin Calculus, Wiener-It Chaos Expansions and Stochastic Geometry*, Springer-Verlag.
- [56] A. Poinas, B. Delyon, and F. Lavancier (2017), Mixing properties and central limit theorem for associated point processes, preprint, arXiv:1705.02276v2.
- [57] R. Pemantle and Y. Peres (2014), Concentration of Lipschitz functionals of determinantal and other strong Rayleigh measures, *Comb. Probab. Comput.*, **23**, 140–160.
- [58] M. D. Penrose (2003), *Random Geometric Graphs*, Oxford Univ. Press, New York.
- [59] M. D. Penrose (2007), Gaussian limits for random geometric measures, *Elec. J. Probab.*, **12**, 989–1035.
- [60] M. D. Penrose (2007), Laws of large numbers in stochastic geometry with statistical applications, *Bernoulli*, **13**, 1124–1150.
- [61] M. D. Penrose and J. E. Yukich (2001), Central limit theorems for some graphs in computational geometry, *Ann. Appl. Probab.*, **11**, 1005–1041.
- [62] M. D. Penrose and J. E. Yukich (2002), Limit theory for random sequential packing and deposition, *Ann. Appl. Probab.*, **12**, 272–301.
- [63] M. D. Penrose and J. E. Yukich (2003), Weak laws of large numbers in geometric probability, *Ann. Appl. Probab.*, **13**, 277–303.
- [64] M. D. Penrose and J. E. Yukich (2005), Normal approximation in geometric probability, *In Stein's Method and Applications*, Lect. Note Series, Inst. Math. Sci., Nat. Univ. of Singapore, **5**, A. Barbour and Louis Chen, Eds., 37–58.
- [65] T. Ram Reddy, S. Vadlamani and D. Yogeshwaran (2017), Central limit theorem for quasi-local statistics of spin models on Cayley graphs, preprint arXiv:1709.10424.
- [66] M. Reitzner and M. Schulte (2013), Central limit theorems for  $U$ -statistics of Poisson point processes, *Ann. Probab.*, **41**, No. 6, 3879–3909.
- [67] R. Schneider (1993), *Convex Bodies: the Brunn-Minkowski Theory*, Cambridge University Press, Cambridge.
- [68] R. Schneider and W. Weil (2008), *Stochastic and Integral geometry*, Probability and its Applications (New York), Springer-Verlag, Berlin.
- [69] T. Schreiber and J. E. Yukich (2013), Limit theorems for geometric functionals of Gibbs point processes, *Ann. Inst. Henri Poincaré Probab. Stat.*, **49**, 4, 1158 - 1182.
- [70] M. Schulte and C. Thäle (2016), Central limit theorems for the radial spanning tree. *Rand. Struct. Alg.*, online version.
- [71] T. Shirai and Y. Takahashi (2003), Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point processes, *J. Funct. Anal.*, **205**, 2, 414–463.
- [72] A. Soshnikov (2002), Gaussian limit for determinantal random point fields, *Ann. Probab.*, **30**, 171–187.
- [73] A. Xia and J. E. Yukich (2015), Normal approximation for statistics of Gibbsian input in geometric probability, *Adv. Appl. Probab.*, **47**, 934–972.
- [74] D. Yogeshwaran and R. J. Adler (2015), On the topology of random complexes built over stationary point processes, *Ann. Appl. Probab.*, **25**, No. 6, 3338–3380.
- [75] J. E. Yukich (1998), *Probability Theory of Classical Euclidean Optimization Problems*, Lecture Notes in Mathematics, vol. 1675, Springer-Verlag.
- [76] J. E. Yukich (2013), Limit theorems in discrete stochastic geometry. In *Stochastic Geometry, Spatial Statistics and Random Fields : Asymptotic methods*. Ed. Spodarev, E., Lect. Notes Math., Springer, vol. 2068.
- [77] A. Zomorodian (2012), Topological data analysis. In *Advances in Applied and Computational Topology*, Proc. Sympos. Appl. Math., Amer. Math. Soc., **70**, 1–39.

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