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A bivariate life distribution and notions of negative dependence

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Bivariate life distributions used to model negative dependence typically possess certain limitations; in particular, the correlation coefficient takes values in a restricted subrange of $[-1, 0]$. We construct a new bivariate life distribution to remedy this. Properties of the proposed distribution are studied. It is shown that the distribution satisfies most of the popular notions of negative dependence prevalent in the literature. Stress–strength reliability bounds are obtained, and parameter estimation methodology has been discussed. Performance of the estimators are compared through a simulation study.

KEYWORDS

conditional method of moments, likelihood ratio dependent, quadrant dependent, regression dependent, stress–strength

1 | INTRODUCTION

Bivariate and multivariate life distributions are extensively used for modelling various natural and physical phenomena in the fields of reliability engineering (Fiondella, 2010), hydrology (Genest, Favre, Béliveau, & Jacques, 2007; Phatarford, 1976; Zhang & Singh, 2007), environmental science (Crovelli, 1973; Gumbel & Mustafi, 1967), economics (Wrigley & Dunn, 1984), medical science (Crowder, 1985; Cui & Sun, 2004; Hougaard, 1986), psephology (Hoyer & Mayer, 1972), and so forth. Traditionally, bivariate distributions have been proposed either by specifying two conditional distributions or through a simple specification of one marginal and one conditional distribution. A typical mechanism of generating bivariate distributions in recent times is by using a variety of copulas (Bairamov & Kotz, 2003; Lai & Xie, 2000; Mohtashami-Borzadaran, Amini, & Ahmadi, 2019). The structural properties of such distributions have been studied in detail (Balakrishana & Lai, 2009, Chapters 1 and 2). They provide details of historical developments, genesis, theoretical properties, and various areas of application of bivariate life distributions. Interestingly, most such life distributions available in the literature are positively correlated. However, in many real-life scenarios, paired observations of non-negative variables are negatively correlated. For example, rainfall intensity and duration are jointly modelled incorporating their negative dependence for the study of derived flood frequency distribution (Kurothe, Goel, & Mathur, 1997). Gumbel (1960) proposed two different bivariate exponential distributions with correlation coefficient lying in the intervals $[-0.4, 0]$ and $[-0.25, 0.25]$, respectively. Freund (1961) proposed another bivariate extension of the exponential distribution where the lower bound of the correlation coefficient is restricted to $-1/3$. The conventional mechanism for inducing dependence between two non-negative random variables is designed to incorporate positive association. Following similar constructions, one cannot guarantee that the resulting random variables are non-negative in nature while being negatively dependent. In an attempt to manufacture non-negative dependent random variables, it is also not easy to ensure that the correlation coefficient takes any value in the interval $[-1, 0]$.

Lehmann (1966) introduced various concepts of negative dependence for bivariate distributions. Later, Esary and Lehmann (1972) and Yanagimoto (1972) developed stronger notions of bivariate negative dependence. These results related to negative dependence are useful in deriving reliability bounds. In this context, Ghosh (1981) introduced various notions of negative dependence and discussed associated properties in the multivariate set-up. However, only a few bivariate models satisfying such nice properties have been developed for practitioners. Farlie-Gumble-Morgensterm (FGM) family of distributions exhibits negative dependence among the component variables in a very strong sense. However, the correlation coefficient for this family lies within the interval $[-1/3, 1/3]$ (Schucany, Parr, & Boyer, 1978). Bairamov and Kotz (2000) proposed a four-parameter extension of the FGM family with correlation coefficient that lies within the interval $[-0.48, 0.50]$. Similarly, Bekrizadeh, Parham, and Zadkarmi (2012) proposed a three-parameter extension with correlation coefficient that lies within the interval $[-0.5, 0.43]$. The aforementioned models are based on a more general family of FGM extensions introduced by Sarmanov (1996), which fails to provide an admissible parameter space for the dependence parameter to have closed-form bounds on the corresponding correlation coefficient. To address this issue, Amblard and Girard (2009) proposed another extension, but its application is limited because of a singular component concentrated on the diagonal. Some other extensions of FGM coupla are discussed in Ahn (2015) and Bekrizadeh and Jamshidi (2017). As mentioned before, it is a challenging problem to develop a bivariate life distribution with the flexibility that the correlation coefficient takes any value in the interval $[-1, 0]$. The complexity of the problem increases if strong notions of negative dependence are desired in addition to the aforementioned virtue.

In this paper, we propose a negatively dependent bivariate life distribution that possesses nice closed-form expressions not only for the joint, conditional, and marginal distributions and densities but also for the maximum likelihood estimates (MLEs) of the model parameters. Most importantly, the correlation coefficient of the proposed distribution may take any value in the interval $[-1, 0]$. It also exhibits various strong notions of negative dependence available in the literature. In Section 2, we provide the genesis of the proposed distribution and its associated properties including moments and conditional distributions. Next, we discuss various notions of negative dependence in the context of the proposed bivariate distribution in Section 3. Section 4 provides a stress–strength reliability bound under the proposed model. In Section 5, we provide estimation methodology for the associated parameters. An illustrative data analysis is presented in Section 6. We provide some concluding remarks in Section 7.

2 | THE BIVARIATE DISTRIBUTION

There are different methods available in the literature for the construction of bivariate life distributions. Suppose Z_1, Z_2, and Z_3 are three independent non-negative random variables. Then, one can generate dependent random variables by taking $X = \min(Z_1, Z_3)$ and $Y = \min(Z_2, Z_3)$. Alternatively, a popular choice is to take $X = \max(Z_1, Z_3)$ and $Y = \max(Z_2, Z_3)$ for the same purpose. These constructions provide positively dependent random variables. However, it does not seem feasible to incorporate negative dependence using similar mechanisms. In an attempt to induce negative correlation, Arnold and Strauss (1988) consider both the conditional densities to be exponential and exploit the differencing argument. In this case, the lower bound of correlation coefficient is restricted to -0.32. As discussed before, similar restrictions have been observed for the bivariate life distributions generated by Farlie-Gumbel-Morgenstern copula. Another mechanism of generating positively dependent random variables is to consider a pair $X = Z_1 + Z_3$ and $Y = Z_2 + Z_3$, which can be tweaked to incorporate negative dependence by the modification $X = Z_1 + Z_3$ and $Y = Z_2 - Z_3$. Unfortunately, this does not guarantee that the resulting random variables both X and Y are non-negative. We overcome all the above difficulties by considering the following construction.

Suppose Z_1 and Z_2 are independent exponential random variables with mean $1/\lambda$ and $1/\mu$, respectively. Now, we define the paired random variable (X, Y) as

$$\begin{aligned} X &= Z_1 \\ \text{and} \quad Y &= e^{Z_2 - Z_1}. \end{aligned}$$

Using standard transformation of variables, one can obtain the joint probability density function of (X, Y) as

$$f(x,y) = \begin{cases} \lambda\mu e^{-(\lambda+\mu)x} y^{-(1+\mu)}, & 0 < x < \infty, \\ & e^{-x} < y < \infty \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

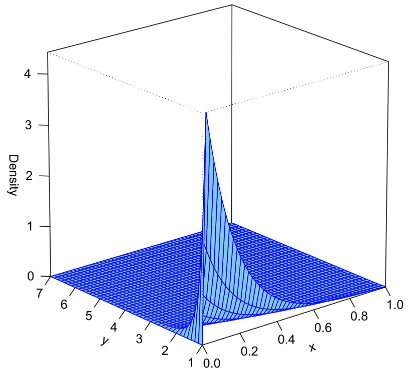
and the corresponding cumulative distribution function (cdf) is given by

$$F(x,y) = \begin{cases} y^{\lambda} - e^{-\lambda x} + \frac{\lambda}{(\lambda+\mu)y^{\mu}}\left(e^{-(\lambda+\mu)x} - y^{\lambda+\mu}\right), \\ \qquad 0 < y < 1, \quad x > -\ln y, \\ 1 - e^{-\lambda x} - \frac{\lambda}{(\lambda+\mu)y^{\mu}}\left(1 - e^{-(\lambda+\mu)x}\right) \\ \qquad y > 1, x > 0. \end{cases} \tag{2}$$

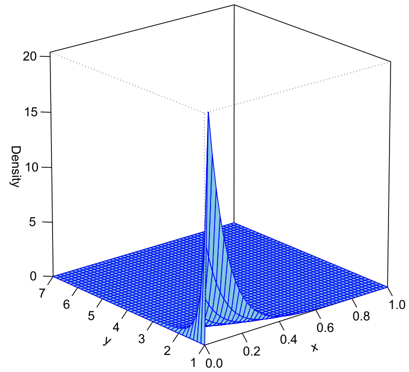
Figure 1a–c provides graphical representations of the density function (1) for three different sets of parameter choices. The characteristic function of the above distribution is given by

$$\Psi(t_1, t_2) = \lambda\mu[\text{ExpIntegralE}\quad[1+\mu, -it_2] + (\Gamma[\lambda - it_1] - \Gamma[\lambda - it_1, -it_2])(-it_2)^{-\lambda+it_1}] / \left[\lambda + \mu - it_1\right],$$

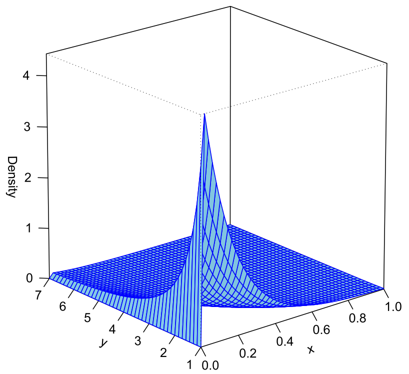
where ExpIntegralE $[n, Z] = \int_1^{\infty} \frac{e^{-zt}}{t^n} dt$ and $\Gamma[a, z] = \int_z^{\infty} t^{a-1} e^{-t} dt$.



(a) Density plot for $\lambda = 1$ and $\mu = 5$



(b) Density plot for $\lambda = 5$ and $\mu = 5$



(c) Density plot for $\lambda = 5$ and $\mu = 1$

FIGURE 1 Plots of the density for different choices of λ and μ: (a) density plot for $\lambda = 1$ and $\mu = 5$, (b) density plot for $\lambda = 5$ and $\mu = 5$, and (c) density plot for $\lambda = 5$ and $\mu = 1$

2.1 | Marginal and conditional distributions

Note that the marginal distribution of X, by definition, is exponential with mean $1/\lambda$. From (1), it is easy to derive the marginal density of Y, which is given by

$$f_Y(y) = \begin{cases} \int_{-\log_e y}^{\infty} \lambda\mu e^{-(\lambda+\mu)x} y^{-(1+\mu)} \, dx = \frac{\lambda\mu}{\lambda+\mu} y^{\lambda-1}, \\ \qquad 0 < y < 1 \\ \int_0^{\infty} \lambda\mu e^{-(\lambda+\mu)x} y^{-(1+\mu)} \, dx = \frac{\lambda\mu}{\lambda+\mu} y^{-(\mu+1)}, \\ \qquad 1 < y < \infty. \end{cases}$$

One can directly find the marginal distribution function of Y as

$$F_Y(y) = F(\infty, y) = \begin{cases} \frac{\mu}{(\lambda+\mu)} y^{\lambda}, & 0 < y < 1, \\ 1 - \frac{\lambda}{(\lambda+\mu) y^{\mu}}, & y > 1, \end{cases}$$

from (2). Now, the conditional density function of $Y|X = x$ turns out to be

$$f_{Y|X}(y|x) = \begin{cases} \mu e^{-\mu x} y^{-(1+\mu)}, & y > e^{-x}, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, the conditional density of $X|Y = y$ is as follows:

For $0 < y < 1$,

$$f_{X|Y}(x|y) = (\lambda + \mu) e^{-(\lambda+\mu)x} y^{-(\lambda+\mu)}, \quad x > -\ln y,$$

whereas for $y > 1$

$$f_{X|Y}(x|y) = (\lambda + \mu) e^{-(\lambda+\mu)x}, \quad x > 0.$$

Remark 1. For $y > 1$, the conditional distribution of $X|Y = y$ is exponential with mean $1/(\lambda + \mu)$.

2.2 | Moments and correlation

First, we provide the moments of the marginal distributions. Note that X follows exponential distribution and hence has moments of all orders (see Johnson and Kotz, 1970, p. 498, for more details). In order to find the rth-order moment of Y, one needs to evaluate

$$E(Y^r) = \int_0^1 \frac{\lambda\mu}{\lambda+\mu} y^{\lambda+r-1} dy + \int_1^\infty \frac{\lambda\mu}{\lambda+\mu} y^{r-(\mu+1)} dy. \tag{3}$$

The second integral in (3) exists if $\mu > r$ and so $E(Y^r)$ exists for all $r < \mu$, and in this case,

$$E(Y^r) = \frac{\lambda\mu}{(\lambda+r)(\mu-r)}.$$

Therefore, variance of Y exists if $\mu > 2$ and given by

$$Var(Y) = \frac{\lambda\mu(\lambda^2+\mu^2+2\lambda-2\mu+1)}{(\lambda+2)(\mu-2)(\lambda+1)^2(\mu-1)^2}.$$

As discussed before, the main motivation for the proposed bivariate distribution is that the correlation coefficient between X and Y can take any value in the interval $[-1, 0]$ unlike the existing negatively dependent bivariate distributions. Now, we derive the product moment correlation coefficient of X and Y and demonstrate the aforementioned property. Using the independence of Z_1 and Z_2, we get

$$\begin{aligned} Cov(X,Y) &= E(XY) - E(X)E(Y) \\ &= E(Z_1 e^{-Z_1})E(e^{Z_2}) - E(Z_1)E(e^{Z_2})E(e^{-Z_1}). \end{aligned}$$

Because $E(e^{Z_2})$ exists for $\mu > 1$, $Cov(X, Y)$ is equal to $\frac{\mu}{(\lambda+1)^2(1-\mu)}$ for $\mu > 1$. As mentioned before, the variance of Y exists for $\mu > 2$; hence, the product moment correlation coefficient of X and Y is equal to

$$\rho(X,Y) = -\frac{\lambda}{\lambda+1}\sqrt{\frac{\mu(\lambda+2)(\mu-2)}{\lambda(\lambda^2+\mu^2+2\lambda-2\mu+1)}}$$

for $\mu > 2$.

Theorem 1. *The correlation coefficient between X and Y can take any value between −1 and 0.*

Proof. Observe that $\rho(X, Y)$ can be written as

$$\rho(X,Y) = -\frac{\sqrt{\lambda(\lambda+2)}}{\lambda+1}\sqrt{\frac{\mu(\mu-2)}{(\lambda+1)^2+\mu(\mu-2)}}.$$

Note that for any fixed $\mu > 2$, one can make $\rho(X, Y)$ arbitrarily close to zero by choosing λ sufficiently small. As $\sqrt{\lambda(\lambda+2)}/(\lambda+1) \to 1$ as $\lambda \to \infty$, for $\delta > 0$, we can choose λ_0, sufficiently large, so that

$$\frac{\sqrt{\lambda(\lambda+2)}}{\lambda+1} > 1 - \frac{\delta}{2} \quad \forall\ \lambda \geq \lambda_0.$$

Now, for any fixed $\lambda \geq \lambda_0$, choose $\mu = n\lambda$. For these choices,

$$\sqrt{\frac{\mu(\mu-2)}{(\lambda+1)^2+\mu(\mu-2)}} = \sqrt{\frac{n^2\lambda^2-2n\lambda}{n^2\lambda^2-2n\lambda+(\lambda+1)^2}} = \gamma_n, \text{ say.}$$

Now, $\gamma_n \to 1$ as $n \to \infty$. So $\exists$ an integer $N_0 \geq 1 \ni \gamma_n > 1 - \frac{\delta}{2}\ \forall\, n \geq N_0$. Thus, for all $\lambda \geq \lambda_0$ and $\mu = n\lambda$ where $n \geq N_0$,

$$\frac{\sqrt{\lambda(\lambda+2)}}{\lambda+1}\sqrt{\frac{\mu(\mu-2)}{(\lambda+1)^2+\mu(\mu-2)}} > \left(1-\frac{\delta}{2}\right)\left(1-\frac{\delta}{2}\right) > 1-\delta.$$

This establishes that there exists λ and μ such that $\rho(X, Y)$ can be made arbitrarily close to −1. □

Now, we provide the expectations and variances of the conditional distributions. The conditional expectation and variance of $Y|X = x$ are given by

$$E[Y|X = x] = \mu e^{-\mu x} \int_{e^{-x}}^{\infty} y^{-\mu} dy = \frac{\mu}{\mu - 1} e^{-x}, \quad \text{for} \quad \mu > 1, \tag{4}$$

and

$$Var(Y|X) = \mu e^{-2x} \left[\frac{1}{\mu - 2} - \frac{\mu}{(\mu - 1)^2} \right], \quad \text{for} \quad \mu > 2,$$

respectively.

Remark 2. The regression of Y on X is log-linear in the sense that $\ln(E[Y|X = x])$ is a linear function of x. Also, it is interesting to note that the $Var(Y|X)$ is a decreasing function of x and it is bounded above by $\mu \left[\frac{1}{\mu-2} - \frac{\mu}{(\mu-1)^2} \right]$.

Similarly, the conditional expectation and variance of $X|Y = y$ are given by

$$E[X|Y = y] = \begin{cases} \int_{-\ln y}^{\infty} x(\lambda + \mu) e^{-(\lambda+\mu)x} y^{-(\lambda+\mu)} dx & \\ = \frac{1}{\lambda+\mu} - \ln y, & 0 < y < 1 \\ \frac{1}{\lambda+\mu}, & 1 < y < \infty, \end{cases} \tag{5}$$

and

$$Var(X|Y) = \frac{1}{(\lambda + \mu)^2},$$

respectively.

Remark 3. Even though the conditional distributions of $X|Y = y$ are different for $0 < y < 1$ and $y > 1$, the latter distribution is exponential with mean $1/(\lambda + \mu)$, whereas the former is not. Interestingly, the variances are not only identical but also independent of y; that is, the conditional distribution of $X|Y = y$ is homoscedastic.

3 | CONNECTIONS WITH NOTIONS OF NEGATIVE DEPENDENCE

The product moment correlation coefficient measures only linear relationship between two random variables. However, it is possible that two random variables may have strong linear relation but possess weak association with respect to different notions of dependence or vice versa. In this section, we discuss several such notions of negative dependence, namely, *quadrant dependence*, *regression dependence*, and *likelihood ratio dependence*, and investigate whether these properties hold for the proposed bivariate distribution.

Definition 1. Negatively (positively) quadrant dependent: Let $G(x, y)$ be the distribution function of the pair of random variables (X, Y) with marginal cdfs $G_1(x)$ and $G_2(y)$. The pair (X, Y) (or its cdf G) is said to be negatively quadrant dependent if

$$G(x, y) \leq G_1(x)G_2(y) \quad \text{for all} \quad x, \quad y. \tag{6}$$

The dependence is strict if the inequality holds for at least some pair (x, y). Similarly, the pair (X, Y) (or G) is positively quadrant dependent if the direction of the inequality in (6) is reversed (see Lehmann, 1966, for more details).

Definition 2. Negatively (positively) regression dependent: Lehmann (1966) has called $G(x, y)$ negatively regression dependent if and only if

$$G(y|x) \leq G(y|x') \quad \text{for almost all } y \text{ and almost all} \quad x < x',$$

where $G(y|x) = P[Y \leq y|X = x]$. Alternatively, $G(x, y)$ is positively regression dependent if and only if the reverse inequality holds.

Definition 3. Negatively likelihood ratio dependent: Two random variables X and Y are said to be negatively likelihood ratio dependent if their joint density function $g(x, y)$ is reversely regular of order two (Karlin, 1968, p. 12); that is, $g(x, y)$ satisfies

$$g(x_1, y_1)g(x_2, y_2) \leq g(x_1, y_2)g(x_2, y_1)$$

for all $x_1 \leq x_2$ and $y_1 \leq y_2$.

Proposition 1. *The distribution function $F(x, y)$ given in (2) is negatively quadrant dependent.*

Proof.

Case 1. $0 < y < 1$ and $x > -\ln y$.

$$
\begin{aligned}
&F(x,y) - F_1(x)F_2(y)\\
&= y^{\lambda} - e^{-\lambda x} + \frac{\lambda}{(\lambda+\mu)y^{\mu}}\left(e^{-(\lambda+\mu)x} - y^{\lambda+\mu}\right) - (1-e^{-\lambda x})\left(\frac{\mu}{\lambda+\mu}y^{\lambda}\right)\\
&= \frac{e^{-\lambda x}}{y^{\mu}}\left[\frac{\mu}{\lambda+\mu}y^{\lambda+\mu} - y^{\mu} + \frac{\lambda}{\lambda+\mu}e^{-\mu x}\right] \le 0.
\end{aligned}
$$

The last inequality holds using the facts $0 < y < 1$ and $e^{-\mu x} \le y^{\mu}$.

Case 2. $y > 1$ and $x > 0$.

$$
\begin{aligned}
&F(x,y) - F_1(x)F_2(y)\\
&= 1 - e^{-\lambda x} - \frac{\lambda}{(\lambda+\mu)y^{\mu}}(1 - e^{-(\lambda+\mu)x}) - (1-e^{-\lambda x})\left[\frac{\mu}{(\lambda+\mu)y^{\mu}}\right]\\
&= \frac{\lambda}{(\lambda+\mu)y^{\mu}}e^{-\lambda x}(e^{-\mu x} - 1) < 0.
\end{aligned}
$$

□

Remark 4. It is simple to note that the proposed distribution also satisfies the property of strong negative orthant dependence (see Ghosh,1981, for more details).

Proposition 2. *The distribution function $F(x,y)$ given in (2) is negatively regression dependent.*

Proof. For $x < x'$, we have $e^{-\mu x} > e^{-\mu x'}$
$\Rightarrow 1 - e^{-\mu x}y^{-\mu} < 1 - e^{-\mu x'}y^{-\mu}$
$\Rightarrow F(y|x) \ge F(y|x')$
$\Rightarrow F(x,y)$ is negatively regression dependent. □

Proposition 3. *The density function $f(x,y)$ given in (1) is negatively likelihood ratio dependent.*

Proof. The proof is a direct consequence of the definition. □

4 | STRESS–STRENGTH RELIABILITY BOUND

Reliability of a mechanical system depends on stress–strength interference, and the system survives as long as the strength (X) is greater than the stress (Y). In this context, engineers are interested in computing the reliability function given by $R = P[Y < X]$. Traditionally, stress and strength are assumed to be independent for the sake of simplicity, and ample amount of research material is available in the literature under various parametric and nonparametric set-ups (English, Sargent, & Lander, 1996; Kotz, Lumelskii, & Pensky, 2003, Chapters 3 and 5; Pham & Almhana, 1995; Roy & Dasgupta, 2001). However, in many situations, it is more realistic to assume stress and strength are dependent (Kotz, Lumelskii, & Pensky, 2003, p. 110). In this section, we find reliability of a system under negatively dependent stress–strength interference. Now,

$$
\begin{aligned}
R &= P[X > Y] = 1 - P[Y > X]\\
&= 1 - \int_0^{\infty} P[Y > X|X = x]f_X(x)dx = 1 - \int_0^{\theta} \lambda e^{-\lambda x}dx - \int_{\theta}^{\infty}\left(\int_x^{\infty} f(y|x)dy\right)\lambda e^{-\lambda x}dx\\
&= e^{-\lambda\theta} - \lambda\int_{\theta}^{\infty} e^{-(\lambda+\mu)x}x^{-\mu}dx = e^{-\lambda\theta} - \frac{\lambda}{(\lambda+\mu)^{1-\mu}}\int_{(\lambda+\mu)\theta}^{\infty}\frac{e^{-y}}{y^{\mu}}dy\\
&= e^{-\lambda\theta} - \frac{\lambda}{(\lambda+\mu)^{1-\mu}}\Gamma[1-\mu,(\lambda+\mu)\theta] \le e^{-\lambda\theta}
\end{aligned}
\quad , \tag{7}
$$

where θ is the solution of the equation $\theta + \log_e\theta = 0$, $0 < \theta < 1$. The method of bisection yields the approximate value of θ as 0.567. It is interesting to note that this upper bound of R depends on λ only. Alternatively, if Y is interpreted as strength and X as stress, then $1 - e^{\theta\lambda}$ would provide a lower bound of the reliability R.

In many situations, it is not possible to control the stress associated with the operating conditions of the system; however, one can possibly achieve higher reliability simply by ensuring greater average strength (i.e., a small value of λ). Figure 2 shows a plot of maximal possible reliability against the average strength of the system. It is seen that the maximal reliability of a system improves rapidly as the average strength increases up to 5 units. However, the improvement is very slow beyond 10 units of average strength. Engineers may use this result for the cost benefit analysis during product development.

Next, we focus on the problem of finding a lower bound for R.

$$
\begin{aligned}
R &= P[X > Y] \\
&= \int_0^\infty P[X > Y|Y = y] f_Y(y) dy \\
&= \int_0^\theta \frac{\lambda\mu}{\lambda+\mu} y^{\lambda-1} dy + \int_\theta^1 \left(\int_y^\infty f(x|y) dx \right) \frac{\lambda\mu}{\lambda+\mu} y^{\lambda-1} dy + \int_1^\infty \left(\int_y^\infty f(x|y) dx \right) \frac{\lambda\mu}{\lambda+\mu} y^{-(1+\mu)x} dy \\
&= \frac{\mu}{\lambda+\mu}\theta^\lambda + \frac{\lambda\mu}{\lambda+\mu} \int_\theta^\infty e^{-(\lambda+\mu)y} y^{-(1+\mu)} dy \\
&\geq \frac{\mu}{\lambda+\mu}\theta^\lambda .
\end{aligned} \tag{8}
$$

In contrast to the upper bound obtained in (7), the lower bound in (8) involves both of the parameters λ and μ. A reversal of the roles of X and Y would, in this case, yield $1 - \frac{\mu}{\lambda+\mu}\theta^\lambda$ as an upper bound of the reliability R.

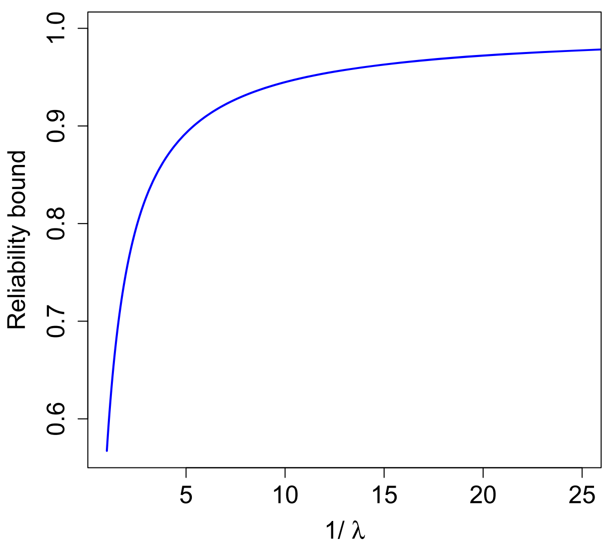


FIGURE 2 Maximal possible reliability with respect to average strength

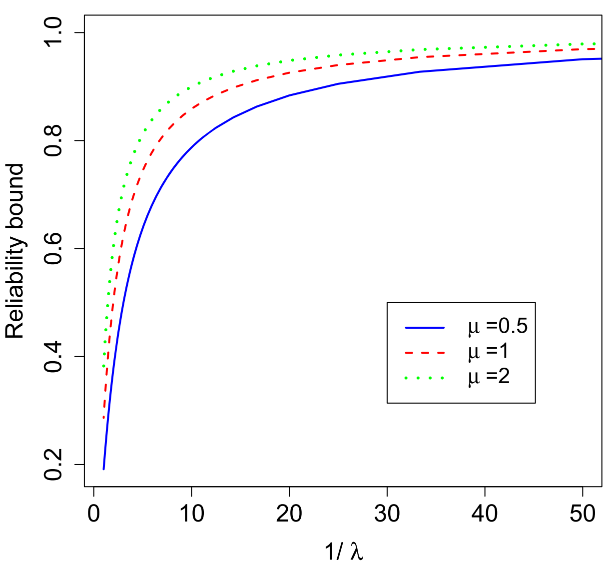


FIGURE 3 Minimal possible reliability with respect to average strength for different values of μ

Figure 3 shows a plot of the lower bound of reliability against the average strength of the system for different choices of μ. Here also, the minimal reliability of a system improves rapidly as the average strength increases up to 5 units. However, the improvement is slow beyond 10 units of average strength. As expected, the lower bound of R increases as μ decreases, and these differences are more prominent when the average strength lies between 5 units and 15 units.

5 | ESTIMATION METHODOLOGY

In this section, we provide estimation of the parameters associated with the proposed distribution. First, we derive MLEs of the unknown parameters λ and μ based on a bivariate random sample $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ from (2). The log-likelihood function of (λ, μ) can be written as

$$L(\lambda, \mu) = n \ln \lambda + n \ln \mu - (\lambda + \mu) \sum_{i=1}^{n} x_i - (1 + \mu) \sum_{i=1}^{n} \ln y_i. \tag{9}$$

Now, maximizing the log-likelihood function provided in (9) with respect to (λ, μ), the MLEs of λ and μ are obtained as

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i} \text{ and } \hat{\mu} = \frac{n}{\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \ln y_i},$$

respectively. Note that the MLE of λ is a function of the x_i's only and is the same as that of the MLE of the scale parameter from an exponential distribution. Also, the method of moments estimate (MME) of λ coincides with its MLE $\hat{\lambda}$. However, the MME of μ exists if $\mu > 1$ and is given by

$$\hat{\hat{\mu}} = \max\left\{0, \left[1 - \frac{n\hat{\lambda}}{(\hat{\lambda} + 1)\sum_{i=1}^{n} y_i}\right]^{-1}\right\}.$$

Similar to the MMEs, we also provide another set of estimators

$$\tilde{\lambda} = \max\left\{0, \frac{n}{\sum\limits_{i=1}^{n} \left[x_i + \ln y_i I(y_i < 1)\right]} - \tilde{\mu}\right\}$$

and

$$\tilde{\mu} = \max\left\{0, \frac{1}{2}\left[\frac{\sum_{i=1}^{n} y_i e^{x_i} + n}{\sum_{i=1}^{n} y_i e^{x_i} - n} + 1\right]\right\},$$

based on the expressions of conditional expectations (4) and (5), which exist for $\mu > 1$. This estimation methodology is referred to as conditional MME (CMME). Note that all the moments based estimators presented above are modified for necessary boundary corrections. In order to compare the performance of these estimators, we simulate data with three different choices of (λ, μ) for $n = 20, 50$, and 200 and calculate bias, standard error (SE), and 95% confidence interval based on 10,000 replications. The results are presented in Tables 1, 2 and 3. The MLE or MME of λ performs better compared with CMME with respect to both bias and SE. Similarly, the MLE of μ performs better compared with both MME and CMME. Interestingly enough, the bias and SE of the CMME of μ are marginally smaller than the corresponding MME. As expected, the performance of all the estimators improves as the sample size increases.

TABLE 1 Results of the simulation study with $n = 20$

		MLE			MME			CMME		
Parameter	True	Bias	SE	CI	Bias	SE	CI	Bias	SE	CI
λ	1	0.055	0.244	[0.679, 1.607]	0.055	0.244	[0.679, 1.607]	−0.065	0.623	[0.041, 2.458]
μ	2	0.108	0.494	[1.345, 3.280]	0.366	0.911	[1.402, 4.12]	0.223	0.502	[1.422, 3.395]
λ	3	0.162	0.747	[2.020, 4.917]	0.162	0.747	[2.020, 4.917]	0.044	1.189	[1.213, 5.882]
μ	2	0.108	0.497	[1.347, 3.281]	0.248	0.526	[1.427, 3.477]	0.228	0.504	[1.424, 3.414]
λ	2	0.104	0.493	[1.350, 3.285]	0.104	0.493	[1.350, 3.285]	-	-	-
μ	0.5	0.029	0.125	[0.343, 0.823]	-	-	-	-	-	-

Abbreviations: CI, confidence interval; CMME, conditional method of moments estimate; MLE, maximum likelihood estimate; SE, standard error.

6 | DATA ANALYSIS

For an illustration of the proposed model and estimation methodology, we consider a data set on daily air quality measurements for New York Metropolitan Area from May 1, 1973 to September 30, 1973. Information on six variables, including average wind speed (in miles per hour) and mean ozone level (in parts per billion), was obtained from the New York State Department of Conservation and the National Weather Service. See Chambers, Cleveland, Kleiner, and Tukey (1983), Chapters 2–5, for details. Ozone in the upper atmosphere helps to protect the Earth from the sun's harmful rays. On the contrary, exposure to ozone also can be harmful to both human health and some plants in the lower atmosphere. Variations in weather conditions play an important role in determining ozone levels (Khiem et al., 2010; Topcu, Anteplioglu, & Incecik, 2003). In general, wind can affect both the location and concentration of ozone level. High winds tend to disperse pollutants which in turn dilute the concentration of ozone level. However, stagnant conditions or light winds allow pollution levels to build up, and thereby, ozone level too becomes more concentrated. Meteorologists are interested in studying the effect of wind speed on distribution patterns of ozone (Gorai et al., 2015). On the basis of the observed data, we find that Spearman's and Pearsons' correlation coefficients between wind speed and ozone levels are −0.59 and −0.60, respectively, which indicate strong negative dependence. To analyse this phenomenon, we fit the proposed model, given by (1) and (2), using the maximum likelihood method. We find that the choice of wind speed as X and mean ozone as Y fits the data better compared with the reverse choice of X and Y with respect to the Akaike information criterion. Under this set-up, the MLEs of the model parameters

TABLE 2 Results of the simulation study with $n = 50$

		MLE			MME			CMME		
Parameter	True	Bias	SE	CI	Bias	SE	CI	Bias	SE	CI
λ	1	0.020	0.146	[0.775, 1.344]	0.020	0.146	[0.775, 1.344]	−0.0542	0.396	[0.308, 1.853]
μ	2	0.043	0.292	[1.546, 2.676]	0.161	0.386	[1.507, 3.021]	0.114	0.318	[1.520, 2.774]
λ	3	0.059	0.444	[2.320, 4.058]	0.059	0.444	[2.320, 4.058]	−0.010	0.724	[1.772, 4.626]
μ	2	0.044	0.297	[1.542, 2.690]	0.122	0.328	[1.525, 2.839]	0.113	0.318	[1.532, 2.792]
λ	2	0.045	0.294	[1.555, 2.701]	0.045	0.294	[1.555, 2.701]	-	-	-
μ	0.5	0.010	0.074	[0.385, 0.673]	-	-	-	-	-	-

Abbreviations: CI, confidence interval; CMME, conditional method of moments estimate; MLE, maximum likelihood estimate; SE, standard error.

TABLE 3 Results of the simulation study with $n = 200$

		MLE			MME			CMME		
Parameter	True	Bias	SE	CI	Bias	SE	CI	Bias	SE	CI
λ	1	0.005	0.071	[0.877, 1.152]	0.005	0.071	[0.877, 1.152]	−0.022	0.214	[0.616, 1.451]
μ	2	0.009	0.142	[1.751, 2.300]	0.048	0.202	[1.653, 2.446]	0.034	0.176	[1.674, 2.362]
λ	3	0.014	0.214	[2.619, 3.463]	0.014	0.214	[2.619, 3.463]	−0.016	0.364	[2.321, 3.760]
μ	2	0.010	0.142	[1.756, 2.306]	0.041	0.181	[1.671, 2.388]	0.038	0.177	[1.677, 2.378]
λ	2	0.010	0.144	[1.750, 2.316]	0.010	0.144	[1.750, 2.316]	-	-	-
μ	0.5	0.003	0.036	[0.437, 0.578]	-	-	-	-	-	-

Abbreviations: CI, confidence interval; CMME, conditional method of moments estimate; MLE, maximum likelihood estimate; SE, standard error.

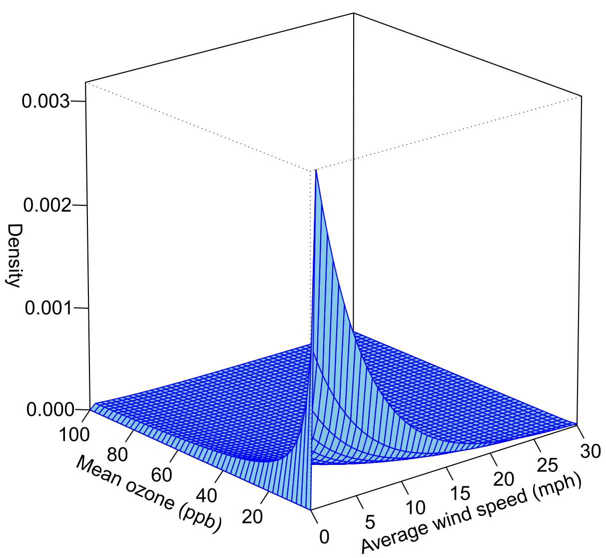


FIGURE 4 Estimated joint density of wind speed and mean ozone

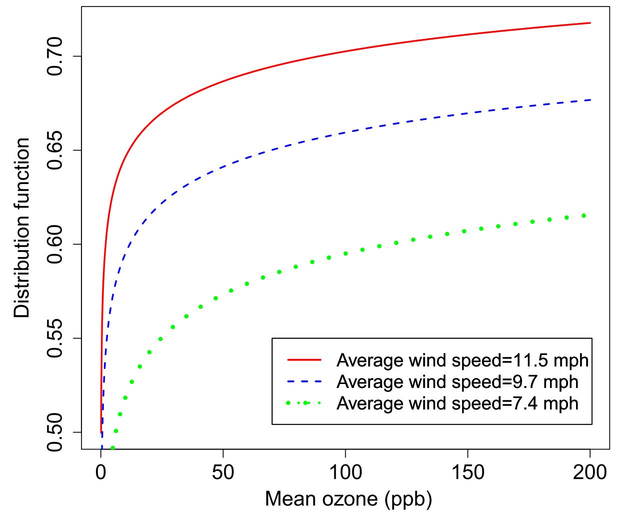


FIGURE 5 Effect of wind speed on the distribution of mean ozone

are $\hat{\lambda} = 0.101$ and $\hat{\mu} = 0.075$ with 95% confidence interval as $[0.085, 0.122]$ and $[0.063, 0.091]$, respectively. The estimated joint density is graphically presented in Figure 4. The estimated conditional distribution of mean ozone level under the proposed model keeping the wind speed fixed at the empirical first (7.4 mph), second (9.7 mph), and third (11.5 mph) quartiles is presented in Figure 5. It is easy to see that the distribution of mean ozone level decreases stochastically as wind speed increases. The advantage of fitting the proposed bivariate model is that it gives an estimated conditional distribution of mean ozone level given the wind speed, including the mean, median, and different quantiles.

7 | CONCLUDING REMARKS

To the best of the authors' knowledge, the proposed bivariate life distribution is perhaps the first of its kind without any restriction on the correlation coefficient and also satisfies most of the popular notions of negative dependence available in the literature. In this context, stress–strength reliability bounds have been obtained and estimation methodology for the model parameters proposed. In many real-life scenarios, stress and strength typically depend on external factors such as temperature, pressure, and humidity. It would be interesting to incorporate this phenomenon by introducing covariates in the stress–strength reliability modelling. Devising tests of dependence under this set-up might be a challenging problem as well. Multivariate extensions of the proposed model can also be taken up in future.

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DATA ACCESSIBILITY

The data that support the findings of this study are available in Chambers, Cleveland, Kleiner, and Tukey (1983).

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