

1-1-2019

## Gelfand-kirillov dimension of the quantized algebra of regular functions on homogeneous spaces

Partha Sarathi Chakraborty  
*Indian Statistical Institute, Kolkata*

Bipul Saurabh  
*Indian Institute of Technology Gandhinagar*

Follow this and additional works at: <https://digitalcommons.isical.ac.in/journal-articles>

---

### Recommended Citation

Chakraborty, Partha Sarathi and Saurabh, Bipul, "Gelfand-kirillov dimension of the quantized algebra of regular functions on homogeneous spaces" (2019). *Journal Articles*. 1016.  
<https://digitalcommons.isical.ac.in/journal-articles/1016>

This Research Article is brought to you for free and open access by the Scholarly Publications at ISI Digital Commons. It has been accepted for inclusion in Journal Articles by an authorized administrator of ISI Digital Commons. For more information, please contact [ksatpathy@gmail.com](mailto:ksatpathy@gmail.com).

See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/320075115>

# Gelfand–Kirillov dimension of the algebra of regular functions on quantum groups

Article · September 2017

---

CITATIONS

2

---

READS

51

2 authors:



[Partha Sarathi Chakraborty](#)

37 PUBLICATIONS 344 CITATIONS

[SEE PROFILE](#)



[Bipul Saurabh](#)

Indian Institute of Technology Gandhinagar

14 PUBLICATIONS 15 CITATIONS

[SEE PROFILE](#)

Some of the authors of this publication are also working on these related projects:



spectral triples [View project](#)

# Gelfand-Kirillov dimension of the algebra of regular functions on quantum groups

PARTHA SARATHI CHAKRABORTY, BIPUL SAURABH

September 28, 2017

## Abstract

Let  $G_q$  be the  $q$ -deformation of a simply connected simple compact Lie group  $G$  of type  $A$ ,  $C$  or  $D$  and  $\mathcal{O}_q(G)$  be the algebra of regular functions on  $G_q$ . In this article, we prove that the Gelfand-Kirillov dimension of  $\mathcal{O}_q(G)$  is equal to the dimension of real manifold  $G$ .

**AMS Subject Classification No.:** 16P90, 17B37, 20G42

**Keywords.** Quantized function algebra, Weyl group, Gelfand Kirillov dimension.

## 1 Introduction

Motivated by the isomorphism theorem of Weyl algebras, Gelfand and Kirillov [9] introduced a measure namely, Gelfand Kirillov dimension (abbreviated as GKdim), of growth of an algebra. For finitely generated commutative algebra  $\mathcal{A}$ , the Gelfand Kirillov dimension is same as the Krull dimension of  $\mathcal{A}$  and for commutative domains, it equals the transcendence degree of its fraction field (see chapter 4, [9]). In order to give a precise estimate for the growth exponent, Banica and Vergnioux [1] proved that for a connected simply connected compact real Lie group  $G$ , GKdim of the Hopf-algebra  $\mathcal{O}(G)$  generated by matrix co-efficients of all finite dimensional unitary representations of  $G$  is same as manifold dimension of  $G$ . In the same article, they mentioned that they do not have any other example of Hopf algebra having polynomial growth. Later D'Andrea, Pinzari and Rossi ([4]) extended their result to compact Lie groups (see Theorem 3.1, [4]). But apart from these commutative examples, not much is known about the growth of other Hopf algebras. For many noncommutative Hopf algebras, even the question whether they have a polynomial growth remain unanswered. Therefore it is worthwhile to investigate in this direction. The most natural candidate to investigate is the Hopf algebra of finite dimensional unitary representations of a compact quantum group. In this paper, we take the case of  $q$ -deformation of a classical Lie group of type  $A$ ,  $C$  and  $D$  and extend the result of Banica and Vergnioux to the noncommutative Peter-Weyl algebra associated with these compact quantum groups.

Let  $G$  be a semisimple simply connected compact Lie group of rank  $n$  and  $\mathfrak{g}$  be the Lie algebra of  $G$ . The algebra of functions  $\mathcal{O}_q(G)$  on its  $q$ -deformation  $G_q$  can be defined as the subalgebra of the dual algebra of quantized universal enveloping algebra  $U_q(\mathfrak{g})$  generated by matrix co-efficients of all finite dimensional admissible representations of  $U_q(\mathfrak{g})$ . In this article, we are mainly interested in the computation of GKdim for  $\mathcal{O}_q(G)$ . In commutative case i.e. for  $q = 1$ , Banica and Vergnioux [1] proved that the GKdim of polynomial algebra  $\mathcal{O}(G)$  is equal to the dimension of  $G$  as a real manifold. Here we show that in noncommutative case i.e. for  $0 < q < 1$ , the canonical Hopf  $*$ -algebra  $\mathcal{O}_q(G)$  has GKdim equal to the dimension of real manifold  $G$  provided  $G$  is of type  $A$ ,  $C$  and  $D$ . Moreover, we prove that similar results hold for certain quotient spaces of  $G_q$  in type  $A$  and  $C$ . It answers the query of Banica and Vergnioux by providing examples of noncommutative noncocommutative Hopf algebras having polynomial growth.

Let us give a sketch of the proof. We will assume  $G$  to be of type  $A_n$ ,  $C_n$  or  $D_n$ . Let  $\ell(\omega_n)$  be the length of the longest element  $\omega_n$  of the Weyl group  $W_n$  of  $G$ . Let  $\mathcal{P}_q(\mathcal{S})$  be the algebra of endomorphisms on  $c_{00}(\mathbb{N})$  generated by the endomorphisms  $\sqrt{1 - q^{2N+2}}S$ ,  $\sqrt{1 - q^{2N}}S^*$  and  $\beta := q^N$  where  $S$  is the left shift operator,  $S^*$  is the right shift operator and  $N$  is the number operator. Further let  $\mathcal{P}(C(\mathbb{T}))$  be the algebra of endomorphisms  $c_{00}(\mathbb{Z})$  generated by left shift operator  $S$  and right shift operator  $S^*$ . It can be shown that the algebra  $\mathcal{O}_q(G)$  can be embedded as a subalgebra of  $\mathcal{P}(C(\mathbb{T}))^{\otimes n} \otimes \mathcal{P}_q(\mathcal{S})^{\otimes \ell(\omega_n)}$ . We show that the algebra generated by the left shift and the right shift in  $c_{00}(\mathbb{N})$  has GKdim 2 and the algebra generated by the left shift and the right shift in  $c_{00}(\mathbb{Z})$  has GKdim 1. It proves that GKdim of  $\mathcal{O}_q(G)$  is less than  $2\ell(\omega_n) + n$ . Next, we write  $\omega_n$  as a product of  $n$  elements in a certain manner and then using recursion we produce enough number of linealy independent endomorphisms in  $\mathcal{O}_q(G)$  to get GKdim of  $\mathcal{O}_q(G)$  to be equal to  $2\ell(\omega_n) + n$ . This completes the proof as the dimension of  $G$  as a real manifold is same as  $2\ell(\omega_n) + n$ .

Organisation of this paper is as follows. Next section is dedicated to the preliminaries on representation theory of the Hopf  $*$ -algebra  $\mathcal{O}_q(G)$ . In the third section, we compute GKdim of the algebra  $\mathcal{O}_q(G)$  and prove our main result. In the final section, we prove similar results for Peter Weyl algebra of some quotient spaces.

Throughout the paper algebras are assumed to be unital and over the field  $\mathbb{C}$ . Elements of the Weyl group will be called Weyl words. We denote by  $\ell(w)$  the length of the Weyl word  $w$ . We used  $SP(2n)$  instead of more commonly used notation  $SP(n)$  for symplectic group of rank  $n$  and hence quantum symplectic group is denoted by  $SP_q(2n)$ . Let us denote by  $\{e_n : n \in \mathbb{N}\}$  and  $\{e_n : n \in \mathbb{Z}\}$  the standard bases of the vector spaces  $c_{00}(\mathbb{N})$  and  $c_{00}(\mathbb{Z})$  respectively. The map  $e_n \mapsto e_{n-1}$  will be denoted by  $S$  and the map  $e_n \mapsto e_{n+1}$  will be denoted by  $S^*$ . The map  $e_n \mapsto ne_n$  will be called the number operator  $N$ . We denote by  $\prod_{i=1}^n a_i$  the element  $a_n a_{n-1} \cdots a_1$ . Let  $T$  and  $T'$  be two endomorphisms of the vector space  $c_{00}(\mathbb{Z})^{\otimes l} \otimes c_{00}(\mathbb{N})^{\otimes k}$  and  $V$  be a subspace of  $c_{00}(\mathbb{Z})^{\otimes l} \otimes c_{00}(\mathbb{N})^{\otimes k}$ . We say that  $T \sim T'$  on  $V$  if there exist natural

numbers  $m_1, m_2, \dots, m_k$  and a nonzero constant  $C$  such that

$$T = CT' \underbrace{(1 \otimes 1 \otimes \dots \otimes 1)}_{l \text{ copies}} \otimes q^{m_1 N} \otimes q^{m_2 N} \otimes \dots \otimes q^{m_k N}$$

on  $V$ . Throughout this paper,  $q$  will denote a real number in the interval  $(0, 1)$  and  $C$  is used to denote a generic constant.

## 2 Quantized algebra of regular functions

In this section, we recall the definition of quantized algebra of regular functions on a simply connected semisimple compact Lie group  $G$  and give a faithful homomorphism of this algebra in order to find a new set of generators consisting of endomorphisms of a vector space. For a detailed treatment, we refer the reader to ([7], Chapter 3 in [8]). Let  $G$  be a simply connected semisimple compact Lie group of rank  $n$  and  $\mathfrak{g}$  be its complexified Lie algebra. Fix a nondegenerate symmetric ad-invariant form  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$  such that its restriction to the real Lie algebra of  $G$  is negative definite. Let  $\Pi := \{\alpha_1, \alpha_2, \dots, \alpha_n\}$  be the set of simple roots. For simplicity, we write the root  $\alpha_i$  as  $i$  and the reflection  $s_{\alpha_i}$  defined by the root  $\alpha_i$  as  $s_i$ . The Weyl group  $W_n$  of  $G$  can be described as the group generated by the reflections  $\{s_i : 1 \leq i \leq n\}$ .

**Definition 2.1.** Let  $U_q(\mathfrak{g})$  be the quantized universal enveloping algebra of  $\mathfrak{g}$ . It has a  $*$ -structure corresponding to the compact real form of  $\mathfrak{g}$  (see page 161 and 179, [7]). The Hopf  $*$ -subalgebra of the dual Hopf  $*$ -algebra of  $U_q(\mathfrak{g})$  consisting of matrix co-efficients of finite dimensional unitarizable  $U_q(\mathfrak{g})$ -modules is called the quantized algebra of regular functions on  $G$  (see page 96 – 97, [8]). It is denoted by  $\mathcal{O}_q(G)$ .

Let  $((u_{j,\mathfrak{g}}^i))$  be the defining corepresentation of  $\mathcal{O}_q(G)$  if  $G$  is of type  $A_n$  and  $C_n$  and the irreducible corepresentation of  $\mathcal{O}_q(G)$  corresponding to the highest weight  $(1, 0, 0, \dots, 0)$  if  $G$  is of type  $D_n$ . In first case, entries of the matrix  $((u_{j,\mathfrak{g}}^i))$  generate the Hopf  $*$ -algebra  $\mathcal{O}_q(G)$ . In latter case, they generate a proper Hopf  $*$ -subalgebra of  $\mathcal{O}_q(\text{Spin}(2n))$  which we denote as  $\mathcal{O}_q(SO(2n))$ . The generators of  $\mathcal{O}_q(\text{Spin}(2n))$  are the matrix entries of the corepresentation  $((z_j^i))$  of  $\mathcal{O}_q(\text{Spin}(2n))$  with highest weight  $(1/2, 1/2, \dots, 1/2)$ . We denote the dimension of the corepresentation  $((u_{j,\mathfrak{g}}^i))$  by  $N_n$ . We will drop the subscript  $\mathfrak{g}$  in  $((u_{j,\mathfrak{g}}^i))$  whenever the Lie algebra  $\mathfrak{g}$  is clear from the context. Using a result of Korogodski and Soibelman ([8]), we will now describe all simple unitarizable  $\mathcal{O}_q(G)$ -modules.

**Elementary simple unitarizable  $\mathcal{O}_q(G)$ -modules:** Let  $d_i = \langle \alpha_i, \alpha_i \rangle / 2$  and  $q_i = q^{d_i}$  for  $1 \leq i \leq n$ . Define  $\phi_i : U_{q_i}(\mathfrak{sl}(2)) \longrightarrow U_q(\mathfrak{g})$  be a  $*$ -homomorphism given on the generators of  $U_{q_i}(\mathfrak{sl}(2))$  by,

$$K \longmapsto K_i, \quad E \longmapsto E_i, \quad F \longmapsto F_i.$$

By duality, it induces an epimorphism

$$\phi_i^* : \mathcal{O}_q(G) \longrightarrow \mathcal{O}_{q_i}(SU(2)).$$

We will use this map to get all elementary simple unitarizable modules of  $\mathcal{O}_q(G)$ . Denote by  $\Psi$  the following action of  $\mathcal{O}_q(SU(2))$  on  $c_{00}(\mathbb{N})$  (see Proposition 4.1.1, [8]);

$$\Psi(u_l^k)e_p = \begin{cases} \sqrt{1 - q^{2p}}e_{p-1} & \text{if } k = l = 1, \\ \sqrt{1 - q^{2p+2}}e_{p+1} & \text{if } k = l = 2, \\ -q^{p+1}e_p & \text{if } k = 1, l = 2, \\ q^pe_p & \text{if } k = 2, l = 1, \\ \delta_{kl}e_p & \text{otherwise .} \end{cases} \quad (2.1)$$

For each  $1 \leq i \leq n$ , define an action  $\pi_{s_i}^n := \Psi \circ \phi_i^*$  of  $\mathcal{O}_q(G)$ . Each  $\pi_{s_i}^n$  gives rise to an elementary simple  $\mathcal{O}_q(G)$ -module  $V_{s_i}$ . Also, for each  $t \in \mathbb{T}^n$ , there are one dimensional  $\mathcal{O}_q(G)$ -module  $V_t$  with the action  $\{\tau_t^n\}$ . Given two actions  $\varphi$  and  $\psi$  of  $\mathcal{O}_q(G)$ , define an action  $\varphi * \psi := (\varphi \otimes \psi) \circ \Delta$ . Similarly for any two  $\mathcal{O}_q(G)$ -module  $V_\varphi$  and  $V_\psi$ , define  $V_\varphi \otimes V_\psi$  as  $\mathcal{O}_q(G)$ -module with  $\mathcal{O}_q(G)$  action coming from  $\varphi * \psi$ . For  $w \in W_n$  such that  $s_{i_1}s_{i_2}\dots s_{i_k}$  is a reduced expression for  $w$  and  $t \in \mathbb{T}^n$ , define an action  $\pi_{t,w}^n$  by  $\tau_t^n * \pi_{s_{i_1}}^n * \pi_{s_{i_2}}^n * \dots * \pi_{s_{i_k}}^n$  and denote the corresponding  $\mathcal{O}_q(G)$ -module by  $V_{t,w}$ . If  $t = 1$ , we write the action  $\pi_{t,w}^n$  as  $\pi_w^n$  and the associated module  $V_{t,w}$  by  $V_w$ . We refer the reader to ([8], page 121) for the following theorem.

**Theorem 2.2.** *The set  $\{V_{t,w}; t \in \mathbb{T}^n, w \in W_n\}$  is a complete set of mutually inequivalent simple unitarizable left  $\mathcal{O}_q(G)$ -module.*

Define the endomorphisms  $\alpha := \sqrt{1 - q^{2N+2}}S$ ,  $\alpha^* := \sqrt{1 - q^{2N}}S^*$  and  $\beta := q^N$  acting on the vector space  $c_{00}(\mathbb{N})$ . Let  $\mathcal{P}_q(\mathcal{T}) \subset \text{END}(c_{00}(\mathbb{N}))$  be the algebra generated by  $\alpha$ ,  $\alpha^*$  and  $\beta$  and  $\mathcal{P}(C(\mathbb{T})) \subset \text{END}(c_{00}(\mathbb{Z}))$  be the algebra generated by  $S$  and  $S^*$ . Given a Weyl word  $w$  of length  $\ell(w)$ , we define a homomorphism  $\chi_w^n : \mathcal{O}_q(G) \longrightarrow \mathcal{P}(C(\mathbb{T}))^{\otimes n} \otimes \mathcal{P}_q(\mathcal{T})^{\otimes \ell(w)}$  such that  $\chi_w^n(a)(t) = \pi_{t,w}^n(a)$  for all  $a \in \mathcal{O}_q(G)$ .

**Theorem 2.3.** *Let  $\omega_n$  be the longest word of the Weyl group of  $G$ . Then the homomorphism*

$$\chi_{\omega_n}^n : \mathcal{O}_q(G) \longrightarrow \mathcal{P}(C(\mathbb{T}))^{\otimes n} \otimes \mathcal{P}_q(\mathcal{T})^{\otimes \ell(\omega_n)}$$

*is faithful.*

*Proof:* Consider the enveloping  $C^*$ -algebra  $C(G_q)$  of the Hopf  $*$ -algebra  $\mathcal{O}_q(G)$ . For each  $w \in W_n$  and  $t \in \mathbb{T}^n$ , one can extend the irreducible representation  $\pi_{t,w}^n$  and homomorphism  $\chi_w^n$  to the  $C^*$ -algebra  $C(G_q)$  which we will denote by the same symbols. It follows from [10] that the set  $\{\pi_{t,w}^n; t \in \mathbb{T}^n, w \in W_n\}$  is a complete set of mutually inequivalent irreducible representations

of  $C(G_q)$ . It is not difficult to show that if  $w'$  is a subword of  $w$  then the representation  $\pi_{t,w'}^n$  factors through the homomorphism  $\chi_w^n$ . Since  $\omega_n$  is the longest word of  $W_n$ , it follows that each irreducible representation factors through  $\chi_{\omega_n}^n$ . As a consequence, the homomorphism  $\chi_{\omega_n}^n : C(G_q) \rightarrow C(\mathbb{T}^n) \otimes \mathcal{F}^{\otimes \ell(\omega_n)}$  is faithful. Restricting this homomorphism to the subalgebra  $\mathcal{O}_q(G)$  proves the claim.  $\square$

Consider the action  $\chi_e^n$  of  $\mathcal{O}_q(G)$  on the vector space  $c_{00}(\mathbb{Z})^{\otimes n}$ . It is not difficult to see that  $\chi_e^n(a)(t) = \tau_t(a)$  for all  $a \in \mathcal{O}_q(G)$ . Therefore for any  $w \in W$ , we have  $\chi_w^n = \chi_e^n * \pi_w^n$ . We will explicitly write down the endomorphisms  $\chi_e^n(u_j^i)$  of  $\mathcal{O}_q(G)$  for type  $A_n$ ,  $C_n$  or  $D_n$ .

For  $\mathcal{O}_q(G) = \mathcal{O}_q(SU(n+1))$ ,

$$\chi_e^n(u_j^i) = \begin{cases} \delta_{ij} 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{S^*}_{n+2-i \text{th place}} \otimes 1 \otimes \cdots \otimes 1 & \text{if } i \neq 1, \\ \delta_{ij} S \otimes S \otimes \cdots \otimes S & \text{if } i = 1. \end{cases}$$

For  $\mathcal{O}_q(G) = \mathcal{O}_q(SP(2n))$  or  $\mathcal{O}_q(\text{Spin}(2n))$ ,

$$\chi_e^n(u_j^i) = \begin{cases} \delta_{ij} 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{S^*}_{2n+1-i \text{th place}} \otimes 1 \otimes \cdots \otimes 1 & \text{if } i > n, \\ \delta_{ij} 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{S}_i \otimes 1 \otimes \cdots \otimes 1 & \text{if } i \leq n. \end{cases}$$

Looking at the expression of  $\chi_e^n(u_j^i)$ , it follows that

$$\chi_w^n(u_j^i) = (\chi_e^n \otimes \pi_w^n)(\Delta(u_j^i)) = (\chi_e^n \otimes \pi_w^n)\left(\sum_{k=1}^{N_n} u_k^i \otimes u_j^k\right) = \chi_e^n(u_i^i) \otimes \pi_w^n(u_j^i). \quad (2.2)$$

### 3 Main result

In the present section, we show that Gelfand-Kirillov dimension of quantized algebra of regular functions on a simply connected simple compact Lie group  $G$  of type  $A$ ,  $C$  or  $D$  is equal to the dimension of  $G$  as a real manifold. Unless otherwise specified, we denote by  $\mathcal{O}_q(G)$  one of the Hopf  $*$ -algebras  $\mathcal{O}_q(SU(n+1))$ ,  $\mathcal{O}_q(SP(2n))$  or  $\mathcal{O}_q(\text{Spin}(2n))$ .

**Definition 3.1.** ([9]) Let  $A$  be a unital algebra. The Gelfand-Kirillov dimension of  $A$  is given by

$$\text{GKdim}(A) = \sup_V \overline{\lim} \frac{\ln \dim(V^k)}{\ln k}$$

where the supremum is taken over all finite dimensional subspace  $V$  of  $A$  containing 1. If  $A$  is a finitely generated unital algebra then

$$\text{GKdim}(A) = \sup_{\xi} \overline{\lim} \frac{\ln \dim(\xi^k)}{\ln k}$$

where the supremum is taken over all finite sets  $\xi$  containing 1 that generates  $A$ .

**Remark 3.2.** The quantity “ $\overline{\lim} \frac{\ln \dim(\xi^k)}{\ln k}$ ,” does not depend on particular choices of  $\xi$  and hence one can choose a fixed (but finite) set of generators of  $A$ .

We state some properties of Gelfand-Kirillov dimension omitting their straightforward proofs.

- If  $B$  is a finitely generated unital subalgebra of  $A$  then  $GKdim(B) \leq GKdim(A)$  (see [11]).
- $GKdim(A \otimes B) \leq GKdim(A) + GKdim(B)$ .

**Proposition 3.3.**  $GKdim(\mathcal{P}(C(\mathbb{T}))) = 1$  and  $GKdim(\mathcal{P}_q(\mathcal{T})) = 2$ .

*Proof:* Clearly  $\{1, S, S^*\}^m = \text{span}\{S^k : -m \leq k \leq m\}$  and hence  $GKdim(\mathcal{P}(C(\mathbb{T}))) = 1$ . To show the other claim, take the generating set of  $\mathcal{P}_q(\mathcal{T})$  to be  $F = \{1, \alpha, \alpha^*, \beta\}$ . From the commutation relations  $q\beta\alpha = \alpha\beta$  and  $\alpha\alpha^* - \alpha^*\alpha = (1 - q^2)\beta^2$ , it is easy to see that

$$F^m = \text{span}\{(\alpha^*)^{m_1} \beta^{m_2} \alpha^{m_3} : m_1 + m_2 + m_3 \leq m\}.$$

Since  $\beta^2 = 1 - \alpha^*\alpha$ , we get

$$F^m = \text{span}\left\{ \{(\alpha^*)^{m_1} \beta \alpha^{m_3} : m_1 + m_3 < m\} \cup \{(\alpha^*)^{m_1} \alpha^{m_3} : m_1 + m_3 \leq m\} \right\}.$$

Hence the dimension of  $F^m$  is less than or equal to  $(m+1)^2$ . Since  $\{(\alpha^*)^{m_1} \alpha^{m_3} : m_1 + m_3 \leq m\}$  are linearly independent set of endomorphisms, we conclude that the dimension of  $F^m$  is greater than or equal to  $\binom{m+1}{2}$ . Putting together, we get  $GKdim(\mathcal{P}_q(\mathcal{T})) = 2$ .  $\square$

**Lemma 3.4.** *Let  $\omega_n$  be the longest element of the Weyl group of  $G$ . Then one has*

$$GKdim(\mathcal{O}_q(G)) \leq 2\ell(\omega_n) + n.$$

*Proof:* By Theorem 2.3, the algebra  $\mathcal{O}_q(G)$  can be viewed as a subalgebra of  $\mathcal{P}(C(\mathbb{T}))^{\otimes n} \otimes \mathcal{P}_q(\mathcal{T})^{\otimes \ell(\omega_n)}$ . Using the properties of Gelfand-Kirillov dimension mentioned above and Proposition 3.3, we have

$$GKdim(\mathcal{O}_q(G)) \leq GKdim(\mathcal{P}(C(\mathbb{T}))^{\otimes n} \otimes \mathcal{P}_q(\mathcal{T})^{\otimes \ell(\omega_n)}) \leq 2\ell(\omega_n) + n.$$

This settles the claim.  $\square$

In what follows, we will show that equality holds in Lemma 3.4. Our strategy is similar to that given in [3] with some modifications. First we need the following result.

**Lemma 3.5.** *Let  $\alpha_d$  be the endomorphism  $S\sqrt{1 - q^{2dN}}$  of the vector space  $c_{00}(\mathbb{N})$ . For any fixed  $j, k \in \mathbb{N}$  and  $0 \leq i \leq j$ , let  $T_i \sim \alpha_d^i (\alpha_d^*)^{i+k}$  on  $c_{00}(\mathbb{N})$ . Then elements of the set  $\{T_i : 0 \leq i \leq j\}$  are linearly independent endomorphisms.*



*Proof:* Enough to prove for  $k = 0$ . Consider the set  $\{q^{a_r N} : 1 \leq r \leq s, \text{ no two } a_r \text{'s are same}\}$ . Let  $V$  be the following Vandermonde matrix;

$$V = \begin{bmatrix} 1 & q^{a_1} & q^{2a_1} & \dots & q^{(s-1)a_1} \\ 1 & q^{a_2} & q^{2a_2} & \dots & q^{(s-1)a_2} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 1 & q^{a_s} & q^{2a_s} & \dots & q^{(s-1)a_s} \end{bmatrix}.$$

Since  $\det(V) = \prod_{p \neq r} (q^{a_p} - q^{a_r}) \neq 0$ , it follows that elements of the set  $\{q^{a_r N} : 1 \leq r \leq s, \text{ no two } a_r \text{'s are same}\}$  are linealy independent. Next, we have

$$\begin{aligned} T_i &\sim \alpha_d^i (\alpha_d^*)^i \sim (1 - q^{2dN+2d})(1 - q^{2dN+4d}) \dots (1 - q^{2dN+2dm}), \\ \Rightarrow T_i &= C_i q^{m_i N} (1 - q^{2dN+2d}) \dots (1 - q^{2dN+2dm}) \end{aligned}$$

for some  $m_i \in \mathbb{N}$  and nonzero constant  $C_i$ . If we expand the right hand side, we get  $2^m$  terms of the form  $q^{b_r} q^{a_r N}$  such that all  $a_r$ 's are different. Let us assume that  $\sum_{i=1}^j c_i T_i = 0$ . Since  $T_j$  has  $2^j$  terms of the form  $q^{a_r N}$  and  $2^j > \sum_{i=1}^{j-1} 2^i$ , we get  $c_j q^{sN} = 0$  for some  $s \in \mathbb{N}$  which further implies that  $c_j = 0$ . Repeating the same argument, we get  $c_i = 0$  for all  $1 \leq i \leq j$  and this completes the proof.  $\square$

We will recall from [3] some results that will be needed to prove our main claim.

**Lemma 3.6.** *Let  $w \in W_n$ . Then there exist polynomials  $p_1^{(w,n)}, p_2^{(w,n)}, \dots, p_{\ell(w_n)}^{(w,n)}$  with non-commuting variables  $\pi_w^n(u_j^{N_j})$ 's and a permutation  $\sigma$  of  $\{1, 2, \dots, \ell(w_n)\}$  such that for all  $r_1^n, r_2^n, \dots, r_{\ell(w_n)}^n \in \mathbb{N}$ , one has*

$$p_j^{(w,n)} \sim 1^{\otimes \sum_{i=1}^{n-1} \ell(w_i)} \otimes 1^{\otimes \sigma(j)-1} \otimes \sqrt{1 - q^{2d\sigma(j)N}} S^* \otimes 1^{\otimes \ell(w_n) - \sigma(j)} \quad (3.1)$$

on the subspace generated by standard basis elements having  $e_0$  at  $(\sum_{i=1}^{n-1} \ell(w_i) + \sigma(k))^{th}$  place for  $k < j$ .

*Proof:* See the proof of the Lemma 3.4 in [3].  $\square$

An element  $w$  of  $W_n$  can be written in a reduced form as:  $\psi_{1,k_1}^{(\epsilon_1)}(w) \psi_{2,k_2}^{(\epsilon_2)}(w) \dots \psi_{n,k_n}^{(\epsilon_n)}(w)$  for some choices of  $\epsilon_1, \epsilon_2, \dots, \epsilon_n$  and  $k_1, k_2, \dots, k_n$  where  $\epsilon_r \in \{0, 1, 2\}$  and  $n - r + 1 \leq k_r \leq n$  with the convention that,

**Case 1:**  $\mathfrak{sl}(n+1)$

$$\psi_{r,k_r}^\epsilon(w) = \begin{cases} s_r s_{r-1} \dots s_{n-k_r+1} & \text{if } \epsilon = 1, 2 \\ \text{empty string} & \text{if } \epsilon = 0. \end{cases}$$

**Case 2:  $\mathfrak{sp}(2n)$**

$$\psi_{r,k_r}^\epsilon(w) = \begin{cases} s_{n-r+1}s_{n-r+2} \cdots s_{k_r} & \text{if } \epsilon = 1, \\ s_{n-r+1}s_{n-r+2} \cdots \dots s_{n-1}s_n s_{n-1} \cdots s_{k_r} & \text{if } \epsilon = 2, \\ \text{empty string} & \text{if } \epsilon = 0. \end{cases}$$

**Case 3:  $\mathfrak{so}(2n)$**

$$\psi_{r,k_r}^\epsilon(w) = \begin{cases} s_{n-r+1}s_{n-r+2} \cdots s_{k_r} & \text{if } \epsilon = 1, \\ s_{n-r+1}s_{n-r+2} \cdots \dots s_{n-1}s_n s_{n-2}s_{n-3} \cdots s_{k_r} & \text{if } \epsilon = 2, \\ \text{empty string} & \text{if } \epsilon = 0. \end{cases}$$

For details, we refer the reader to ([6] or subsection 2.2 in [3]). We call the word  $\psi_{r,k_r}^{(\epsilon_r)}(w)$  the  $r^{\text{th}}$  part  $w_r$  of  $w$ . For type  $C_n$  and  $D_n$ , let  $M_n^i = n - i + 1$  and  $N_n^i = N_n - n + i$  and for type  $A_n$ , let  $M_n^i = 1$  and  $N_n^i = i + 1$ .

**Lemma 3.7.** *Let  $w \in W_n$  be of the form  $w = w_{i+1}w_{i+2} \cdots w_l$ ,  $l \leq n$  and let  $V_w$  be the associated  $\mathcal{O}_q(G)$ -module. Then for each  $M_n^i \leq k \leq N_n^i$ , there exists unique  $r_w(k) \in \{M_n^{i+l}, \dots, N_n^{i+l}\}$  such that*

$$\pi_w^n(u_{r_w(k)}^k)(e_0 \otimes e_0 \otimes \cdots \otimes e_0) = C e_0 \otimes e_0 \otimes \cdots \otimes e_0$$

where  $C$  is a nonzero real number. Moreover,

1.  $\pi_w^n(u_{r_w(k)}^j)(e_0 \otimes e_0 \otimes \cdots \otimes e_0) = 0$  for  $j \in \{M_n^i, \dots, N_n^i\} \setminus \{k\}$ .
2.  $\pi_w^n((u_{r_w(k)}^k)^*)(e_0 \otimes e_0 \otimes \cdots \otimes e_0) = C e_0 \otimes e_0 \otimes \cdots \otimes e_0$ .
3.  $\pi_w^n((u_{r_w(k)}^j)^*)(e_0 \otimes e_0 \otimes \cdots \otimes e_0) = 0$  for  $j \in \{M_n^i, \dots, N_n^i\} \setminus \{k\}$ .

*Proof:* In Lemma 3.6 in [3], explicit description of the function  $r_w(k)$  is given. Using that and diagram representation, one can verify the claim.  $\square$

The following lemma gives some linearly independent endomorphisms which when applied to a fixed vector of the form  $e_0 \otimes v \otimes e_0^{\otimes \ell(w_n)}$  give all matrix units of the the first component and the components in the  $n^{\text{th}}$  part. More precisely,

**Lemma 3.8.** *Let  $w \in W_n$ . Then there exist a permutation  $\sigma$  of  $\{1, 2, \dots, \ell(w_n)\}$  and polynomials  $g_0^{(w,n)}, g_1^{(w,n)}, \dots, g_{\ell(w_n)}^{(w,n)}$  and  $g_{1*}^{(w,n)}, \dots, g_{\ell(w_n)*}^{(w,n)}$  with variables  $\chi_w^n(u_j^{N_n})$ 's and  $\chi_w^n((u_j^{N_n})^*)$ 's such that*

1.

$$\begin{aligned} & (g_0^{(w,n)})_{r_0^n} (g_{1*}^{(w,n)})_{s_{\sigma(1)}} (g_1^{(w,n)})_{r_{\sigma(1)}} (g_{2*}^{(w,n)})_{s_{\sigma(2)}} (g_2^{(w,n)})_{r_{\sigma(2)}} \dots (g_{\ell(w_n)*}^{(w,n)})_{s_{\sigma(\ell(w_n))}} \\ & (g_{\ell(w_n)}^{(w,n)})_{r_{\sigma(\ell(w_n))}} (e_0 \otimes v \otimes e_0^{\otimes \ell(w_n)}) = C e_{r_0^n} \otimes v \otimes e_{r_1^n - s_1^n} \otimes \cdots \otimes e_{r_{\ell(w_n)}^n - s_{\ell(w_n)}^n} \end{aligned}$$

where  $v \in c_{00}(\mathbb{Z})^{\otimes n-1} \otimes c_{00}(\mathbb{N})^{\otimes \sum_{k=1}^{n-1} \ell(w_k)}$ ,  $r_0^n \in \mathbb{N}$ ,  $r_i^n, s_i^n \in \mathbb{N}$  and  $r_i^n \geq s_i^n$  for  $1 \leq i \leq \ell(w_n)$ .

2. The elements of the set

$$\left\{ (g_0^{(w,n)})^{r_0^n} (g_{1*}^{(w,n)})^{s_{\sigma(1)}^n} (g_1^{(w,n)})^{r_{\sigma(1)}^n} (g_{2*}^{(w,n)})^{s_{\sigma(2)}^n} (g_2^{(w,n)})^{r_{\sigma(2)}^n} \dots \right. \\ \left. \dots (g_{\ell(w_n)*}^{(w,n)})^{s_{\sigma(\ell(w_n))}^n} (g_{\ell(w_n)}^{(w,n)})^{r_{\sigma(\ell(w_n))}^n} : r_0^n \in \mathbb{N}, r_i^n, s_i^n \in \mathbb{N}, r_i^n \geq s_i^n \in \mathbb{N} \right\}$$

are linearly independent endomorphisms.

*Proof:* Define

$$g_0^{(w,n)} := \chi_w^n(u_{N_n - \ell(w_n)}^{N_n}).$$

Since the endomorphism  $\pi_w^n(u_{N_n - \ell(w_n)}^{N_n})$  is of the form  $1^{\otimes \sum_{k=1}^{n-1} \ell(w_k)} \otimes q^{m_1 N} \otimes q^{m_2 N} \otimes \dots \otimes q^{m_{\ell(w_n)} N}$ , we get

$$g_0^{(w,n)} = \chi_w^n(u_{N_n - \ell(w_n)}^{N_n}) = \chi_e^n * \pi_w^n(u_{N_n - \ell(w_n)}^{N_n}) = \chi_e^n(u_{N_n}^{N_n}) \otimes \pi_w^n(u_{N_n - \ell(w_n)}^{N_n}) \\ \sim \underbrace{S^* \otimes 1 \otimes \dots \otimes 1}_{n \text{ times}} \otimes 1^{\otimes \sum_{k=1}^n \ell(w_k)} \quad (3.2)$$

on the whole vector space. For  $1 \leq j \leq \ell(w_n)$ , let  $h_j^{(w,n)}$  be the polynomial obtained by replacing the action  $\pi_w^n$  in the polynomial  $p_j^{(w,n)}$  given in Lemma 3.6 with  $\chi_w^n$ . Define

$$g_j^{(w,n)} := (g_0^{(w,n)*})^s h_j^{(w,n)}$$

where  $s = \text{degree of } p_j^{(w,n)}$ . Hence we have

$$g_j^{(w,n)} = (g_0^{(w,n)*})^s h_j^{(w,n)} \sim \underbrace{1 \otimes 1 \otimes \dots \otimes 1}_{n \text{ times}} \otimes p_j^{(w,n)} \quad (3.3)$$

on the subspace generated by standard orthonormal basis elements having  $e_0$  at  $(n + \sum_{i=1}^{n-1} \ell(w_i) + \sigma(k))^{th}$  place for  $k < j$ . Define  $g_{j*}^{(w,n)} = (g_j^{(w,n)*})^*$ . Now part (1) of the claim follows from Lemma 3.6. Further it follows from first part of the claim that possible dependency can occur among the elements of the type:

$$\left\{ (g_0^{(w,n)})^{r_0^n} (g_{1*}^{(w,n)})^{p_{\sigma(1)}^n} (g_1^{(w,n)})^{r_{\sigma(1)}^n} (g_{2*}^{(w,n)})^{p_{\sigma(2)}^n} (g_2^{(w,n)})^{r_{\sigma(2)}^n} \dots \right. \\ \left. \dots (g_{\ell(w_n)*}^{(w,n)})^{p_{\sigma(\ell(w_n))}^n} (g_{\ell(w_n)}^{(w,n)})^{r_{\sigma(\ell(w_n))}^n} : r_i^n - p_i^n = c_i^n \right\}.$$

where  $c_i^n$  are arbitrary but fixed natural number. From Lemma 3.6, it follows that

$$(g_j^{(w,n)})^r = (g_0^{(w,n)*})^s h_j^{(w,n)} \\ \sim 1^{\otimes (n + \sum_{i=1}^{n-1} \ell(w_i))} \otimes 1^{\otimes (\sigma(j) - 1)} \otimes (\sqrt{1 - q^{2d_i N}} S^*)^r \otimes 1^{\otimes (\ell(w_n) - \sigma(j))}.$$

on the vector subspace generated by standard orthonormal basis elements having  $e_0$  at  $(n + \sum_{i=1}^{n-1} \ell(w_i) + \sigma(k))^{th}$  place for each  $k < j$ . Employing this and Lemma 3.5, we get the claim.  $\square$

If one counts the number of linearly independent endomorphisms given in part (2) of the lemma, one can show that  $GKdim(\mathcal{O}_q(G)) \geq 2\ell(n) + 1$ . To get the upper bound, we need to extend this result.

**Lemma 3.9.** *Let  $w \in W$  and  $\chi_w^n$  be the associated action of  $\mathcal{O}_q(G)$  in the vector space  $c_{00}(\mathbb{Z})^{\otimes n} \otimes c_{00}(\mathbb{N})^{\otimes \ell(w)}$ . Then for each  $1 \leq i < n$ , there exist endomorphisms  $P_1^i, P_2^i, \dots, P_{N_i}^i$  and  $R_1^i, R_2^i, \dots, R_{N_i}^i$  in  $C(G_q)$  such that for all  $1 \leq j \leq N_i$ , one has*

$$\begin{aligned} P_j^i(v \otimes e_0 \otimes \dots \otimes e_0) &= C(\chi_{w_1 w_2 \dots w^i}^i(u_j^{N_i})v) \otimes e_0 \otimes \dots \otimes e_0 \\ R_j^i(v \otimes e_0 \otimes \dots \otimes e_0) &= C((\chi_{w_1 w_2 \dots w^i}^i(u_j^{N_i}))^*v) \otimes e_0 \otimes \dots \otimes e_0 \end{aligned}$$

where  $v \in c_{00}(\mathbb{Z})^{\otimes n} \otimes c_{00}(\mathbb{N})^{\otimes \sum_{j=1}^i \ell(w_j)}$  and  $C$  is a nonzero constant.

*Proof:* Define the endomorphisms

$$\begin{aligned} P_j^i &:= \chi_w^n(u_{r_{w^{i+1} w^{i+2} \dots w^n}(j)}^{N_i}) \\ R_j^i &:= (\chi_w^n(u_{r_{w^{i+1} w^{i+2} \dots w^n}(j)}^{N_i}))^* \end{aligned}$$

for  $1 \leq j \leq N_i$ . By applying Lemma 3.7, the claim follows immediately.  $\square$

**Lemma 3.10.** *Let  $w \in W$  and  $\chi_w^n$  be the associated action of  $\mathcal{O}_q(G)$  in the vector space  $c_{00}(\mathbb{Z})^{\otimes n} \otimes c_{00}(\mathbb{N})^{\otimes \ell(w)}$ . Then for each  $1 \leq i \leq n$ , there exist permutations  $\sigma_i$  of  $\{1, 2, \dots, \ell(w_i)\}$  and polynomials  $g_0^{(w,i)}, g_1^{(w,i)}, g_2^{(w,i)}, \dots, g_{\ell(w_i)}^{(w,i)}$ ;  $g_{1*}^{(w,i)}, \dots, g_{\ell(w_i)*}^{(w,i)}$  with noncommutative variables  $\chi_w^n(u_s^n)$ 's and  $\chi_w^n((u_s^n)^*)$ 's such that*

1.

$$\begin{aligned} &\prod_{i=1}^{\overleftarrow{n}} (g_{1*}^{(w,i)})^{p_{\sigma_i(1)}^i} (g_1^{(w,n)})^{r_{\sigma_i(1)}^i} (g_{2*}^{(w,i)})^{p_{\sigma_i(2)}^i} (g_2^{(w,i)})^{r_{\sigma_i(2)}^i} \dots (g_{\ell(w_i)*}^{(w,i)})^{p_{\sigma_i(\ell(w_i))}^i} \\ &(g_{\ell_i}^{(w,i)})^{r_{\sigma_i(\ell_i)}^i} \prod_{i=1}^{\overleftarrow{n}} (g_0^{(w,i)})^{r_0^i} (e_0 \otimes \dots \otimes e_0) = C e_{r_0^n} \otimes e_{r_0^{n-1}} \otimes \dots \otimes e_{r_0^1} \otimes e_{r_1^{n-1}} \otimes e_{r_2^{n-1}} \otimes \dots \\ &\dots \otimes e_{r_{\ell(w_1)}^1} \otimes \dots \otimes e_{r_1^n} \otimes e_{r_2^n} \otimes \dots \otimes e_{r_{\ell(w_n)}^n} \end{aligned}$$

where  $r_0^j \in \mathbb{N}$ ,  $r_j^i, p_j^i \in \mathbb{N}$  and  $r_j^i \geq p_j^i$  for  $1 \leq j \leq \ell(w_i)$  and  $1 \leq i \leq n$ .

2. The elements of the set

$$\begin{aligned} &\left\{ \prod_{i=1}^{\overleftarrow{n}} (g_{1*}^{(w,i)})^{p_{\sigma_i(1)}^i} (g_1^{(w,n)})^{r_{\sigma_i(1)}^i} (g_{2*}^{(w,i)})^{p_{\sigma_i(2)}^i} (g_2^{(w,i)})^{r_{\sigma_i(2)}^i} \dots (g_{\ell_i*}^{(w,i)})^{p_{\sigma_i(\ell_i)}^i} \right. \\ &\left. (g_{\ell_i}^{(w,i)})^{r_{\sigma_i(\ell_i)}^i} \prod_{i=1}^{\overleftarrow{n}} (g_0^{(w,i)})^{r_0^i} : r_0^i \in \mathbb{N}, r_j^i \geq p_j^i \text{ for } 1 \leq j \leq \ell(w_i) \text{ and } 1 \leq i \leq n \right\} \end{aligned}$$

are linearly independent endomorphisms.

*Proof:* For  $0 \leq j \leq \ell(w_n)$ , let  $g_j^{(w,n)}$  and  $g_{j*}^{(w,n)}$  be the polynomials as given in Lemma 3.8 and  $\sigma_n$  be the associated permutation. To define permutations  $\sigma_i$  and polynomials  $g_j^{(w,i)}$  and  $g_{j*}^{(w,i)}$  for  $0 \leq j \leq \ell(w_i)$  and  $1 \leq i < n$  we view  $w_1 w_2 \cdots w_i$  as an element of Weyl group of  $G$  of rank  $i$ . Therefore we can define polynomial  $g_j^{(w_1 w_2 \cdots w_i, i)}$  and the permutation  $\sigma_i$  from Proposition 3.8. Replace the variables  $\chi_{w_1 w_2 \cdots w_i}^i(u_k^{N_i})$  with  $P_k^i$  and  $\chi_{w_1 w_2 \cdots w_i}^i((u_k^{N_i})^*)$  with  $R_k^i$  for  $1 \leq k \leq N_i$  in the polynomials  $g_j^{(w_1 w_2 \cdots w_i, i)}$  and  $g_{j*}^{(w_1 w_2 \cdots w_i, i)}$  to define the polynomial  $g_j^{(w,i)}$  and  $g_{j*}^{(w,i)}$  respectively for all  $0 \leq j \leq \ell(w_i)$ . Now both parts of the claim follows from Lemma 3.8 and Lemma 3.9.  $\square$

Define

$$\xi_{G_q} = \begin{cases} \{u_j^i : 1 \leq i, j \leq N_n\} \cup \{1\} & \text{for } \mathcal{O}_q(G) = \mathcal{O}(SU_q(n+1)) \text{ or } \mathcal{O}_q(SP(2n)), \\ \{u_j^i : 1 \leq i, j \leq N_n\} \cup \{z_j^i : 1 \leq i, j \leq 2^n\} \cup \{1\} & \text{for } \mathcal{O}_q(G) = \mathcal{O}(\text{Spin}_q(n)). \end{cases}$$

Then  $\xi_{G_q}$  is a generating set of  $\mathcal{O}_q(G)$  containing 1. We will now prove our main result.

**Theorem 3.11.** *Let  $\omega_n$  be the longest element of the Weyl group of  $G$ . Then one has*

$$\text{GKdim}(\mathcal{O}_q(G)) = 2\ell(\omega_n) + n.$$

*Proof:* From Lemma 3.4, it is enough to show that  $\text{GKdim}(\mathcal{O}_q(G)) \geq 2\ell(\omega_n) + n$ . Since the homomorphism  $\chi_{\omega_n}^n$  is faithful, we will without loss of generality work with the algebra  $\chi_{\omega_n}^n(\mathcal{O}_q(G))$ . Take the generating set  $\mathcal{Y}$  to be  $\chi_{\omega_n}^n(\xi_{G_q}) \cup \chi_{\omega_n}^n(\xi_{G_q}^*)$ . Define

$$M_0 := \max\{\text{degree of } g_j^{(\omega_n, i)} : 0 \leq j \leq \ell((\omega_n)_i), 1 \leq i \leq n\}.$$

Then by part (2) of Lemma 3.10, we have

$$\text{GKdim}(\mathcal{O}_q(G)) \geq \overline{\lim} \frac{\ln \dim(\mathcal{Y}^{M_0 k})}{\ln M_0 k} \geq \overline{\lim} \frac{\ln \frac{\binom{k+2\ell(\omega_n)+n-1}{k}}{2}}{\ln M_0 k} = 2\ell(\omega_n) + n.$$

This completes the proof.  $\square$

**Remark 3.12.** The proof we have given here is very rigid in the sense that it largely depends upon a particular way of representing the algebra. There must be a canonical way of computing  $\text{GKdim}$  of these algebras. In our view, the main obstruction is to get Lemma 3.7 in a more general set up.

**Corollary 3.13.** *One has*

- $\text{GKdim}(\mathcal{O}_q(SU(n+1))) = n^2 + 2n = \dim(SU(n+1)).$
- $\text{GKdim}(\mathcal{O}_q(SP(2n))) = 2n^2 + n = \dim(SP(2n)).$

- $GKdim(\mathcal{O}_q(SO(2n))) = GKdim(\mathcal{O}_q(Spin(2n))) = 2n^2 - n = \dim(SO(2n))$ .

*Proof:* Let  $\omega_n$  be the longest word of  $\mathcal{O}_q(G)$ .

**Case 1:**  $\mathcal{O}_q(SU(n+1))$ .

In this case, the  $r^{th}$ -part of  $\omega_n$  is  $s_r s_{r-1} \cdots s_1$  for  $1 \leq r \leq n$ . Hence  $\ell(\omega_n) = n(n+1)/2$ . Therefore by Theorem 3.11, we have

$$GKdim(\mathcal{O}_q(SU(n+1))) = \frac{2n(n+1)}{2} + n = n^2 + 2n.$$

**Case 2:**  $\mathcal{O}_q(SP(2n))$ .

For each  $1 \leq r \leq n$ , the  $r^{th}$ -part of  $\omega_n$  is  $s_r s_{r+1} \cdots s_{n-1} s_n s_{n-1} \cdots s_{r+1} s_r$ . Hence by applying Theorem 3.11, we get

$$GKdim(\mathcal{O}_q(SP(2n))) = 2\ell(\omega_n) + n = 2n^2 + n.$$

**Case 3:**  $\mathcal{O}_q(Spin(2n))$ .

For each  $1 \leq r \leq n$ , the  $r^{th}$ -part of  $\omega_n$  is  $s_r s_{r+1} \cdots s_{n-2} s_{n-1} s_n s_{n-2} \cdots s_{r+1} s_r$ . Hence  $\ell(\omega_n) = n^2 - n$ . Therefore from Theorem 3.11, we get

$$GKdim(\mathcal{O}_q(Spin(2n))) = 2\ell(\omega_n) + n = 2n^2 - n.$$

Moreover since the polynomials  $g_0^{(\omega_n, k)}$ ,  $g_j^{(\omega_n, k)}$  and  $g_{j^*}^{(\omega_n, k)}$  given in Lemma 3.10 involve variables  $u_j^i$ 's which are in  $\mathcal{O}(SO_q(2n))$ , we get

$$GKdim(\mathcal{O}_q(SO(2n))) = GKdim(\mathcal{O}_q(Spin(2n))).$$

This completes the proof. □

## 4 Quotient spaces

Fix a subset  $S \subset \Pi$  and a subgroup  $L$  of  $\mathbb{T}^{\#S^c}$ . Let  $\mathcal{O}(G_q/K_q^{S,L})$  be the quotient Hopf  $*$ -subalgebra of  $\mathcal{O}_q(G)$  (see page 5, [10]). If  $S$  is the empty set  $\phi$ , define  $W_\phi = \{id\}$ . For a nonempty set  $S$ , define  $W_S$  to be the subgroup of  $W_n$  generated by the simple reflections  $s_\alpha$  with  $\alpha \in S$ . Let

$$W^S := \{w \in W_n : \ell(s_\alpha w) > \ell(w) \quad \forall \alpha \in S\}.$$

Define the algebra  $\mathcal{P}(C(L))$  to be the quotient of  $\mathcal{P}(C(\mathbb{T}^m))$  by the ideal consisting of polynomials vanishing in  $L$ .

**Theorem 4.1.** *Let  $\omega_n^S$  be the longest word of  $W^S$ ,  $m$  be the cardinality of  $S$  and  $k$  be the rank of  $L$ . Then the homomorphism*

$$\chi_{\omega_n^S}^n : \mathcal{O}(G_q/K_q^{S,L}) \longrightarrow \mathcal{P}(C(\mathbb{T}))^{\otimes m} \otimes \mathcal{P}_q(\mathcal{T})^{\otimes \ell(\omega_n^S)}$$

*is faithful. Moreover, the image  $\chi_{\omega_n^S}^n(\mathcal{O}(G_q/K_q^{S,L}))$  is contained in the algebra  $\mathcal{P}(C(L)) \otimes \mathcal{P}_q(\mathcal{T})^{\otimes \ell(\omega_n^S)}$ .*

**Lemma 4.2.** *Let  $L$  be a subgroup of  $\mathbb{T}^m$  of rank  $k$ . Then there exists an algebra isomorphism*

$$\Phi : \mathcal{P}(C(L)) \rightarrow \mathcal{P}(C(\mathbb{T}))^{\otimes k}.$$

*Proof:* Using Fourier transform, one can identify dual group  $\hat{L}$  as a subgroup of  $\mathbb{Z}^m$  isomorphic to  $\mathbb{Z}^k$ . Fix a linear isomorphism  $\phi : \mathbb{Z}^k \rightarrow \hat{L}$ . Applying inverse Fourier transform and using Pontriagin duality, one can identify points of  $\mathbb{T}^k$  with points of  $L$  via the monomial map  $\hat{\phi}$ . This induces an isomorphism

$$\Phi : \mathcal{P}(C(L)) \rightarrow \mathcal{P}(C(\mathbb{T}))^{\otimes k}$$

such that  $\Phi(g)(t_1, t_2, \dots, t_k) = g(\hat{\phi}(t_1, t_2, \dots, t_k))$ . □

**Remark 4.3.** Suppose that the polynomials  $\{g_i(t_1, t_2, \dots, t_m) : 1 \leq i \leq m\}$  generate the algebra  $\mathcal{P}(C(L))$ . Then from the Lemma 4.2, there exist monomials  $\{h_i(g_1, g_2, \dots, g_m) : 1 \leq i \leq k\}$  such that  $h_i(g_1, g_2, \dots, g_m)(t_1, t_2, \dots, t_m) = t_i$  for  $1 \leq i \leq k$ . Hence the elements of the set

$$\left\{ \prod_{i=1}^n (h_i(g_1, g_2, \dots, g_m))^{r_i} : r_i \in \mathbb{N}, 1 \leq i \leq k \right\}$$

are linearly independent polynomials.

Let  $S_1$  be the empty subset of  $\Pi$ . For  $2 \leq m \leq n$ , define  $S_m$  to be the set  $\{1, 2, \dots, m-1\}$  if  $\mathcal{O}_q(G) = \mathcal{O}(SU_q(n+1))$  and  $\{n-m+2, \dots, n\}$  if  $\mathcal{O}_q(G) = \mathcal{O}_q(SP(2n))$ . If  $L = \mathbb{T}^m$  then  $\mathcal{O}(G_q/K_q^{S_{n-m+1}, L})$  is same as  $\mathcal{O}(SU_q(n+1)/SU_q(n+1-m))$  or  $\mathcal{O}(SP_q(2n)/SP_q(2n-2m))$  if  $\mathcal{O}_q(G)$  is  $\mathcal{O}_q(SU(n+1))$  or  $\mathcal{O}_q(SP(2n))$  respectively.

**Theorem 4.4.** *Let  $\omega_n^{S_{n-m+1}}$  be the longest element of the  $W^{S_{n-m+1}}$  and  $k$  be the rank of  $L$ . Then one has*

$$\text{GKdim } \mathcal{O}(G_q/K_q^{S_{n-m+1}, L}) = 2\ell(\omega_n^{S_{n-m+1}}) + k.$$

*Proof:* By Theorem 4.1, the algebra  $\mathcal{O}(G_q/K_q^{S_{n-m+1}, L})$  can be viewed as a subalgebra of  $\mathcal{P}(C(\mathbb{T}))^{\otimes k} \otimes \mathcal{P}_q(\mathcal{T})^{\otimes \ell(\omega_n^{S_{n-m+1}})}$ . Hence using properties of GKdim, we get

$$\text{GKdim } \mathcal{O}(G_q/K_q^{S_{n-m+1}, L}) \leq 2\ell(\omega_n^{S_{n-m+1}}) + k.$$

To show the equality, observe that

1. for an element  $w \in W^{S_{n-m+1}}$  and  $1 \leq r \leq n-m$ , the  $r^{\text{th}}$ -part  $w_r = \psi_{r, k_r}^{(\epsilon_r)}(w)$  is identity element of  $W_n$ . Hence  $w$  can be written uniquely as  $w = w_{n-m+1} w_{n-m+2} \cdots w_n$ .
2. It follows from the definition that entries of last  $m$  rows of  $((u_j^i))$  is in the quotient algebra  $\mathcal{O}(G_q/K_q^{S_{n-m+1}, L})$ .

3. The polynomials  $g_j^{(w,k)}$  and  $g_{j*}^{(w,k)}$  for  $1 \leq j \leq \ell_k$  and  $n - m + 1 \leq k \leq n$  involve variables consisting of entries of last  $m$  rows of  $((u_j^i))$ .

4. It follows from equation (3.2) that for  $1 \leq i \leq m$ , we have

$$g_0^{(w,i)} \sim g_i(t_1, t_2, \dots, t_m) \otimes 1^{\otimes \ell(w)}$$

where  $g_i \in \mathcal{P}(C(L))$  is the projection function  $t_i$  on  $i^{\text{th}}$  co-ordinate restricted to  $L$ . Moreover,  $\{g_i(t_1, t_2, \dots, t_m) : 1 \leq i \leq m\}$  generate the algebra  $\mathcal{P}(C(L))$ .

5. Using remark (4.3) and Lemma 3.10, one can show that the elements of the set

$$\left\{ \prod_{i=n-m+1}^{\overleftarrow{n}} (g_{1*}^{(w,i)})^{p_{\sigma_i(1)}^i} (g_1^{(w,n)})^{r_{\sigma_i(1)}^i} (g_{2*}^{(w,i)})^{p_{\sigma_i(2)}^i} (g_2^{(w,i)})^{r_{\sigma_i(2)}^i} \dots (g_{\ell(w_i)*}^{(w,i)})^{p_{\sigma_i(\ell(w_i))}^i} \right. \\ \left. (g_{\ell(w_i)}^{(w,i)})^{r_{\sigma_i(\ell(w_i))}^i} \prod_{i=1}^{\overleftarrow{n}} (h_i(g_0^{(w,1)}, g_0^{(w,2)}, \dots, g_0^{(w,k)}))^{r_0^i} : r_0^i \in \mathbb{N}, r_j^i \geq p_j^i \right. \\ \left. \text{for } 1 \leq j \leq \ell(w_i) \text{ and } n - m + 1 \leq i \leq n \right\}$$

are linearly independent endomorphisms.

With these facts, the same arguments used in Theorem 3.11 will prove the claim.  $\square$

**Corollary 4.5.** *One has*

- $GKdim(\mathcal{O}(SU_q(n+1)/SU_q(n+1-m))) = dim(SU(n+1)/SU(n+1-m))$ .
- $GKdim(\mathcal{O}(SP_q(2n)/SP_q(2n-2m))) = dim(SP(2n)/SP(2n-2m))$ .

*Proof:* Let  $\omega_n^{S_{n-m+1}}$  be the longest word of  $\mathcal{O}_q(G)$ .

**Case 1:**  $\mathcal{O}(SU_q(n+1)/SU_q(n+1-m))$ .

In this case, the  $r^{\text{th}}$ -part of  $\omega_n^{S_{n-m+1}}$  is  $s_r s_{r-1} \dots s_1$  for  $n - m + 1 \leq r \leq n$ . Hence  $\ell(\omega_n^{S_{n-m+1}}) = \frac{n(n+1) - (n-m)(n-m+1)}{2}$ . Therefore by Theorem 3.11, we have

$$\begin{aligned} GKdim(\mathcal{O}_q(SU(n+1))) &= n(n+1) - (n-m)(n-m+1) + m \\ &= n(n+1) + n - (n-m)(n-m+1) + (n-m) \\ &= dim(SU(n+1)) - dim(SU(n+1-m)) \text{ (by Corollary 3.13)} \\ &= dim(SU(n+1)/SU(n+1-m)). \end{aligned}$$

**Case 2:**  $\mathcal{O}(SP_q(2n)/SP_q(2n-2m))$ .

In this case, the  $r^{\text{th}}$ -part of  $\omega_n^{S_{n-m+1}}$  is  $s_r s_{r+1} \dots s_{n-1} s_n s_{n-1} \dots s_{r+1} s_r$  for  $n - m + 1 \leq r \leq n$ . Hence by applying Theorem 3.11 and following the steps of part (1), we get the claim.  $\square$



## References

- [1] Tedor Banica and Roland Vergnioux. Growth estimates for discrete quantum groups. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 12 (2009), no. 2, 321-340.
- [2] Partha Sarathi Chakraborty and Arup Kumar Pal. Characterization of spectral triples: A combinatorial approach. arXiv:math.OA/0305157, 2003.
- [3] Partha Sarathi Chakraborty and Bipul Saurabh. Gelfand-Kirillov dimension of some simple unitarizable modules. arXiv:1709.08586, 2017.
- [4] Alessandro D'Andrea, Claudia Pinzari and Stefano Rossi. Polynomial growth for compact quantum groups, topological dimension and \*-regularity of the Fourier algebra. arXiv:1602.07496v2, 2016.
- [5] Izrail M. Gelfand and Alexander A. Kirillov. Sur les corps lis aux algbres enveloppantes des algbres de Lie. (*French*) *Inst. Hautes tudes Sci. Publ. Math.* No. 31, 1966, 5-19.
- [6] James E. Humphreys. *Reflection groups and Coxeter groups*. Cambridge Studies in Advanced Mathematics, 29. Cambridge University Press, Cambridge, 1990. xii+204 pp. ISBN:
- [7] Anatoli Klimyk and Konrad Schmüdgen. *Quantum groups and their representations*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997.
- [8] Leonid I. Korogodski and Yan S. Soibelman. *Algebras of functions on quantum groups. Part I*, volume 56 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.
- [9] G R Krause and T H Lenagen. *Growth of algebras and Gelfand-Kirillov dimension*. Revised edition. Graduate Studies in Mathematics, 22. American Mathematical Society, Providence, RI, 2000. x+212 pp. ISBN: 0-8218-0859-1.
- [10] Sergey Neshveyev and Lars Tuset. Quantized algebras of functions on homogeneous spaces with Poisson stabilizers. *Comm. Math. Phys.*, 312(1):223-250, 2012.
- [11] Louis H. Rowen. *Ring theory* Student edition. Academic Press, Inc., Boston, MA, 1991.
- [12] Jasper V. Stokman and Mathijs S. Dijkhuizen. Quantized flag manifolds and irreducible -representations. *Comm. Math. Phys.* 203 (1999), no. 2, 297-324.

PARTHA SARATHI CHAKRABORTY ([parthac@imsc.res.in](mailto:parthac@imsc.res.in))

Institute of Mathematical Sciences (HBNI), CIT Campus, Taramani, Chennai, 600113, INDIA

BIPUL SAURABH ([saurabhbipul2@gmail.com](mailto:saurabhbipul2@gmail.com))

Institute of Mathematical Sciences (HBNI), CIT Campus, Taramani, Chennai, 600113, INDIA