

1-1-2020

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Recommended Citation

Balodi, Mamta; Banerjee, Abhishek; and Naolekar, Anita, "BV-operators and the secondary Hochschild complex" (2020). *Journal Articles*. 447.
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Académie des sciences

Comptes Rendus

Mathématique


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Volume 358, issue 11-12 (2020), p. 1239-1258.

<<https://doi.org/10.5802/crmath.157>>

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Algebra, Geometry and Topology / Algèbre, Géométrie et Topologie

BV-operators and the secondary Hochschild complex

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Abstract. We introduce the notion of a BV-operator $\Delta = \{\Delta^n : V^n \rightarrow V^{n-1}\}_{n \geq 0}$ on a homotopy G -algebra V^\bullet such that the Gerstenhaber bracket on $H(V^\bullet)$ is determined by Δ in a manner similar to the BV-formalism. As an application, we produce a BV-operator on the cochain complex defining the secondary Hochschild cohomology of a symmetric algebra A over a commutative algebra B . In this case, we also show that the operator Δ^\bullet corresponds to Connes' operator.

2020 Mathematics Subject Classification. 16E40.

Funding. AB was partially supported by SERB Matrics fellowship MTR/2017/000112.

Manuscript received 14th October 2020, revised 21st November 2020, accepted 23rd November 2020.

1. Introduction

A Gerstenhaber algebra (see [3]) consists of a graded vector space $W^\bullet = \bigoplus_{n \geq 0} W^n$ equipped with the following two structures:

- A dot product $x \cdot y$ of degree zero making W^\bullet into an associative graded commutative algebra.
- A bracket $[x, y]$ of degree -1 making W^\bullet into a graded Lie algebra satisfying the compatibility property that

$$[x, y \cdot z] = [x, y] \cdot z + (-1)^{(\deg(x)-1)\deg(y)} y \cdot [x, z].$$

Gerstenhaber algebra structures appear in a variety of situations, from Hochschild cohomology of algebras to the exterior algebra of a Lie algebra and the algebra of differential forms on a Poisson manifold.

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An operator $\partial = \{\partial^n : W^n \rightarrow W^{n-1}\}_{n \geq 0}$ on W^\bullet of degree -1 is said to generate the Gerstenhaber bracket (see Koszul [7, §2] and also [6, Definition 3.2]) if it satisfies

$$[x, y] = (-1)^{(\deg(x)-1)\deg(y)}(\partial(x) \cdot y + (-1)^{\deg(x)}x \cdot \partial(y) - \partial(x \cdot y))$$

In particular, a Batalin–Vilkovisky algebra (or BV-algebra) consists of a Gerstenhaber algebra along with a generator ∂ for the bracket such that $\partial^2 = 0$.

In [4], [5], Gerstenhaber and Voronov introduced the notion of a homotopy G -algebra, which is a brace algebra equipped with a differential of degree 1 and a dot product of degree 0 satisfying certain conditions. In particular, the cohomology groups $H(V^\bullet)$ of a homotopy G -algebra V^\bullet carry the structure of a Gerstenhaber algebra.

In this paper, we introduce the notion of a BV-operator $\Delta = \{\Delta^n : V^n \rightarrow V^{n-1}\}_{n \geq 0}$ on a homotopy G -algebra V^\bullet such that the Gerstenhaber bracket on $H(V^\bullet)$ is determined by Δ in a manner similar to the BV-formalism. More explicitly, for classes $\bar{f} \in H^n(V^\bullet)$ and $\bar{g} \in H^m(V^\bullet)$, we have

$$[\bar{f}, \bar{g}] = (-1)^{(n-1)m} \overline{(\Delta(f) \cdot g + (-1)^n f \cdot \Delta(g) - \Delta(f \cdot g))} \in H^{m+n-1}(V^\bullet)$$

where $f \in Z^n(V^\bullet)$, $g \in Z^m(V^\bullet)$ are cocycles representing \bar{f} and \bar{g} respectively. We note that Δ need not be a morphism of cochain complexes and therefore may not induce any operator on $H(V^\bullet)$. As such, Δ may not descend to a generator for the Gerstenhaber bracket on $H(V^\bullet)$.

Our motivation is to introduce a BV-operator on the cochain complex defining the secondary Hochschild cohomology of a symmetric algebra A over a commutative algebra B . For a datum (A, B, ε) consisting of an algebra A , a commutative algebra B and an extension of rings $\varepsilon : B \rightarrow A$ such that $\varepsilon(B) \subseteq Z(A)$, the secondary Hochschild cohomology $H^*(A, B, \varepsilon)$ was introduced by Staic [9] in order to study deformations of algebras $A[[t]]$ having a B -algebra structure. More generally, Staic [9] introduced the secondary Hochschild complex $C^*((A, B, \varepsilon); M)$ with coefficients in an A -bimodule M .

In [10], Staic and Stancu showed that the secondary Hochschild complex $C^*(A, B, \varepsilon) := C^*((A, B, \varepsilon); A)$ with coefficients in A is a non-symmetric operad with multiplication, giving it the structure of a homotopy G -algebra. Hence, the secondary cohomology $H^*(A, B, \varepsilon)$ is equipped with a graded commutative cup product and a Lie bracket which makes it a Gerstenhaber algebra. For more on the secondary cohomology, the reader may see, for instance, [1], Corrigan-Salter and Staic [2], Laubacher, Staic and Stancu [8].

Let k be a field. It is well known (see Tradler [11]) that the Hochschild cohomology of a finite dimensional k -algebra A equipped with a symmetric, non-degenerate, invariant bilinear form $\langle \cdot, \cdot \rangle : A \times A \rightarrow k$ carries the structure of a BV-algebra. For the terms $C^n((A, B, \varepsilon)) = \text{Hom}_k(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, A)$ in the secondary Hochschild complex, we define the BV-operator $\Delta = \sum_{i=1}^{n+1} (-1)^{in} \Delta_i : C^{n+1}(A, B, \varepsilon) \rightarrow C^n(A, B, \varepsilon)$ by the condition (see Section 3)

$$\begin{aligned} \left\langle \Delta_i f \left(\otimes \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & b_{n-1,n} \\ 1 & 1 & 1 & \dots & a_n \end{pmatrix}, a_{n+1} \right) \right. \\ \left. = \left\langle f \left(\otimes \begin{pmatrix} a_i & b_{i,i+1} & b_{i,i+2} & \dots & b_{i,n} & 1 & b_{1,i} & b_{2,i} & \dots & b_{i-1,i} \\ 1 & a_{i+1} & b_{i+1,i+2} & \dots & b_{i+1,n} & 1 & b_{1,i+1} & b_{2,i+1} & \dots & b_{i-1,i+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & a_n & 1 & b_{1,n} & b_{2,n} & \dots & b_{i-1,n} \\ 1 & 1 & \dots & 1 & a_{n+1} & 1 & \dots & \dots & 1 \\ 1 & 1 & \dots & 1 & 1 & a_1 & b_{1,2} & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & \dots & \dots & \dots & \dots & \dots & a_{i-1} \end{pmatrix}, 1 \right) \right\rangle \quad (1) \end{aligned}$$

We then show that the Gerstenhaber bracket on the secondary Hochschild cohomology of (A, B, ε) is determined by Δ in a manner similar to the BV-formalism.

From Tradler [11], we also know that the operator $\Delta^\bullet : C^\bullet(A, A) \rightarrow C^{\bullet-1}(A, A)$ on usual Hochschild cochains inducing the BV-structure on $H^\bullet(A, A)$ corresponds to the operator Ns on duals of Hochschild chains, where N is the “norm operator” and s is the “extra degeneracy” (see (16)). The isomorphism between the two complexes is induced by the k -module isomorphism $A^* \cong A$ determined by the non-degenerate bilinear form $\langle \cdot, \cdot \rangle : A \times A \rightarrow k$. If we pass to the cohomology and take the normalized Hochschild complex which is a quasi-isomorphic subcomplex of $C^\bullet(A, A)$, it follows that Tradler’s Δ^\bullet operator corresponds to Connes’ operator on Hochschild cohomology with coefficients in A .

However, in the case of secondary cohomology, we have mentioned that the operator Δ^\bullet defined in (1) is not a morphism of complexes and we cannot pass to cohomology. Accordingly, we show that the operator Δ^\bullet defined in (1) fits into a commutative diagram (see Theorem 10)

$$\begin{array}{ccc}
 \overline{\overline{C}}^\bullet(A, B, \varepsilon) & \xrightarrow{B} & \overline{\overline{C}}^{\bullet-1}(A, B, \varepsilon) \\
 \downarrow & & \downarrow \\
 C^\bullet((A, B, \varepsilon); A) & \xrightarrow{\Delta^\bullet} & C^{\bullet-1}((A, B, \varepsilon); A)
 \end{array} \tag{2}$$

where B is Connes’ operator. Here, $\overline{\overline{C}}^\bullet(A, B, \varepsilon)$ is the normalization of the co-simplicial module $\overline{C}^\bullet(A, B, \varepsilon)$ introduced by Laubacher, Staic and Stancu [8], which is used to compute the secondary Hochschild cohomology associated to the triple (A, B, ε) . It should be noted (see [8, Remark 4.7]) that despite similar names, the complex $\overline{\overline{C}}^\bullet(A, B, \varepsilon)$ cannot be expressed as a secondary Hochschild complex with coefficients in some A -bimodule. The vertical morphisms in (2) are induced by composing the canonical morphisms $(A \otimes B^{\otimes n})^* \rightarrow A^*$ for each $n \geq 0$, the isomorphism $A^* \cong A$ as well as the inclusion of the quasi-isomorphic subcomplex $\overline{\overline{C}}^\bullet(A, B, \varepsilon) \hookrightarrow \overline{C}^\bullet(A, B, \varepsilon)$.

2. Main Result: BV-operator on homotopy G -algebra

We begin by recalling the notion of a homotopy G -algebra from [5]. A brace algebra (see [5, Definition 1]) is a graded vector space $V = \bigoplus_{n \geq 0} V^n$ with a collection of multilinear operators (braces) $x\{x_1, \dots, x_n\}$ satisfying the following conditions (with $x\{\}$ understood to be x):

- (1) $\deg(x\{x_1, \dots, x_n\}) = \deg(x) + \sum_{i=1}^n \deg(x_i) - n$
- (2) For homogeneous elements $x, x_1, \dots, x_m, y_1, \dots, y_n$, we have

$$\begin{aligned}
 &x\{x_1, \dots, x_m\}\{y_1, \dots, y_n\} \\
 &= \sum_{0 \leq i_1 \leq j_1 \leq i_2 \leq \dots \leq i_m \leq j_m \leq n} (-1)^\varepsilon x\{y_1, \dots, y_{i_1}, x_1\{y_{i_1+1}, \dots, y_{j_1}\}, y_{j_1+1}, \dots, y_{i_m}, \\
 & \hspace{15em} x_m\{y_{i_m+1}, \dots, y_{j_m}\}, y_{j_m+1}, \dots, y_n\}
 \end{aligned}$$

where $\varepsilon = \sum_{p=1}^m |x_p|(\sum_{q=1}^{i_p} |y_q|)$ and $|x| := \deg(x) - 1$.

Definition 1 (see [5, Definition 2]). A homotopy G -algebra consists of the following data:

- (1) A brace algebra $V = \bigoplus_{n \geq 0} V^n$.
- (2) A dot product of degree zero

$$V^m \otimes V^n \longrightarrow V^{m+n} \quad x \otimes y \longmapsto x \cdot y$$

for all $m, n \geq 0$.

- (3) A differential $d : V^\bullet \rightarrow V^{\bullet+1}$ of degree one making V into a DG-algebra with respect to the dot product.

(4) *The dot product satisfies the following compatibility conditions*

$$(x_1 \cdot x_2)\{y_1, \dots, y_n\} = \sum_{k=0}^n (-1)^{\epsilon_k} (x_1\{y_1, \dots, y_k\}) \cdot (x_2\{y_{k+1}, \dots, y_n\})$$

where $\epsilon_k = |x_2| \sum_{p=1}^k |y_p|$ and

$$\begin{aligned} d(x\{x_1, \dots, x_{n+1}\}) - (dx)\{x_1, \dots, x_{n+1}\} - (-1)^{|x|} \sum_{i=1}^{n+1} (-1)^{|x_1|+\dots+|x_{i-1}|} x\{x_1, \dots, dx_i, \dots, x_{n+1}\} \\ = (-1)^{|x||x_1|+1} x_1 \cdot x\{x_2, \dots, x_{n+1}\} + (-1)^{|x|} \sum_{i=1}^n (-1)^{|x_1|+\dots+|x_{i-1}|} x\{x_1, \dots, x_i \cdot x_{i+1}, \dots, x_{n+1}\} - x\{x_1, \dots, x_n\} \cdot x_{n+1} \end{aligned}$$

In particular, a homotopy G -algebra is equipped with a graded Lie bracket which descends to the cohomology of the corresponding cochain complex (V^\bullet, d) (see [5])

$$[\cdot, \cdot] : H^m(V^\bullet) \otimes H^n(V^\bullet) \longrightarrow H^{m+n-1}(V^\bullet) \tag{3}$$

The dot product also descends to the cohomology and the bracket with an element becomes a graded derivation for the induced dot product on $H(V^\bullet) = \bigoplus_{n \geq 0} H^n(V^\bullet)$. In other words, the cohomology $(H(V^\bullet), [\cdot, \cdot], \cdot)$ of a homotopy G -algebra V^\bullet is canonically equipped with the structure of a Gerstenhaber algebra.

We now introduce the notion of a BV-operator on a homotopy G -algebra.

Definition 2. Let $V^\bullet = \bigoplus_{n \geq 0} V^n$ be a homotopy G -algebra, let $d : V^\bullet \rightarrow V^{\bullet+1}$ be its differential and let $[\cdot, \cdot] : V^n \otimes V^m \rightarrow V^{m+n-1}$ be its Lie bracket. We will say that a family $\Delta = \{\Delta^n : V^n \rightarrow V^{n-1}\}_{n \geq 0}$ is a BV-operator on V^\bullet if it satisfies

$$[f, g] - (-1)^{(n-1)m} (\Delta(f) \cdot g + (-1)^n f \cdot \Delta(g) - \Delta(f \cdot g)) \in d(V^{m+n-2})$$

for any cocycles $f \in Z^n(V^\bullet)$, $g \in Z^m(V^\bullet)$.

If V^\bullet is a homotopy G -algebra equipped with a BV-operator Δ , we now show that the bracket on the Gerstenhaber algebra $H(V^\bullet)$ is determined by Δ in a manner similar to the BV-formalism.

Theorem 3. Let $V^\bullet = \bigoplus_{n \geq 0} V^n$ be a homotopy G -algebra equipped with a BV-operator $\Delta = \{\Delta^n : V^n \rightarrow V^{n-1}\}_{n \geq 0}$. Consider $\bar{f} \in H^n(V^\bullet)$ and $\bar{g} \in H^m(V^\bullet)$ and choose cocycles $f \in Z^n(V^\bullet)$ and $g \in Z^m(V^\bullet)$ corresponding respectively to \bar{f} and \bar{g} . Then, we have

$$(\Delta(f) \cdot g + (-1)^n f \cdot \Delta(g) - \Delta(f \cdot g)) \in Z^{m+n-1}(V^\bullet)$$

The Gerstenhaber bracket on the cohomology of V^\bullet is now determined by

$$[\bar{f}, \bar{g}] = (-1)^{(n-1)m} \overline{(\Delta(f) \cdot g + (-1)^n f \cdot \Delta(g) - \Delta(f \cdot g))} \in H^{m+n-1}(V^\bullet)$$

In particular, the right hand side does not depend on the choice of representatives f and g .

Proof. We know that $f \in Z^n(V^\bullet)$ and $g \in Z^m(V^\bullet)$. Since the bracket $[\cdot, \cdot] : V^n \otimes V^m \rightarrow V^{m+n-1}$ descends to a bracket on the cohomology, it follows that $[f, g] \in Z^{m+n-1}(V^\bullet)$. Since $\Delta = \{\Delta^n : V^n \rightarrow V^{n-1}\}_{n \geq 0}$ is a BV-operator, it follows from Definition 2 that

$$[f, g] - (-1)^{(n-1)m} (\Delta(f) \cdot g + (-1)^n f \cdot \Delta(g) - \Delta(f \cdot g)) \in d(V^{m+n-2}) \tag{4}$$

Let us put $z_1 = [f, g]$ and $z_2 = (-1)^{(n-1)m} (\Delta(f) \cdot g + (-1)^n f \cdot \Delta(g) - \Delta(f \cdot g))$. Since $z_1 - z_2 \in d(V^{m+n-2})$, we must have $z_1 - z_2 \in Z^{m+n-1}(V^\bullet)$. We have already seen that $z_1 \in Z^{m+n-1}(V^\bullet)$. Hence, $z_2 \in Z^{m+n-1}(V^\bullet)$. By (4), we know that $z_1 - z_2$ is a coboundary and hence the cohomology classes $\bar{z}_1 = \bar{z}_2$. The result is now clear. \square

3. Application : BV-operator on secondary Hochschild cohomology

Let k be a field and A be an algebra over k . Let B be a commutative k -algebra and $\varepsilon : B \rightarrow A$ be a morphism of k -algebras such that $\varepsilon(B) \subseteq Z(A)$, where $Z(A)$ denotes the center of A . Let M be an A -bimodule such that $\varepsilon(b)m = m\varepsilon(b)$ for all $b \in B$ and $m \in M$. Following [9, §4.2], we consider the complex $(C^\bullet((A, B, \varepsilon); M), \delta^\bullet)$ whose terms are given by

$$C^n((A, B, \varepsilon); M) = \text{Hom}_k\left(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, M\right)$$

An element in $A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}$ will be expressed as a “tensor matrix” of the form

$$\otimes \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n-1} & b_{1,n} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n-1} & b_{2,n} \\ 1 & 1 & a_3 & \dots & b_{3,n-1} & b_{3,n} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a_{n-1} & b_{n-1,n} \\ 1 & 1 & 1 & \dots & 1 & a_n \end{pmatrix}$$

where $a_i \in A$ and $b_{i,j} \in B$. The differentials

$$\delta^n : C^n((A, B, \varepsilon); M) \longrightarrow C^{n+1}((A, B, \varepsilon); M)$$

may be described as follows

$$\begin{aligned} \delta^n(f) & \left(\otimes \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n-1} & b_{1,n} & b_{1,n+1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n-1} & b_{2,n} & b_{2,n+1} \\ 1 & 1 & a_3 & \dots & b_{3,n-1} & b_{3,n} & b_{3,n+1} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1 & a_n & b_{n,n+1} \\ 1 & 1 & 1 & \dots & 1 & 1 & a_{n+1} \end{pmatrix} \right) \\ & = a_1 \varepsilon(b_{1,2} b_{1,3} \dots b_{1,n+1}) f \left(\otimes \begin{pmatrix} a_2 & b_{2,2} & \dots & b_{2,n} & b_{2,n+1} \\ 1 & a_3 & \dots & b_{3,n} & b_{3,n+1} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & \dots & a_n & b_{n,n+1} \\ 1 & 1 & \dots & 1 & a_{n+1} \end{pmatrix} \right) \\ & + \sum_{i=1}^n (-1)^i f \left(\otimes \begin{pmatrix} a_1 & b_{1,2} & \dots & b_{1,i} b_{1,i+1} & \dots & b_{1,n} & b_{1,n+1} \\ 1 & a_2 & \dots & b_{2,i} b_{2,i+1} & \dots & b_{2,n} & b_{2,n+1} \\ \vdots & \vdots & \dots & \dots & \dots & \vdots & \vdots \\ 1 & 1 & \dots & \varepsilon(b_{i,i+1}) a_i a_{i+1} & \dots & b_{i,n} b_{i+1,n} & b_{i,n+1} b_{i+1,n+1} \\ \vdots & \vdots & \dots & \dots & \dots & \vdots & \vdots \\ 1 & 1 & \dots & \dots & \dots & a_n & b_{n,n+1} \\ 1 & 1 & \dots & \dots & \dots & 1 & a_{n+1} \end{pmatrix} \right) \\ & + (-1)^{n+1} f \left(\otimes \begin{pmatrix} a_1 & b_{1,2} & \dots & b_{1,n-1} & b_{1,n} \\ 1 & a_2 & \dots & b_{2,n-1} & b_{2,n} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & \dots & a_{n-1} & b_{n-1,n} \\ 1 & 1 & \dots & 1 & a_n \end{pmatrix} \right) \varepsilon(b_{1,n+1} b_{2,n+1} \dots b_{n,n+1}) a_{n+1} \end{aligned}$$

for $f \in C^n((A, B, \varepsilon); M)$, $a_i \in A$, $b_{i,j} \in B$. The cohomology groups of $(C^*((A, B, \varepsilon); M), \delta^*)$ are known as the secondary Hochschild cohomologies $H^n((A, B, \varepsilon); M)$ of the triple (A, B, ε) with coefficients in M (see [9]).

From [10, Proposition 3.1], we know that the secondary Hochschild complex $C^*(A, B, \varepsilon) := C^*((A, B, \varepsilon); A)$ carries the structure of a homotopy G -algebra. This induces a graded Lie bracket

$$[\cdot, \cdot] : H^m(A, B, \varepsilon) \otimes H^n(A, B, \varepsilon) \longrightarrow H^{m+n-1}(A, B, \varepsilon) \tag{5}$$

on the secondary cohomology. It follows (see [10, Corollary 3.2]) that the secondary cohomology $H^*(A, B, \varepsilon)$ carries the structure of a Gerstenhaber algebra in the sense of [3].

From now onwards, we always let A be a finite dimensional k -algebra equipped with a symmetric, non-degenerate, invariant bilinear form $\langle \cdot, \cdot \rangle : A \times A \rightarrow k$. In particular, $\langle a_1, a_2 \rangle = \langle a_2, a_1 \rangle$, $\langle a_1 a_2, a_3 \rangle = \langle a_1, a_2 a_3 \rangle$ for any $a_1, a_2, a_3 \in A$. For $i \in \{1, \dots, n + 1\}$, we define the maps $\Delta_i : C^{n+1}(A, B, \varepsilon) \rightarrow C^n(A, B, \varepsilon)$ as follows:

$$\begin{aligned} & \left\langle \Delta_i f \otimes \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & b_{n-1,n} \\ 1 & 1 & 1 & \dots & a_n \end{pmatrix}, a_{n+1} \right\rangle \\ &= \left\langle f \otimes \begin{pmatrix} a_i & b_{i,i+1} & b_{i,i+2} & \dots & b_{i,n} & 1 & b_{1,i} & b_{2,i} & \dots & b_{i-1,i} \\ 1 & a_{i+1} & b_{i+1,i+2} & \dots & b_{i+1,n} & 1 & b_{1,i+1} & b_{2,i+1} & \dots & b_{i-1,i+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & a_n & 1 & b_{1,n} & b_{2,n} & \dots & b_{i-1,n} \\ 1 & 1 & \dots & 1 & a_{n+1} & 1 & \dots & \dots & 1 \\ 1 & 1 & \dots & 1 & 1 & a_1 & b_{1,2} & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & \dots & \dots & \dots & \dots & \dots & a_{i-1} \end{pmatrix}, 1 \right\rangle \end{aligned}$$

To clarify the above operator, let us express

$$\otimes \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & b_{n-1,n} \\ 1 & 1 & 1 & \dots & a_n \end{pmatrix} = \begin{pmatrix} U(i-1) & X_{12} \\ 1 & U(n-i-1) \end{pmatrix}$$

where $U(k)$ is a square matrix of dimension k . Then, we have

$$\left\langle \Delta_i f \begin{pmatrix} U(i-1) & X_{12} \\ 1 & U(n-i-1) \end{pmatrix}, a_{n+1} \right\rangle = \left\langle f \begin{pmatrix} U(n-i-1) & 1 & X_{12}^t \\ 1 & a_{n+1} & 1 \\ 1 & 1 & U(i) \end{pmatrix}, 1 \right\rangle$$

where X_{12}^t denotes the transpose of X_{12} . The operator $\Delta : C^{n+1}(A, B, \varepsilon) \rightarrow C^n(A, B, \varepsilon)$ is then defined as

$$\Delta := \sum_{i=1}^{n+1} (-1)^{in} \Delta_i.$$

Following [10, §3], we know that the complex $C^*(A, B, \varepsilon)$ carries a dot product of degree 0, i.e., for $f \in C^n(A, B, \varepsilon)$, $g \in C^m(A, B, \varepsilon)$, we have $f \cdot g \in C^{m+n}(A, B, \varepsilon)$. We also consider the operations

$$\circ_i : C^n(A, B, \varepsilon) \otimes C^m(A, B, \varepsilon) \longrightarrow C^{m+n-1}(A, B, \varepsilon)$$

and set $f \circ g := \sum_{i=1}^n (-1)^{(i-1)(m-1)} f \circ_i g$ as in [10, §3]. We also set

$$\rho^1, \rho^2 : C^n(A, B, \varepsilon) \otimes C^m(A, B, \varepsilon) \longrightarrow C^{n+m-1}(A, B, \varepsilon)$$

$$\rho^1(f \otimes g) := \sum_{i=1}^m (-1)^{i(n+m-1)} \Delta_i(f \cdot g) \quad \rho^2(f \otimes g) := \sum_{i=m+1}^{m+n} (-1)^{i(n+m-1)} \Delta_i(f \cdot g)$$

for $f \in C^n(A, B, \varepsilon)$, $g \in C^m(A, B, \varepsilon)$. It is clear that $\rho^1(f \otimes g) + \rho^2(f \otimes g) = \Delta(f \cdot g)$.

Lemma 4. $\rho^1(f \otimes g) = (-1)^{nm} \rho^2(g \otimes f)$ for all $f \in C^n(A, B, \varepsilon)$ and $g \in C^m(A, B, \varepsilon)$.

Proof. This may be verified by direct computation. □

Lemma 5. Let $f \in Z^n(A, B, \varepsilon)$, $g \in Z^m(A, B, \varepsilon)$. Then $f \circ g - (-1)^{(n-1)m} \Delta(f) \cdot g + (-1)^{(n-1)m} \rho^2(f \otimes g)$ is a coboundary. In fact, if we define H

$$H = \sum_{i,j \geq 1, i+j \leq n} (-1)^{(j-1)(m-1)+i(n+m)+1} \Delta_i(f \circ_j g),$$

then,

$$\delta H = f \circ g - (-1)^{(n-1)m} \Delta(f) \cdot g + (-1)^{(n-1)m} \rho^2(f \otimes g).$$

Proof. We set, for $k \geq 0$, $p \geq 0$:

$$T_{k+p}^k = \otimes \begin{pmatrix} a_{k+1} & \dots & b_{k+1, k+p} \\ \vdots & & \vdots \\ 1 & \dots & a_{k+p} \end{pmatrix}$$

We see that

$$\left\langle \delta(\Delta_i(f \circ_j g)) \otimes \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1, n+m-1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2, n+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & b_{n+m-2, n+m-1} \\ 1 & 1 & 1 & \dots & a_{n+m-1} \end{pmatrix}, a_{n+m} \right\rangle$$

$$= \left\langle f \begin{pmatrix} a_{i+1} \dots b_{i+1, i+j-1} \prod_{k=0}^{m-1} b_{i+1, i+j+k} & \dots & 1 & b_{2, i+1} & \dots & b_{i-1, i+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & a_{i+j-1} & \prod_{k=0}^{m-1} b_{i+j-1, i+j+k} & \dots & 1 & b_{2, i+j-1} & \dots & b_{i-1, i+j-1} \\ 1 & \dots & 1 & g(T_{i+j+m-1}^{i+j-1}) & \prod_{k=0}^{m-1} b_{i+j+k, i+j+m} & 1 & \dots & \dots & \prod_{k=0}^{m-1} b_{i-1, i+j+k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & 1 & 1 & 1 & \alpha & 1 & \dots & 1 \\ 1 & \dots & 1 & 1 & 1 & 1 & a_2 & \dots & b_{2, i} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & 1 & 1 & 1 & 1 & 1 & \dots & a_i \end{pmatrix}, 1 \right\rangle$$

$$+ \sum_{\lambda=1}^{i-1} (-1)^\lambda \left\langle f \begin{pmatrix} a_{i+1} \dots \prod_{k=0}^{m-1} b_{i+1, i+j+k} & \dots & 1 & b_{1, i+1} & b_{\lambda+1, i+1, i+1} & b_{i, i+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & \prod_{k=0}^{m-1} b_{i+j-1, i+j+k} & \dots & 1 & b_{1, i+j-1} & b_{\lambda, i+j-1, \lambda+1, i+j-1} & b_{i, i+j-1} \\ 1 & \dots & g(T_{i+j+m-1}^{i+j-1}) & \prod_{k=0}^{m-1} b_{i+j+k, i+j+m} & \dots & \prod_{k=0}^{m-1} b_{1, i+j+k} & \prod_{k=0}^{m-1} b_{\lambda, i+j+k, \lambda+1, i+j+k} & \prod_{k=0}^{m-1} b_{i, i+j+k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & \dots & a_{n+m} & 1 & \dots & 1 \\ 1 & 1 & \dots & \dots & 1 & a_1 & \dots & b_{1, i} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 1 & 1 & \beta_\lambda & \dots & b_{\lambda, i} b_{\lambda+1, i} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 1 & 1 & 1 & \dots & a_i \end{pmatrix}, 1 \right\rangle$$

$$\begin{aligned}
 & + \sum_{\lambda=i}^{i+j-2} (-1)^\lambda \left\langle f \left(\begin{array}{cccccccc} a_i & b_{i,i+1} & b_{i,\lambda} b_{i,\lambda+1} & \prod_{k=0}^{m-1} b_{i,i+j+k} & \dots & 1 & b_{1,i} & \dots & b_{i-1,i} \\ 1 & a_{i+1} & b_{i+1,\lambda} b_{i+1,\lambda+1} & \prod_{k=0}^{m-1} b_{i+1,i+j+k} & \dots & 1 & b_{1,i+1} & \dots & b_{i-1,i+1} \\ 1 & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & \beta_\lambda & \prod_{k=0}^{m-1} b_{\lambda,i+j+k} b_{\lambda+1,i+j+k} & \dots & 1 & b_{1,\lambda} & \dots & b_{i-1,\lambda} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & \dots & \prod_{k=0}^{m-1} b_{i+j-1,i+j+k} & \dots & 1 & b_{1,i+j-1} & \dots & b_{i-1,i+j-1} \\ 1 & \dots & \dots & g(T_{i+j+m-1}^{i+j-1}) & \prod_{k=0}^{m-1} b_{i+j+k,i+j+m} & 1 & \dots & \dots & \prod_{k=0}^{m-1} b_{i-1,i+j+k} \\ 1 & \dots & \dots & 1 & 1 & a_{n+m} & 1 & \dots & 1 \\ 1 & \dots & \dots & 1 & 1 & 1 & a_1 & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & \dots & 1 & \dots & 1 & 1 & \dots & a_{i-1} \end{array} \right) , 1 \rangle \\
 & + \sum_{\lambda=i+j-1}^{i+j+m-2} (-1)^\lambda \left\langle f \left(\begin{array}{cccccccc} a_i & \dots & \prod_{k=-1}^{m-2} b_{i,i+j+k} & \dots & 1 & b_{1,i} & \dots & b_{i-1,i} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_{i+j-2} & \prod_{k=-1}^{m-2} b_{i+j-2,i+j+k} & \dots & 1 & b_{1,i+j-2} & \dots & b_{i-1,i+j-2} \\ 1 & 1 & g(T_{i+j+m-2}^{i+j-2}) & \prod_{k=-1}^{m-2} b_{i+j+k,i+j+m-1} & 1 & \dots & \dots & \prod_{k=-1}^{m-2} b_{i-1,i+j+k} \\ \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a_{n+m} & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 & a_1 & b_{1,2} & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & 1 & \dots & 1 & 1 & \dots & a_{i-1} \end{array} \right) , 1 \rangle \\
 & + \sum_{\lambda=i+j+m-1}^{n+m-2} (-1)^\lambda \left\langle f \left(\begin{array}{cccccccc} a_i & b_{i,i+1} & \dots & b_{i,i+j-2} & \prod_{k=-1}^{m-2} b_{i,i+j+k} & \dots & 1 & \dots & b_{i-1,i} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots \\ 1 & \dots & \dots & a_{i+j-2} & \prod_{k=-1}^{m-2} b_{i+j-2,i+j+k} & \dots & 1 & \dots & b_{i-1,i+j-2} \\ 1 & \dots & \dots & g(T_{i+j+m-2}^{i+j-2}) & \prod_{k=-1}^{m-2} b_{i+j+k,i+j+m-1} & \dots & 1 & \dots & \prod_{k=-1}^{m-2} b_{i-1,i+j+k} \\ 1 & \dots & \dots & 1 & 1 & \beta_\lambda & 1 & \dots & b_{i-1,\lambda} b_{i-1,\lambda+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots \\ 1 & \dots & \dots & \dots & \dots & \dots & a_{n+m} & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & \dots & \dots & a_1 & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & \dots & \dots & 1 & \dots & a_{i-1} \end{array} \right) , 1 \rangle \\
 & + (-1)^{n+m-1} \left\langle f \left(\begin{array}{cccccccc} a_i & b_{i,i+1} & \dots & b_{i,i+j-2} & \prod_{k=-1}^{m-2} b_{i,i+j+k} & \dots & 1 & \dots & b_{i-1,i} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots \\ 1 & \dots & \dots & a_{i+j-2} & \prod_{k=-1}^{m-2} b_{i+j-2,i+j+k} & \dots & 1 & \dots & b_{i-1,i+j-2} \\ 1 & \dots & \dots & g(T_{i+j+m-2}^{i+j-2}) & \prod_{k=-1}^{m-2} b_{i+j+k,i+j+m-1} & \dots & 1 & \dots & \prod_{k=-1}^{m-2} b_{i-1,i+j+k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots \\ 1 & \dots & \dots & \dots & \dots & \dots & \gamma & 1 & \dots & 1 \\ 1 & \dots & \dots & \dots & \dots & \dots & a_1 & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & \dots & \dots & \dots & \dots & 1 & \dots & a_{i-1} \end{array} \right) , 1 \rangle \tag{6}
 \end{aligned}$$

where $\alpha := \varepsilon(b_{1,2} \dots b_{1,n+m-1}) a_{n+m} a_1$, $\gamma := \varepsilon(b_{1,n+m-1} \dots b_{n+m-2,n+m-1}) a_{n+m} a_{n+m-1}$, $\beta_\lambda := \varepsilon(b_{\lambda,\lambda+1}) a_\lambda a_{\lambda+1}$ for $1 \leq \lambda \leq n+m-2$ and

$$\bar{T}_{i+j+m-2}^{i+j-2} := \begin{pmatrix} a_{i+j-1} & b_{i+j-1,i+j} & \dots & b_{i+j-1,\lambda} b_{i+j-1,\lambda+1} & & b_{i+j-1,i+j+m-2} \\ 1 & a_{i+j} & \dots & b_{i+j,\lambda} b_{i+j,\lambda+1} & & b_{i+j,i+j+m-2} \\ 1 & 1 & \vdots & & & \\ 1 & 1 & \dots & \varepsilon(b_{\lambda,\lambda+1}) a_\lambda a_{\lambda+1} & \dots & b_{\lambda,i+j+m-2} b_{\lambda+1,i+j+m-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & \dots & a_{i+j+m-2} \end{pmatrix}.$$

We write the entire expression of (6) as

$$\left\langle \delta(\Delta_i(f \circ_j g)) \otimes \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n+m-1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & b_{n+m-2,n+m-1} \\ 1 & 1 & 1 & \dots & a_{n+m-1} \end{pmatrix}, a_{n+m} \right\rangle = E_1 + E_2 + E_3 + E_4 + E_5 + E_6,$$

where E_k denotes the k -th term in the expression.

We set for $i, j \geq 1$ and $i + j \leq n$,

$$A_{i,j} := (-1)^{i+1} \left\langle a'_i f \begin{pmatrix} a_{i+1} \dots b_{i+1,i+j-1} \prod_{k=0}^{m-1} b_{i+1,i+j+k} & \dots & 1 & \dots & b_{i-1,i+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & a_{i+j-1} \prod_{k=0}^{m-1} b_{i+j-1,i+j+k} & \dots & 1 & \dots & b_{i-1,i+j-1} \\ 1 & \dots & g(T_{i+j+m-1}^{i+j-1}) \prod_{k=0}^{m-1} b_{i+j+k,i+j+m} & \dots & 1 & \dots & \prod_{k=0}^{m-1} b_{i-1,i+j+k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & \dots & \dots & a_{n+m} & 1 & \dots & 1 \\ 1 & 1 & \dots & \dots & \dots & a_1 & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & \dots & \dots & 1 & \dots & a_{i-1} \end{pmatrix}, 1 \right\rangle$$

$$+ E_3 + (-1)^{i+j-1} \left\langle f \begin{pmatrix} a_i \dots b_{i,i+j-2} \prod_{k=-1}^{m-2} b_{i,i+j+k} & \dots & 1 & \dots & b_{i-1,i} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & a_{i+j-2} \prod_{k=-1}^{m-2} b_{i+j-2,i+j+k} & \dots & 1 & \dots & b_{i-1,i+j-2} \\ 1 & \dots & \eta \prod_{k=0}^{m-1} b_{i+j+k,i+j+m} & \dots & 1 & \dots & \prod_{k=0}^{m-1} b_{i-1,i+j+k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & \dots & \dots & a_{n+m} & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & \dots & a_1 & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & \dots & \dots & \dots & a_{i-1} \end{pmatrix}, 1 \right\rangle$$

where $a'_i = a_i \varepsilon(b_{i,i+1} \dots b_{i,n+m-1} b_{1,i} \dots b_{i-1,i})$. We also set

$$B_{i,j} := (-1)^{i+j+m-2} \left\langle f \begin{pmatrix} a_i \dots b_{i,i+j-2} \prod_{k=-1}^{m-2} b_{i,i+j+k} & \dots & 1 & \dots & b_{i-1,i} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & a_{i+j-2} \prod_{k=-1}^{m-2} b_{i+j-2,i+j+k} & \dots & 1 & \dots & b_{i-1,i+j-2} \\ 1 & \dots & \zeta \prod_{k=-1}^{m-2} b_{i+j+k,i+j+m-1} & \dots & 1 & \dots & \prod_{k=-1}^{m-2} b_{i-1,i+j+k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & \dots & \dots & a_{n+m} & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & \dots & a_1 & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & \dots & \dots & \dots & a_{i-1} \end{pmatrix}, 1 \right\rangle$$

$+ E_5 + E_6$

where

$$\eta = a_{i+j-1} \varepsilon(b_{i+j-1,i+j} \dots b_{i+j-1,n+m-1}) g(T_{i+j+m-1}^{i+j-1})$$

$$\zeta = g(T_{i+j+m-2}^{i+j-2}) \varepsilon \left(\prod_{k=-1}^{m-2} b_{i+j+k,i+j+m-1} \right),$$

and

$$C_{i,j} := (-1)^i \left\langle f \left(\begin{array}{cccccccc} a_{i+1} & \dots & b_{i+1,i+j-1} & \prod_{k=0}^{m-1} b_{i+1,i+j+k} & \dots & 1 & \dots & b_{i-1,i+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots \\ 1 & \dots & a_{i+j-1} & \prod_{k=0}^{m-1} b_{i+j-1,i+j+k} & \dots & 1 & \dots & b_{i-1,i+j-1} \\ 1 & \dots & \dots & g(T_{i+j+m-1}^{i+j-1}) & \prod_{k=0}^{m-1} b_{i+j+k,i+j+m} & \dots & 1 & \dots & \prod_{k=0}^{m-1} b_{i-1,i+j+k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots \\ 1 & \dots & \dots & \dots & \dots & a_{n+m} & 1 & 1 \\ 1 & 1 & \dots & \dots & \dots & \dots & a_1 & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & \dots & \dots & \dots & 1 & a_{i-1} \end{array} \right) a'_{i,1} \right\rangle + E_1 + E_2$$

The first term of $A_{i,j}$ and that of $C_{i,j}$ are the same modulo a sign. Using the fact that $\delta g = 0$, the third term of $A_{i,j}$ and the first term of $B_{i,j}$ add up to give E_4 . Thus, we have

$$\left\langle \delta(\Delta_i(f \circ_j g)) \left(\otimes \left(\begin{array}{cccc} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n+m-1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & b_{n+m-2,n+m-1} \\ 1 & 1 & 1 & \dots & a_{n+m-1} \end{array} \right) , a_{n+m} \right\rangle = A_{i,j} + B_{i,j} + C_{i,j} \tag{7}$$

It may be verified that

$$(-1)^{i+1} A_{i,j-1} + (-1)^{i+m} B_{i,j} + (-1)^{i+n} C_{i-1,j} = \left\langle \Delta_i((\delta f) \circ_j g) \left(\otimes \left(\begin{array}{cccc} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n+m-1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a_{m+n-1} \end{array} \right) , a_{n+m} \right\rangle = 0 \tag{8}$$

for $2 \leq i \leq n, 2 \leq j \leq n-1$ and $i+j \leq n$. The second equality in (8) uses the fact that $\delta f = 0$.

For $i, j \in \{1, \dots, n+1\}$, we define

$$A_{i,0} := (-1)^{i+1} \left\langle g(T_{i+m-1}^{i-1}) f \left(\begin{array}{cccccccc} a_{i+m} & b_{i+m,i+m+1} & \dots & b_{i+m,n+m-1} & 1 & b_{1,i+m} & b_{2,i+m} \dots & b_{i-1,i+m} \\ 1 & a_{i+m+1} & \dots & b_{i+m+1,n+m-1} & 1 & b_{2,i+m+1} & b_{2,i+m+1} \dots & b_{i-1,i+m+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & a_{n+m} & 1 & \dots & 1 & \dots \\ 1 & 1 & \dots & 1 & a_1 & b_{1,2} \dots & b_{1,i-1} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & \dots & \dots & \dots & a_{i-1} & \dots \end{array} \right) , 1 \right\rangle,$$

$$C_{0,j} = \left\langle f \left(\begin{array}{cccccccc} a_1 & \dots & b_{1,j-1} & \prod_{k=0}^{m-1} b_{1,j+k} & b_{1,j+m} & \dots & b_{1,n+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & a_{j-1} & \prod_{k=0}^{m-1} b_{j-1,j+k} & b_{j-1,j+m} & \dots & b_{j-1,n+m-1} \\ 1 & \dots & 1 & g(T_{j+m-1}^{j-1}) & \prod_{k=0}^{m-1} b_{j+k,j+m} & \dots & \prod_{k=0}^{m-1} b_{j+k,n+m-1} \\ 1 & \dots & 1 & 1 & a_{j+m} & \dots & b_{j+m,n+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & \dots & \dots & \dots & \dots & a_{n+m-1} \end{array} \right) a_{n+m}, 1 \right\rangle,$$

and for $i \in \{1, \dots, n\}$, define

$$B_{i,n-i+1} := (-1)^{n+m+1} \left\langle f \begin{pmatrix} a_i \dots b_{i,n-1} & \prod_{k=1}^m b_{i,n-1+k} & b_{1,i} & \dots & b_{i-1,i} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 \dots a_{n-1} & \prod_{k=1}^m b_{n-1,n-1+k} & b_{1,n-1} & \dots & b_{i-1,n-1} \\ 1 \dots 1 & g(T_{n+m-1}^{n-1}) \cdot a_{n+m} & \prod_{k=0}^{m-1} b_{1,n+k} & \dots & \prod_{k=0}^{m-1} b_{i-1,n+k} \\ 1 \dots 1 & 1 & a_1 & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 \dots \dots & \dots & \dots & \dots & a_{i-1} \end{pmatrix}, 1 \right\rangle.$$

Thus, $A_{i,j}, B_{i,j+1}, C_{i-1,j}$ are defined for all the values of i, j with $i, j \geq 1$ and $i + j \leq n + 1$. Moreover, it may be verified that

$$A_{1,j-1} + (-1)^{m+1} B_{1,j} + (-1)^{n+1} C_{0,j} = \left\langle (\delta f) \begin{pmatrix} a_1 \dots b_{1,j-1} & \prod_{k=0}^{m-1} b_{1,j+k} & b_{1,j+m} & \dots & b_{1,n+m-1} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 1 \\ 1 \dots a_{j-1} & \prod_{k=0}^{m-1} b_{j-1,j+k} & b_{j-1,j+m} & \dots & b_{j-1,n+m-1} & 1 \\ 1 \dots 1 & g(T_{j+m-1}^{j-1}) & \prod_{k=0}^{m-1} b_{j+k,j+m} & \dots & \prod_{k=0}^{m-1} b_{j+k,n+m-1} & 1 \\ 1 \dots 1 & 1 & a_{j+m} & \dots & b_{j+m,n+m-1} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 1 \\ 1 \dots \dots & \dots & \dots & \dots & a_{n+m-1} & 1 \\ 1 \dots 1 & 1 & 1 & 1 & 1 & a_{n+m} \end{pmatrix}, 1 \right\rangle = 0$$

We also have

$$(-1)^{i+1} A_{i,0} + (-1)^{i+m} B_{i,1} + (-1)^{i+n} C_{i-1,1} = \left\langle (\delta f) \begin{pmatrix} g(T_{i+m-1}^{i-1}) & \prod_{k=0}^{m-1} b_{i+k,i+m} & \dots & \prod_{k=0}^{m-1} b_{i+k,n+m-1} & 1 & \prod_{k=0}^{m-1} b_{1,i+k} & \dots & \prod_{k=0}^{m-1} b_{i-1,i+k} \\ 1 & a_{i+m} & \dots & b_{i+m,n+m-1} & 1 & b_{1,i+m} & \dots & \prod_{k=0}^{m-1} b_{i-1,i+m} \\ \dots & \dots & \dots & \dots & 1 & \dots & \dots & \dots \\ \dots & \dots & \dots & a_{n+m-1} & 1 & b_{1,n+m-1} & \dots & b_{i-1,n+m-1} \\ \dots & \dots & \dots & 1 & a_{n+m} & 1 & \dots & 1 \\ \dots & \dots & \dots & 1 & 1 & a_1 & \dots & b_{1,i-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 1 & 1 & 1 & \dots & a_{i-1} \end{pmatrix}, 1 \right\rangle = 0$$

and

$$(-1)^{i+1} A_{i,n-i} + (-1)^{i+m} B_{i,n-i+1} + (-1)^{i+n} C_{i-1,n-i+1} = \left\langle (\delta f) \begin{pmatrix} a_i \dots \dots b_{i,n-1} & \prod_{k=0}^{m-1} b_{i,n+k} & 1 & b_{1,i} & \dots & b_{i-1,i} \\ 1 & a_{i+1} \dots b_{i+1,n-1} & \prod_{k=0}^{m-1} b_{i+1,n+k} & 1 & b_{1,i+1} & \dots & b_{i-1,i+1} \\ \vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots \\ 1 \dots \dots a_{n-1} & \prod_{k=0}^{m-1} b_{n-1,n+k} & 1 & \dots & \dots & b_{i-1,n-1} \\ 1 \dots \dots & g(T_{n+m-1}^{n-1}) & 1 & \prod_{k=0}^{m-1} b_{1,n+k} & \dots & \prod_{k=0}^{m-1} b_{i-1,n+k} \\ 1 \dots \dots \dots & \dots & a_{n+m} & 1 & \dots & 1 \\ 1 & 1 \dots \dots & \dots & a_1 & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 \dots \dots & \dots & \dots & \dots & a_{i-1} \end{pmatrix}, 1 \right\rangle$$

Thus, we obtain

$$0 = \sum_{1 \leq i \leq n, 1 \leq j \leq n-1, i+j \leq n+1} (-1)^{(j-1)(m-1)+i(n+m-1)} \left((-1)^{i+1} A_{i,j-1} + (-1)^{i+m} B_{i,j} + (-1)^{i+n} C_{i-1,j} \right) \quad (9)$$

Rearranging the terms in the above sum, and using equation (7), we get,

$$\begin{aligned} 0 &= \sum_{1 \leq i \leq n, 1 \leq j \leq n, i+j \leq n} (-1)^{(j-1)(m-1)+i(n+m)+1} (A_{i,j} + B_{i,j} + C_{i,j}) \\ &\quad - \sum_{i=1}^n (-1)^{m-1+i(n+m)} A_{i,0} - \sum_{i=1}^n (-1)^{n(m+1)+i(n+1)} B_{i,n-i+1} - \sum_{j=1}^n (-1)^{(j-1)(m-1)} C_{0,j} \\ &= \left\langle \delta(H) \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n+m-1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & b_{n+m-2,n+m-1} \\ 1 & 1 & 1 & \dots & a_{n+m-1} \end{pmatrix}, a_{n+m} \right\rangle \\ &\quad - (-1)^{m(n+1)} \left\langle (\rho^2(f \otimes g)) \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n+m-1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & b_{n+m-2,n+m-1} \\ 1 & 1 & 1 & \dots & a_{n+m-1} \end{pmatrix}, a_{n+m} \right\rangle \\ &\quad - (-1)^{m(n+1)} \left\langle (\Delta(f) \cdot g) \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n+m-1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & b_{n+m-2,n+m-1} \\ 1 & 1 & 1 & \dots & a_{n+m-1} \end{pmatrix}, a_{n+m} \right\rangle \\ &\quad - \left\langle (f \circ g) \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n+m-1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & b_{n+m-2,n+m-1} \\ 1 & 1 & 1 & \dots & a_{n+m-1} \end{pmatrix}, a_{n+m} \right\rangle. \quad \square \end{aligned}$$

Proposition 6. The family $\Delta = \{\Delta^\bullet : C^\bullet(A, B, \varepsilon) \rightarrow C^{\bullet-1}(A, B, \varepsilon)\}$ determines a BV-operator on the homotopy G-algebra $C^\bullet(A, B, \varepsilon)$.

Proof. We consider $f \in Z^n(A, B, \varepsilon)$ and $g \in Z^m(A, B, \varepsilon)$. By definition (see [10, §3]), we know that

$$[f, g] = f \circ g - (-1)^{(n-1)(m-1)} g \circ f \in C^{m+n-1}(A, B, \varepsilon) \quad (10)$$

Applying Lemma 5, we know that the cochains

$$\begin{aligned} &f \circ g - (-1)^{(n-1)m} \Delta(f) \cdot g + (-1)^{(n-1)m} \rho^2(f \otimes g) \\ &g \circ f - (-1)^{(m-1)n} \Delta(g) \cdot f + (-1)^{(m-1)n} \rho^2(g \otimes f) \end{aligned}$$

are coboundaries. From (10), it now follows that

$$[f, g] - (-1)^{(n-1)m} \Delta(f) \cdot g + (-1)^{(n-1)m} \rho^2(f \otimes g) + (-1)^{(m-1)n} \Delta(g) \cdot f + (-1)^m \rho^2(g \otimes f) \quad (11)$$

is a coboundary. Applying Lemma 4, it follows from (11) that

$$[f, g] - (-1)^{(n-1)m} \Delta(f) \cdot g + (-1)^{m(n-1)} \rho^2(f \otimes g) + (-1)^{(m-1)n} \Delta(g) \cdot f + (-1)^{m(n-1)} \rho^1(f \otimes g)$$

is a coboundary. Since $\rho^1(f \otimes g) + \rho^2(f \otimes g) = \Delta(f \cdot g)$, we get

$$[f, g] - (-1)^{(n-1)m} \Delta(f) \cdot g + (-1)^{m(n-1)} \Delta(f \cdot g) + (-1)^{(m-1)n} \Delta(g) \cdot f$$

is a coboundary. Using the fact that the dot product is graded commutative, we can put $\Delta(g) \cdot f = (-1)^{n(m-1)} f \cdot \Delta(g)$. The result is now clear. \square

Theorem 7. For secondary cohomology classes $\bar{f} \in H^n(A, B, \varepsilon)$ and $\bar{g} \in H^m(A, B, \varepsilon)$, the Gerstenhaber bracket is determined by

$$[\bar{f}, \bar{g}] = (-1)^{(n-1)m} \overline{(\Delta(f) \cdot g + (-1)^n f \cdot \Delta(g) - \Delta(f \cdot g))} \in H^{m+n-1}(A, B, \varepsilon)$$

Here f and g are any cocycles representing the classes \bar{f} and \bar{g} respectively.

Proof. This follows directly by applying Theorem 3 and Proposition 6. \square

It is natural to ask whether the BV-operator determined by $\Delta = \{\Delta^\bullet : C^\bullet(A, B, \varepsilon) \rightarrow C^{\bullet-1}(A, B, \varepsilon)\}$ induces a BV-algebra structure on the secondary Hochschild cohomology $H^\bullet(A, B, \varepsilon)$. In the special case of $B = k$, we are reduced to ordinary Hochschild cohomology and hence Δ determines a BV-algebra structure on $H^\bullet(A, A)$. However, this is not true in general because $\Delta^\bullet : C^\bullet(A, B, \varepsilon) \rightarrow C^{\bullet-1}(A, B, \varepsilon)$ does not commute with the differentials. For instance, we take $n = 2$ and ask when the following diagram commutes:

$$\begin{CD} C^2(A, B, \varepsilon) @>\delta^2>> C^3(A, B, \varepsilon) \\ @V\Delta^2VV @VV-\Delta^3V \\ C^1(A, B, \varepsilon) @>\delta^1>> C^2(A, B, \varepsilon) \end{CD} \tag{12}$$

For any $a_1, a_2, a_3 \in A, b_{1,2} \in B$ and $f \in C^2(A, B, \varepsilon)$, we have

$$\begin{aligned} & \langle (-\Delta^3 \delta^2 f) \left(\begin{matrix} a_1 & b_{1,2} \\ & a_2 \end{matrix}, a_3 \right) \rangle \\ &= -\left\langle (\delta^2 f) \left(\begin{matrix} a_1 & b_{1,2} & 1 \\ & a_2 & 1 \\ & & a_3 \end{matrix}, 1 \right) \right\rangle - \left\langle (\delta^2 f) \left(\begin{matrix} a_2 & 1 & b_{1,2} \\ & a_3 & 1 \\ & & a_1 \end{matrix}, 1 \right) \right\rangle - \left\langle (\delta^2 f) \left(\begin{matrix} a_3 & 1 & 1 \\ & a_1 & b_{1,2} \\ & & a_2 \end{matrix}, 1 \right) \right\rangle \\ &= -\left\langle a_1 \varepsilon(b_{1,2}) f \left(\begin{matrix} a_2 & 1 \\ & a_3 \end{matrix}, 1 \right) \right\rangle + \left\langle f \left(\begin{matrix} a_1 a_2 \varepsilon(b_{1,2}) & 1 \\ & a_3 \end{matrix}, 1 \right) \right\rangle - \left\langle f \left(\begin{matrix} a_1 & b_{1,2} \\ & a_2 a_3 \end{matrix}, 1 \right) \right\rangle + \left\langle f \left(\begin{matrix} a_1 & b_{1,2} \\ & a_2 \end{matrix}, a_3, 1 \right) \right\rangle \\ & \quad - \left\langle a_2 \varepsilon(b_{1,2}) f \left(\begin{matrix} a_3 & 1 \\ & a_1 \end{matrix}, 1 \right) \right\rangle + \left\langle f \left(\begin{matrix} a_2 a_3 & b_{1,2} \\ & a_1 \end{matrix}, 1 \right) \right\rangle - \left\langle f \left(\begin{matrix} a_2 & b_{1,2} \\ & a_3 a_1 \end{matrix}, 1 \right) \right\rangle + \left\langle f \left(\begin{matrix} a_2 & 1 \\ & a_3 \end{matrix}, a_1 \varepsilon(b_{1,2}), 1 \right) \right\rangle \\ & \quad - \left\langle a_3 f \left(\begin{matrix} a_1 & b_{1,2} \\ & a_2 \end{matrix}, 1 \right) \right\rangle + \left\langle f \left(\begin{matrix} a_3 a_1 & b_{1,2} \\ & a_2 \end{matrix}, 1 \right) \right\rangle - \left\langle f \left(\begin{matrix} a_3 & 1 \\ & a_1 a_2 \varepsilon(b_{1,2}) \end{matrix}, 1 \right) \right\rangle + \left\langle f \left(\begin{matrix} a_3 & 1 \\ & a_1 \end{matrix}, a_2 \varepsilon(b_{1,2}), 1 \right) \right\rangle \end{aligned}$$

Now, using the properties of the inner product $\langle \cdot, \cdot \rangle$ on A we see that the first term cancels with the eighth, the fourth term cancels with the ninth, the fifth term cancels with the twelfth. Thus, the above expression reduces to

$$\begin{aligned} & \left\langle f \left(\begin{matrix} a_1 a_2 \varepsilon(b_{1,2}) & 1 \\ & a_3 \end{matrix}, 1 \right) \right\rangle - \left\langle f \left(\begin{matrix} a_1 & b_{1,2} \\ & a_2 a_3 \end{matrix}, 1 \right) \right\rangle + \left\langle f \left(\begin{matrix} a_2 a_3 & b_{1,2} \\ & a_1 \end{matrix}, 1 \right) \right\rangle - \left\langle f \left(\begin{matrix} a_2 & b_{1,2} \\ & a_3 a_1 \end{matrix}, 1 \right) \right\rangle \\ & \quad + \left\langle f \left(\begin{matrix} a_3 a_1 & b_{1,2} \\ & a_2 \end{matrix}, 1 \right) \right\rangle - \left\langle f \left(\begin{matrix} a_3 & 1 \\ & a_1 a_2 \varepsilon(b_{1,2}) \end{matrix}, 1 \right) \right\rangle \end{aligned} \tag{13}$$

On the other hand, we have

$$\begin{aligned}
 & \left\langle (\delta^1 \Delta^2 f) \begin{pmatrix} a_1 & b_{1,2} \\ & a_2 \end{pmatrix}, a_3 \right\rangle \\
 &= \langle a_1 \varepsilon(b_{1,2})(\Delta^2 f)(a_2), a_3 \rangle - \langle (\Delta^2 f)(a_1 a_2 \varepsilon(b_{1,2})), a_3 \rangle + \langle (\Delta^2 f)(a_1) a_2 \varepsilon(b_{1,2}), a_3 \rangle \\
 &= \langle (\Delta^2 f)(a_2), a_3 a_1 \varepsilon(b_{1,2}) \rangle - \langle (\Delta^2 f)(a_1 a_2 \varepsilon(b_{1,2})), a_3 \rangle + \langle (\Delta^2 f)(a_1), a_2 a_3 \varepsilon(b_{1,2}) \rangle \\
 &= -\left\langle f \begin{pmatrix} a_2 & 1 \\ & a_3 a_1 \varepsilon(b_{1,2}) \end{pmatrix}, 1 \right\rangle + \left\langle f \begin{pmatrix} a_3 a_1 \varepsilon(b_{1,2}) & 1 \\ & a_2 \end{pmatrix}, 1 \right\rangle + \left\langle f \begin{pmatrix} a_1 a_2 \varepsilon(b_{1,2}) & 1 \\ & a_3 \end{pmatrix}, 1 \right\rangle \\
 &\quad - \left\langle f \begin{pmatrix} a_3 & 1 \\ & a_1 a_2 \varepsilon(b_{1,2}) \end{pmatrix}, 1 \right\rangle - \left\langle f \begin{pmatrix} a_1 & 1 \\ & a_2 a_3 \varepsilon(b_{1,2}) \end{pmatrix}, 1 \right\rangle + \left\langle f \begin{pmatrix} a_2 a_3 \varepsilon(b_{1,2}) & 1 \\ & a_1 \end{pmatrix}, 1 \right\rangle \tag{14}
 \end{aligned}$$

From the expressions in (13) and (14), it is clear that the diagram (12) does not commute in general, even if we take A to be commutative and $B = A$.

4. Relation with extra degeneracy and norm operator

We continue with A being a finite dimensional k -algebra equipped with a symmetric, non-degenerate and invariant bilinear form $\langle \cdot, \cdot \rangle : A \times A \rightarrow k$ and B a commutative k -algebra with a morphism of k -algebras $\varepsilon : B \rightarrow A$ such that $\varepsilon(B) \subseteq Z(A)$. In particular, the non-degenerate pairing on A induces mutually inverse isomorphisms

$$\phi : A^* \xrightarrow{\cong} A \quad \phi^{-1} : A \xrightarrow{\cong} A^* \tag{15}$$

We let $C^\bullet(A, M)$ denote the ordinary Hochschild complex of A with coefficients in an A -bimodule M and its cohomology by $H^\bullet(A, M)$. In particular, we may set $M = A^*$ equipped with the A -bimodule structure $(a' f a'')(a) := f(a'' a a')$ for $f \in A^* = \text{Hom}(A, k)$ and $a, a', a'' \in A$. In that case, the terms $\{C^n(A, A^*)\}_{n \geq 0}$ in the Hochschild complex $C^\bullet(A, A^*)$ may also be written as $C^n(A, A^*) \cong \text{Hom}(A^{\otimes n+1}, k)$. We denote by $\widehat{C}^\bullet(A)$ the corresponding complex defined by setting $\widehat{C}^n(A) := \text{Hom}(A^{\otimes(n+1)}, k)$ for $n \geq 0$.

From Tradler [11], we know that the operator $\Delta^\bullet : C^\bullet(A, A) \rightarrow C^{\bullet-1}(A, A)$ on Hochschild cochains inducing the BV-structure on $H^\bullet(A, A)$ fits into the following commutative diagram with the duals of Hochschild chains

$$\begin{array}{ccc}
 \widehat{C}^{\bullet+1}(A) & \xrightarrow{Ns} & \widehat{C}^\bullet(A) \\
 \cong \downarrow & & \downarrow \cong \\
 C^\bullet(A, A^*) & & C^{\bullet-1}(A, A^*) \\
 \phi^\bullet \downarrow \cong & & \cong \downarrow \phi^{\bullet-1} \\
 C^\bullet(A, A) & \xrightarrow{\Delta^\bullet} & C^{\bullet-1}(A, A)
 \end{array} \tag{16}$$

Here, each $\phi^\bullet : C^\bullet(A, A^*) \rightarrow C^\bullet(A, A)$ is the isomorphism induced by $\phi : A^* \xrightarrow{\cong} A$, while s and N respectively are the usual extra degeneracy and norm operators given by

$$s : \widehat{C}^{n+1}(A) \longrightarrow \widehat{C}^n(A) \quad (sf)(a_1, \dots, a_n) := f(a_1, \dots, a_n, 1) \tag{17}$$

$$N := 1 + \lambda + \dots + \lambda^n : \widehat{C}^n(A) \longrightarrow \widehat{C}^n(A) \quad (\lambda \cdot f)(a_0, \dots, a_n) := (-1)^n f(a_n, a_0, \dots, a_{n-1})$$

If we pass to the normalized Hochschild complex which is a quasi-isomorphic subcomplex of $C^\bullet(A, A^*)$, then (16) induces the following commutative diagram

$$\begin{array}{ccc}
 H^\bullet(A, A^*) & \xrightarrow{B^\bullet} & H^{\bullet-1}(A, A^*) \\
 \phi^\bullet \downarrow \cong & & \cong \downarrow \phi^{\bullet-1} \\
 H^\bullet(A, A) & \xrightarrow{\Delta^\bullet} & H^{\bullet-1}(A, A)
 \end{array} \tag{18}$$

where $B^\bullet : H^\bullet(A, A^*) \rightarrow H^{\bullet-1}(A, A^*)$ is the standard Connes operator.

In the case of secondary Hochschild cohomology, we have shown in Section 3 that Δ^\bullet is not in general a morphism of complexes, i.e., it does not descend to cohomology. We will now show that the operator Δ^\bullet on secondary Hochschild cohomology $H^\bullet(A, B, \varepsilon)$ fits into a commutative diagram similar to (18).

In [8], Laubacher, Staic and Stancu have introduced a co-simplicial module $\bar{C}^\bullet(A, B, \varepsilon)$ which is used to compute the secondary Hochschild cohomology associated to the triple (A, B, ε) . The terms of this co-simplicial module are given by

$$\bar{C}^\bullet(A, B, \varepsilon) := \left\{ \text{Hom} \left(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, \text{Hom}(A \otimes B^n, k) \right) \right\}_{n \geq 0} \tag{19}$$

It is important to note (see [8, Remark 4.7]) that despite the similar names, the complex $\bar{C}^\bullet(A, B, \varepsilon)$ cannot be expressed as the secondary Hochschild complex of (A, B, ε) with coefficients in some A -bimodule. This is because the ‘‘coefficient module’’ $\text{Hom}(A \otimes B^n, k)$ appearing in (19) varies with n .

In addition, the cosimplicial module $\bar{C}^\bullet(A, B, \varepsilon)$ is equipped with a cyclic operator, which can be used to compute the secondary cyclic cohomology associated to the triple (A, B, ε) . Using the isomorphisms

$$\Psi^n : \text{Hom} \left(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, \text{Hom}(A \otimes B^n, k) \right) \xrightarrow{\cong} \text{Hom} \left(A^{\otimes(n+1)} \otimes B^{\otimes \frac{n(n+1)}{2}}, k \right) \tag{20}$$

given by

$$\begin{aligned} (\Psi^n g) & \left(\otimes \begin{pmatrix} a_0 & b_{0,1} & b_{0,2} & \dots & b_{0,n-1} & b_{0,n} \\ 1 & a_1 & b_{1,2} & \dots & b_{1,n-1} & b_{1,n} \\ 1 & 1 & a_2 & \dots & b_{2,n-1} & b_{2,n} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a_{n-1} & b_{n-1,n} \\ 1 & 1 & 1 & \dots & 1 & a_n \end{pmatrix} \right) \\ & = g \left(\otimes \begin{pmatrix} a_0 & b_{0,1} & \dots & b_{0,n-2} & b_{0,n-1} \\ 1 & a_1 & \dots & b_{1,n-2} & b_{1,n-1} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & \dots & a_{n-2} & b_{n-2,n-1} \\ 1 & 1 & \dots & 1 & a_{n-1} \end{pmatrix} \right) (a_n \otimes b_{0,n} \otimes b_{1,n} \otimes \dots \otimes b_{n-1,n}) \end{aligned} \tag{21}$$

we first transfer the cyclic operator from [8] to a complex $\widehat{C}^\bullet(A, B, \varepsilon)$ whose terms are given by

$$\left\{ \widehat{C}^n(A, B, \varepsilon) := \text{Hom} \left(A^{\otimes(n+1)} \otimes B^{\otimes \frac{n(n+1)}{2}}, k \right) \right\}_{n \geq 0} \tag{22}$$

Lemma 8. *For each $n \geq 0$, there is an action of the cyclic group $\mathbb{Z}_{n+1} = \langle \lambda \rangle$ on the k -space $\widehat{C}^n(A, B, \varepsilon) = \text{Hom} \left(A^{\otimes(n+1)} \otimes B^{\otimes \frac{n(n+1)}{2}}, k \right)$ given by*

$$(\lambda \cdot f) \left(\otimes \begin{pmatrix} a_0 & b_{0,1} & b_{0,2} & \dots & b_{0,n-1} & b_{0,n} \\ 1 & a_1 & b_{1,2} & \dots & b_{1,n-1} & b_{1,n} \\ 1 & 1 & a_2 & \dots & b_{2,n-1} & b_{2,n} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a_{n-1} & b_{n-1,n} \\ 1 & 1 & 1 & \dots & 1 & a_n \end{pmatrix} \right) = (-1)^n f \left(\otimes \begin{pmatrix} a_n & b_{0,n} & b_{1,n} & \dots & b_{n-2,n} & b_{n-1,n} \\ 1 & a_0 & b_{0,1} & \dots & b_{0,n-2} & b_{0,n-1} \\ 1 & 1 & a_1 & \dots & b_{1,n-2} & b_{1,n-1} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a_{n-2} & b_{n-2,n-1} \\ 1 & 1 & 1 & \dots & 1 & a_{n-1} \end{pmatrix} \right)$$

for any $f \in \widehat{C}^n(A, B, \varepsilon)$, $a_i \in A$ and $b_{i,j} \in B$.

Proof. This is clear from the definition in [8, §4.2] and the isomorphisms in (20). □

The norm operator $N : \widehat{C}^n(A, B, \varepsilon) \rightarrow \widehat{C}^n(A, B, \varepsilon)$ is then defined as $N = 1 + \lambda + \dots + \lambda^n$. The extra degeneracy $s : \widehat{C}^n(A, B, \varepsilon) \rightarrow \widehat{C}^{n-1}(A, B, \varepsilon)$ is given by

$$(sf) \left(\otimes \begin{pmatrix} \begin{pmatrix} a_0 & b_{0,1} & b_{0,2} & \dots & b_{0,n-2} & b_{0,n-1} \\ 1 & a_1 & b_{1,2} & \dots & b_{1,n-2} & b_{1,n-1} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a_{n-2} & b_{n-2,n-1} \\ 1 & 1 & 1 & \dots & 1 & a_{n-1} \end{pmatrix} \end{pmatrix} \right) = f \left(\otimes \begin{pmatrix} \begin{pmatrix} a_0 & b_{0,1} & b_{0,2} & \dots & b_{0,n-1} & 1 \\ 1 & a_1 & b_{1,2} & \dots & b_{1,n-1} & 1 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a_{n-1} & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix} \end{pmatrix} \right)$$

for any $f \in \widehat{C}^n(A, B, \varepsilon)$, $a_i \in A$ and $b_{i,j} \in B$.

For $n \geq 0$, let $\alpha^n : (A \otimes B^{\otimes n})^* \rightarrow A^*$ be the map defined by $p \mapsto \bar{p} := \alpha^n(p)$, where $\bar{p}(a) := p(a \otimes 1_B \otimes \dots \otimes 1_B)$. We denote by $\alpha^* : \bar{C}^*(A, B, \varepsilon) \rightarrow C^*((A, B, \varepsilon); A^*)$ the induced map.

We also let $\Phi^* : C^*((A, B, \varepsilon); A^*) \rightarrow C^*((A, B, \varepsilon); A)$ be the map induced by the isomorphism $\phi : A^* \rightarrow A$ and $\Phi'^* : C^*((A, B, \varepsilon); A) \rightarrow C^*((A, B, \varepsilon); A^*)$ be the map induced by ϕ^{-1} . It may also be verified that the inverse $\Psi'_n : \text{Hom}(A^{\otimes(n+1)} \otimes B^{\otimes \frac{n(n+1)}{2}}, k) \rightarrow \text{Hom}(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, \text{Hom}(A \otimes B^n, k))$ of the map in (21) is given by

$$(\Psi'^n f) \left(\otimes \begin{pmatrix} \begin{pmatrix} a_1 & b_{1,2} & \dots & b_{1,n-1} & b_{1,n} \\ 1 & a_2 & \dots & b_{2,n-1} & b_{2,n} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & \dots & a_{n-1} & b_{n-1,n} \\ 1 & 1 & \dots & 1 & a_n \end{pmatrix} \end{pmatrix} \right) \begin{pmatrix} a_{n+1} \\ b_{1,n+1} \\ b_{2,n+1} \\ \vdots \\ b_{n-1,n+1} \\ b_{n,n+1} \end{pmatrix} = f \left(\otimes \begin{pmatrix} \begin{pmatrix} a_1 & b_{1,2} & \dots & b_{1,n-1} & b_{1,n} & b_{1,n+1} \\ 1 & a_2 & \dots & b_{2,n-1} & b_{2,n} & b_{2,n+1} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & a_{n-1} & b_{n-1,n} & b_{n-1,n+1} \\ 1 & 1 & \dots & 1 & a_n & b_{n,n+1} \\ 1 & 1 & \dots & 1 & 1 & a_{n+1} \end{pmatrix} \end{pmatrix} \right) \quad (23)$$

Proposition 9. *Let A be a finite dimensional k -algebra equipped with a symmetric, non-degenerate and invariant bilinear form $\langle \cdot, \cdot \rangle : A \times A \rightarrow k$ and B be a commutative k -algebra with a morphism of k -algebras $\varepsilon : B \rightarrow A$ such that $\varepsilon(B) \subseteq Z(A)$. Then, the following diagram commutes:*

$$\begin{CD} \bar{C}^*(A, B, \varepsilon) @>Ns>> \bar{C}^{*-1}(A, B, \varepsilon) \\ @V\Phi^* \circ \alpha^*VV @VV\Phi^{(*-1)} \circ \alpha^{(*-1)}V \\ C^*((A, B, \varepsilon); A) @>\Delta^*>> C^{*-1}((A, B, \varepsilon); A) \end{CD} \quad (24)$$

Proof. We will show that for any $n \geq 0$, the following digram is commutative:

$$\begin{CD} \text{Hom}(A^{\otimes(n+1)} \otimes B^{\otimes \frac{n(n+1)}{2}}, k) @>Ns>> \text{Hom}(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, k) \\ @V\Psi'^nVV @VV\Psi'^{n-1}V \\ \text{Hom}(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, (A \otimes B^{\otimes n})^*) @>>> \text{Hom}(A^{\otimes(n-1)} \otimes B^{\otimes \frac{(n-1)(n-2)}{2}}, (A \otimes B^{\otimes(n-1)})^*) \\ @V\alpha^nVV @VV\alpha^{n-1}V \\ \text{Hom}(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, A^*) @>>> \text{Hom}(A^{\otimes(n-1)} \otimes B^{\otimes \frac{(n-1)(n-2)}{2}}, A^*) \\ @V\Phi^nVV @VV\Phi^{n-1}V \\ \text{Hom}(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, A) @>\Delta>> \text{Hom}(A^{\otimes(n-1)} \otimes B^{\otimes \frac{(n-1)(n-2)}{2}}, A) \end{CD} \quad (25)$$

Since Φ^{n-1} is an isomorphism, it suffices to check that this diagram is commutative when composed with $\Phi^{n-1} : \text{Hom}\left(A^{\otimes(n-1)} \otimes B^{\otimes \frac{(n-1)(n-2)}{2}}, A\right) \rightarrow \text{Hom}\left(A^{\otimes(n-1)} \otimes B^{\otimes \frac{(n-1)(n-2)}{2}}, A^*\right)$. Let $f \in \text{Hom}\left(A^{\otimes(n+1)} \otimes B^{\otimes \frac{n(n+1)}{2}}, k\right)$. Then, for $a_i \in A$ and $b_{i,j} \in B$, we have

$$\begin{aligned} & \left\langle (\Delta \circ \Phi^n \circ \alpha^n \circ \Psi^m(f)) \otimes \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n-2} & b_{1,n-1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n-2} & b_{2,n-1} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a_{n-2} & b_{n-2,n-1} \\ 1 & 1 & 1 & \dots & 1 & a_{n-1} \end{pmatrix}, a_n \right\rangle \\ &= \sum_{i=1}^n (-1)^{(n-1)i} \left\langle \Phi^n \alpha^n \Psi^m(f) \otimes \begin{pmatrix} a_i & b_{i,i+1} & b_{i,i+2} & \dots & b_{i,n-1} & 1 & b_{1,i} & b_{2,i} & \dots & b_{i-1,i} \\ 1 & a_{i+1} & b_{i+1,i+2} & \dots & b_{i+1,n-1} & 1 & b_{1,i+1} & b_{2,i+1} & \dots & b_{i-1,i+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & a_{n-1} & 1 & b_{1,n-1} & b_{2,n-1} & \dots & b_{i-1,n-1} \\ 1 & 1 & \dots & 1 & a_n & 1 & \dots & \dots & 1 \\ 1 & 1 & \dots & 1 & 1 & a_1 & b_{1,2} & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & \dots & \dots & \dots & \dots & \dots & a_{i-1} \end{pmatrix}, 1 \right\rangle \\ &= \sum_{i=1}^n (-1)^{(n-1)i} \phi^{-1}(\phi \alpha^n \Psi^m(f)) \otimes \begin{pmatrix} a_i & b_{i,i+1} & b_{i,i+2} & \dots & b_{i,n-1} & 1 & b_{1,i} & b_{2,i} & \dots & b_{i-1,i} \\ 1 & a_{i+1} & b_{i+1,i+2} & \dots & b_{i+1,n-1} & 1 & b_{1,i+1} & b_{2,i+1} & \dots & b_{i-1,i+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & a_{n-1} & 1 & b_{1,n-1} & b_{2,n-1} & \dots & b_{i-1,n-1} \\ 1 & 1 & \dots & 1 & a_n & 1 & \dots & \dots & 1 \\ 1 & 1 & \dots & 1 & 1 & a_1 & b_{1,2} & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & \dots & \dots & \dots & \dots & \dots & a_{i-1} \end{pmatrix} \quad (1) \\ &= \sum_{i=1}^n (-1)^{(n-1)i} (\alpha^n \Psi^m(f)) \otimes \begin{pmatrix} a_i & b_{i,i+1} & b_{i,i+2} & \dots & b_{i,n-1} & 1 & b_{1,i} & b_{2,i} & \dots & b_{i-1,i} \\ 1 & a_{i+1} & b_{i+1,i+2} & \dots & b_{i+1,n-1} & 1 & b_{1,i+1} & b_{2,i+1} & \dots & b_{i-1,i+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & a_{n-1} & 1 & b_{1,n-1} & b_{2,n-1} & \dots & b_{i-1,n-1} \\ 1 & 1 & \dots & 1 & a_n & 1 & \dots & \dots & 1 \\ 1 & 1 & \dots & 1 & 1 & a_1 & b_{1,2} & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & \dots & \dots & \dots & \dots & \dots & a_{i-1} \end{pmatrix} \quad (1) \\ &= \sum_{i=1}^n (-1)^{(n-1)i} (\alpha^n \Psi^m(f)) \otimes \begin{pmatrix} a_i & b_{i,i+1} & b_{i,i+2} & \dots & b_{i,n-1} & 1 & b_{1,i} & b_{2,i} & \dots & b_{i-1,i} \\ 1 & a_{i+1} & b_{i+1,i+2} & \dots & b_{i+1,n-1} & 1 & b_{1,i+1} & b_{2,i+1} & \dots & b_{i-1,i+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & a_{n-1} & 1 & b_{1,n-1} & b_{2,n-1} & \dots & b_{i-1,n-1} \\ 1 & 1 & \dots & 1 & a_n & 1 & \dots & \dots & 1 \\ 1 & 1 & \dots & 1 & 1 & a_1 & b_{1,2} & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & \dots & \dots & \dots & \dots & \dots & a_{i-1} \end{pmatrix} \quad (1) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=1}^n (-1)^{(n-1)i} (\Psi'^n(f)) \left(\otimes \begin{pmatrix} a_i & b_{i,i+1} & b_{i,i+2} & \dots & b_{i,n-1} & 1 & b_{1,i} & b_{2,i} & \dots & b_{i-1,i} \\ 1 & a_{i+1} & b_{i+1,i+2} & \dots & b_{i+1,n-1} & 1 & b_{1,i+1} & b_{2,i+1} & \dots & b_{i-1,i+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & & a_{n-1} & 1 & b_{1,n-1} & b_{2,n-1} & \dots & b_{i-1,n-1} \\ 1 & 1 & \dots & & 1 & a_n & 1 & \dots & & 1 \\ 1 & 1 & \dots & & 1 & 1 & a_1 & b_{1,2} & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & & \dots & \dots & \dots & \dots & & a_{i-1} \end{pmatrix} \right) \\
 & \hspace{20em} (1 \otimes 1 \otimes \dots \otimes 1) \\
 &= \sum_{i=1}^n (-1)^{(n-1)i} f \left(\otimes \begin{pmatrix} a_i & b_{i,i+1} & b_{i,i+2} & \dots & b_{i,n-1} & 1 & b_{1,i} & b_{2,i} & \dots & b_{i-1,i} & 1 \\ 1 & a_{i+1} & b_{i+1,i+2} & \dots & b_{i+1,n-1} & 1 & b_{1,i+1} & b_{2,i+1} & \dots & b_{i-1,i+1} & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \dots & & a_{n-1} & 1 & b_{1,n-1} & b_{2,n-1} & \dots & b_{i-1,n-1} & 1 \\ 1 & 1 & \dots & & 1 & a_n & 1 & \dots & & 1 & 1 \\ 1 & 1 & \dots & & 1 & 1 & a_1 & b_{1,2} & \dots & b_{1,i-1} & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \dots & & \dots & \dots & \dots & \dots & & a_{i-1} & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \vdots & & \vdots & \vdots & \vdots & \dots & & 1 & 1 \end{pmatrix} \right)
 \end{aligned}$$

On the other hand, let $g := (Ns)(f)$. Then, we have

$$\begin{aligned}
 &(\alpha^{n-1} \circ \Psi'^{n-1}(g)) \left(\otimes \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n-2} & b_{1,n-1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n-2} & b_{2,n-1} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a_{n-2} & b_{n-2,n-1} \\ 1 & 1 & 1 & \dots & 1 & a_{n-1} \end{pmatrix} \right) (a_n) \\
 &= (\Psi'^{n-1}(g)) \left(\otimes \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n-2} & b_{1,n-1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n-2} & b_{2,n-1} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a_{n-2} & b_{n-2,n-1} \\ 1 & 1 & 1 & \dots & 1 & a_{n-1} \end{pmatrix} \right) (a_n \otimes 1 \otimes \dots \otimes 1) \\
 &= (Ns)(f) \left(\otimes \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n-2} & b_{1,n-1} & 1 \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n-2} & b_{2,n-1} & 1 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a_{n-2} & b_{n-2,n-1} & 1 \\ 1 & 1 & 1 & \dots & 1 & a_{n-1} & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & a_n \end{pmatrix} \right) \\
 &= \sum_{i=1}^n (-1)^{(n-1)i} f \left(\otimes \begin{pmatrix} a_i & b_{i,i+1} & b_{i,i+2} & \dots & b_{i,n-1} & 1 & b_{1,i} & b_{2,i} & \dots & b_{i-1,i} & 1 \\ 1 & a_{i+1} & b_{i+1,i+2} & \dots & b_{i+1,n-1} & 1 & b_{1,i+1} & b_{2,i+1} & \dots & b_{i-1,i+1} & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \dots & & a_{n-1} & 1 & b_{1,n-1} & b_{2,n-1} & \dots & b_{i-1,n-1} & 1 \\ 1 & 1 & \dots & & 1 & a_n & 1 & \dots & & 1 & 1 \\ 1 & 1 & \dots & & 1 & 1 & a_1 & b_{1,2} & \dots & b_{1,i-1} & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \dots & & \dots & \dots & \dots & \dots & & a_{i-1} & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \vdots & & \vdots & \vdots & \vdots & \dots & & 1 & 1 \end{pmatrix} \right)
 \end{aligned}$$

This proves the result. □

We now let $\overline{\overline{C}}^\bullet(A, B, \varepsilon)$ denote the normalized complex associated to the cosimplicial module $\overline{C}^\bullet(A, B, \varepsilon)$ given in (19). Again, using the isomorphisms in (20), the complex $\overline{\overline{C}}^\bullet(A, B, \varepsilon)$ becomes isomorphic to the normalized complex $\widehat{\overline{C}}^\bullet(A, B, \varepsilon)$ whose terms are given by

$$\widehat{\overline{C}}^n(A, B, \varepsilon) := \text{Ker} \left(\widehat{C}^n(A, B, \varepsilon) \xrightarrow{s_j} \bigoplus_{j=0}^{n-1} \widehat{C}^{n-1}(A, B, \varepsilon) \right)$$

where $s_j : \widehat{C}^n(A, B, \varepsilon) \rightarrow \widehat{C}^{n-1}(A, B, \varepsilon)$ for $0 \leq j \leq n - 1$ is the degeneracy

$$(s_j f) \left(\bigotimes \begin{pmatrix} a_1 & b_{1,2} & \dots & b_{1,n-1} & b_{1,n} \\ 1 & a_2 & \dots & b_{2,n-1} & b_{2,n} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & \dots & a_{n-1} & b_{n-1,n} \\ 1 & 1 & \dots & 1 & a_n \end{pmatrix} \right) = f \left(\bigotimes \begin{pmatrix} a_1 & b_{1,2} & \dots & b_{1,j} & 1 & b_{1,j+1} & \dots & b_{1,n-1} & b_{1,n} \\ 1 & a_2 & \dots & b_{2,j} & 1 & b_{2,j+1} & \dots & b_{2,n-1} & b_{2,n} \\ \vdots & \vdots & \dots & \dots & \dots & \dots & \dots & \vdots & \vdots \\ \vdots & \vdots & \dots & a_j & 1 & \dots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & 1 & \dots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & 1 & a_{j+1} & \dots & b_{j+1,n-1} & b_{j+1,n} \\ \vdots & \vdots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & \dots & 1 & \dots & \dots & a_{n-1} & b_{n-1,n} \\ 1 & 1 & \dots & \dots & 1 & \dots & \dots & 1 & a_n \end{pmatrix} \right)$$

Theorem 10. *Let A be a finite dimensional k -algebra equipped with a symmetric, non-degenerate and invariant bilinear form $\langle \cdot, \cdot \rangle : A \times A \rightarrow k$ and B be a commutative k -algebra with a morphism of k -algebras $\varepsilon : B \rightarrow A$ such that $\varepsilon(B) \subseteq Z(A)$. Then, the following diagram commutes:*

$$\begin{CD} \overline{\overline{C}}^\bullet(A, B, \varepsilon) @>B>> \overline{\overline{C}}^{\bullet-1}(A, B, \varepsilon) \\ @VVV @VVV \\ \overline{C}^\bullet(A, B, \varepsilon) @>Ns>> \overline{C}^{\bullet-1}(A, B, \varepsilon) \\ @V{\Phi^\bullet \circ \alpha^\bullet}VV @V{\Phi^{\bullet-1} \circ \alpha^{\bullet-1}}VV \\ C^\bullet((A, B, \varepsilon); A) @>\Delta^\bullet>> C^{\bullet-1}((A, B, \varepsilon); A) \end{CD} \tag{26}$$

where B is Connes' operator.

Proof. The commutativity of the lower square has already been shown in Proposition 9. By definition, Connes' operator on the complex $\overline{C}^\bullet(A, B, \varepsilon)$ is given by $B = Ns(1 - \lambda)$ which reduces to Ns on the normalized complex $\overline{\overline{C}}^\bullet(A, B, \varepsilon)$. Hence, it may be directly verified that the upper diagram commutes. \square

Acknowledgements

The authors wish to thank the referee for a very detailed report and for making suggestions that helped significantly improve the paper.

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