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# Strategy-proof multi-object mechanism design: Ex-post revenue maximization with non-quasilinear preferences ☆

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**Abstract**

A seller is selling multiple objects to a set of agents, who can buy at most one object. Each agent's preference over (object, payment) pairs need not be quasilinear. The seller considers the following desiderata for her mechanism, which she terms *desirable*: (1) *strategy-proofness*, (2) *ex-post individual rationality*, (3) *equal treatment of equals*, (4) *no wastage* (every object is allocated to some agent). The minimum Walrasian equilibrium price (MWEP) mechanism is desirable. We show that at each preference profile, the MWEP mechanism generates more revenue for the seller than any desirable mechanism satisfying no subsidy. Our result works for the quasilinear domain, where the MWEP mechanism is the VCG mechanism, and for various non-quasilinear domains, some of which incorporate positive income effect of agents. We can relax no subsidy to *no bankruptcy* in our result for certain domains with positive income effect.


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## 1. Introduction

One of the most challenging problems in microeconomic theory is the design of a revenue maximizing mechanism in the multi-object allocation problems. A precise description of a revenue maximizing mechanism in such problems remains elusive even today. In this paper, we provide a partial solution to this problem by imposing some additional constraints besides the conventional incentive compatibility and individual rationality conditions. Our additional constraints are consistent with the objectives of many governments allocating public assets, such as fairness and efficiency, besides revenue maximization.[1] Since our main focus is on revenue maximization, we impose only moderate desiderata for other goals on mechanisms.

We study the problem of allocating $m$ indivisible heterogeneous objects to $n > m$ agents, each of whom can be assigned at most one object (unit demand agents) – such unit demand settings are common in allocating houses in public housing schemes (Andersson and Svensson, 2014), selling team franchises in professional sports leagues, and even in selling a small number of spectrum licenses (Binmore and Klemperer, 2002).[2,3] Agents in our model can have non-quasilinear preferences over consumption bundles – (object, payment) pairs.

We briefly describe the additional axioms that we impose for our revenue maximization exercise. *Equal treatment of equals* is a desideratum for fairness, and requires that two agents having identical preferences be assigned consumption bundles to which they are indifferent. *No wastage* is a desideratum for a mild form of efficiency, and requires that every object be allocated to some agent. We term a mechanism *desirable* if it satisfies strategy-proofness, ex-post individual rationality, equal treatment of equals, and no wastage.

The class of desirable mechanisms is large, but one mechanism, which is based on a market clearing idea, stands out. To explain this mechanism, we need to understand the notion of a *Walrasian equilibrium price (WEP) vector*. It is a price vector such that there is an allocation of objects to agents where each agent gets an object from his demand set. Demange and Gale (1985) showed that the set of WEP vectors is always a non-empty compact lattice in our model. This means that there is a unique minimum WEP vector.[4] The *minimum Walrasian equilibrium price (MWEP)* mechanism selects the minimum WEP vector at every profile of preferences and uses a corresponding equilibrium allocation. The MWEP mechanism is desirable (Demange and

1 For example, Klemperer (2002) discusses the list of goals pursued in UK 3G auction conducted in 2000.

2 When a professional cricket league, called the *Indian Premier League (IPL)* was started in India in 2007, professional teams were sold to interested owners (bidders) by an auction. Since it does not make sense for an owner to have two teams, the unit demand assumption is satisfied in this problem. See the Wiki entry of IPL for details: https://en.wikipedia.org/wiki/Indian_Premier_League and a news article here: http://content-usa.cricinfo.com/ipl/content/current/story/333193.html.

3 Although modern spectrum auctions involve sale of *bundles* of spectrum licenses, Binmore and Klemperer (2002) report that one of the biggest spectrum auctions in the UK involved selling a fixed number of licenses to bidders, each of whom can be assigned at most one license. The unit demand setting is also one of the few *computationally* tractable models of combinatorial auction studied in the literature (Blumrosen and Nisan, 2007).

4 Results of this kind were earlier known for quasilinear preferences (Shapley and Shubik, 1971; Leonard, 1983).

Gale, 1985) and satisfies *no subsidy*. No subsidy requires that payment of each agent be non-negative. In the quasilinear domain of preferences, the MWEP mechanism coincides with the VCG mechanism (Leonard, 1983). However, we emphasize that outside the quasilinear domain, a naive generalization of the VCG mechanism to non-quasilinear preferences is not strategy-proof (Morimoto and Serizawa, 2015).[5] This also means that for an arbitrary domain of non-quasilinear preferences, the MWEP mechanism is very different from a generalization of the VCG mechanism.

We show that on a variety of domains (the set of admissible preferences), the *MWEP* mechanism is the unique *ex-post revenue optimal* mechanism among all desirable mechanisms satisfying *no subsidy*, i.e., for each preference profile, the MWEP mechanism generates more revenue for the seller than any desirable mechanism satisfying no subsidy (Theorem 1). Further, we show that if the domain includes all positive income effect preferences, then the MWEP mechanism is ex-post revenue optimal in the class of all desirable and *no bankruptcy* mechanisms (Theorem 2). No bankruptcy is a weaker condition than no subsidy and requires the sum of payments of all agents across all profiles be bounded below.

Our results are more general than the results in the literature in two ways. First, the MWEP mechanism maximizes ex-post revenue. Hence, we can recommend the MWEP mechanism without resorting to any prior-based maximization. Notice that ex-post revenue optimality is much stronger than expected (ex-ante) revenue optimality, and mechanisms satisfying ex-post revenue optimality rarely exist. Second, our results hold on a variety of domains such as the quasilinear domain, the classical domain, the domain of positive income effect preferences, and any superset of these domains.

Ours is the first paper to study revenue maximization in a multi-object allocation problem when preferences of agents are not quasilinear. While quasilinearity is standard and popular in the literature, its practical relevance is debatable in many settings. For instance, bidders need to invest in various supporting products and processes to realize the full value of the object. For instance, cellular companies invest in communication infrastructure development, a sports team owner invests in marketing, and so on. Such ex-post investments cannot be assumed to be independent of the payments in auctions. Further, bidders in large auctions borrow to pay for objects. High interest rates imposed on the larger amount of borrowings make preferences non-quasilinear.[6]

Our contribution is not methodological. It is well known that the main difficulty in extending the Myersonian approach (Myerson, 1981) to multidimensional type spaces is that the binding incentive constraints are difficult to characterize in such problems (Armstrong, 2000). The literature is developing new toolkits to solve these problems (Carroll, 2017; Daskalakis et al., 2017). While we certainly do not introduce any new method to solve this problem, our results show that one can circumvent some of these difficulties by imposing additional axioms.

We briefly discuss the practical relevance of two of our axioms: equal treatment of equals and no wastage. Later, we elaborate the kind of mechanisms we rule out by imposing these axioms. Equal treatment of equals is arguably the weakest fairness axiom in the literature – as Aristotle (1995) writes, justice is considered to mean “equality for those who are equal, and not for all”.[7]

5 See Section 6.2 in Morimoto and Serizawa (2015).

6 We will discuss the effect of borrowing cost on preferences in Subsection 4.1.

7 The quote is from Aristotle’s Book III titled “The Theory of Citizenship and Constitutions”. It can be found in Part C of the book, titled “The Principle of Oligarchy and Democracy and the Nature of Distributive Justice”, in Chapter 9 and paragraph 1280a7.

Sometimes, there are practical implications of violating fairness – for instance, Deb and Pai (2016) cite many legal implications of violating *symmetry* in mechanisms, which is a stronger property than equal treatment of equals.

Efficiency is an important goal for governments. Although Pareto efficiency is a standard efficiency desideratum in the literature, since we focus on revenue maximization, we impose no wastage, a much weaker desideratum. Unlike Pareto efficiency, no wastage is an easily detectable axiom (detecting violation of Pareto efficiency requires the knowledge of preferences). Violation of no wastage in government auctions creates a lot of controversies in the public, and often, the unsold objects are resold.[8] In such environments, governments cannot commit to reserve prices even though expected revenue maximization may require them. Indeed, McAfee and McMillan (1987); Ashenfelter and Graddy (2003); Jehiel and Lamy (2015); Hu et al. (2019) report that many real-life auctions have zero reserve price. While our results do not provide a theory for why the seller should not keep a reserve price, we show that if the seller uses a mechanism satisfying no wastage and other desirable properties, then the MWEP mechanism is ex-post revenue optimal.

Finally, the MWEP mechanism is Pareto efficient and can be implemented as a simultaneous ascending auction (SAA) (Demange et al., 1986; Morimoto and Serizawa, 2015) – see also Zhou and Serizawa (2019), who provide an alternate algorithm that computes the MWEP by finite number of steps in general non-quasilinear environments. SAAs have distinct advantages of practical implementation and are often used in practice to allocate multiple objects. The efficiency foundations for SAAs have been well-established (Ausubel and Milgrom, 2002). Our results provide a *revenue maximization* foundation for SAAs.

## 2. Preliminaries

A seller has $m$ objects to sell, denoted by $M := \{1, \dots, m\}$. There are $n > m$ agents (buyers), denoted by $N := \{1, \dots, n\}$. Each agent can receive at most one object (unit demand preference). Let $L := M \cup \{0\}$, where 0 is the null object, which is assigned to any agent who does not receive any object in $M$ – thus, the null object can be assigned to more than one agent. Note that the unit demand restriction can either be a restriction on preferences or an institutional constraint. For instance, objects may be substitutable when houses are being allocated in a public housing scheme (Andersson and Svensson, 2014). The unit demand restriction can also be institutional as was the case in the spectrum license auction in UK in 2000 (Binmore and Klemperer, 2002) or in the Indian Premier League auction. As long as the mechanism designer restricts messages in the mechanisms to *only* use information on preferences over individual objects, our results apply.

The consumption set of every agent is the set $L \times \mathbb{R}$, where a typical (consumption) bundle $z \equiv (a, t)$ corresponds to object $a \in L$ and payment $t \in \mathbb{R}$. Notice that $t$ denotes the amount *paid* by an agent to the designer. Now, we formally introduce preferences of agents and the notion of a desirable mechanism.

[8] As an example, the Indian spectrum auctions reported a large number of unsold spectrum blocks in 2016, and all of them are supposed to be re-auctioned. See the following news article: http://www.livemint.com/Industry/xt5r4Zs5RmzjdwuLUdwJMI/Spectrum-auction-ends-after-lukewarm-response-from-telcos.html.

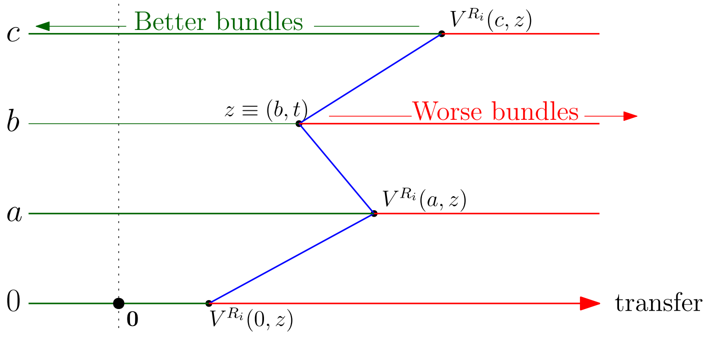


Fig. 1. Valuation at a preference.

### 2.1. *The preferences*

A preference ordering $R_i$ (of agent $i$) over $L \times \mathbb{R}$, with strict part $P_i$ and indifference part $I_i$, is **classical** if it satisfies the following assumptions:

1. **Money monotonicity.** for every $t, t' \in \mathbb{R}$ with $t > t'$ and for every $a \in L$, we have $(a, t')\ P_i\ (a, t)$.
2. **Desirability of objects.** for every $t \in \mathbb{R}$ and for every $a \in M$, $(a, t)\ P_i\ (0, t)$.
3. **Continuity.** for every $z \in L \times \mathbb{R}$, the sets $\{z' \in L \times \mathbb{R} : z'\ R_i\ z\}$ and $\{z' \in L \times \mathbb{R} : z\ R_i\ z'\}$ are closed.
4. **Possibility of compensation.** for every $z \in L \times \mathbb{R}$ and for every $a \in L$, there exists a pair $t, t' \in \mathbb{R}$ such that $z\ R_i\ (a, t)$ and $(a, t')\ R_i\ z$.

A classical preference $R_i$ is *quasilinear* if there exists $v \in \mathbb{R}^{|L|}$ such that for every $a, b \in L$ and $t, t' \in \mathbb{R}$, $(a, t)\ R_i\ (b, t')$ if and only if $v^a - t \geq v^b - t'$. We refer to $v$ as the valuation of the agent, and we normalize $v^0$ to 0. The idea of valuation may be generalized as follows for non-quasilinear preferences.

**Definition 1.** The **valuation** at a classical preference $R_i$ for object $a \in L$ with respect to bundle $z \in L \times \mathbb{R}$ is defined as $V^{R_i}(a, z)$, which uniquely solves $(a, V^{R_i}(a, z))\ I_i\ z$.

Hence, $V^{R_i}(a, z)$ is the amount $t$ agent $i$ is willing to pay so that he is indifferent between $(a, t)$ and $z$. A straightforward consequence of our assumptions is that for every $a \in L$, for every $z \in L \times \mathbb{R}$, and for every classical preference $R_i$, the valuation $V^{R_i}(a, z)$ exists. For any $R$ and for any $z \in L \times \mathbb{R}$, the valuations at $R$ with respect to $z$ is a vector in $\mathbb{R}^{|L|}$.

An illustration of the valuation is shown in Fig. 1. In the figure, the horizontal lines correspond to objects: $L = \{0, a, b, c\}$. The horizontal lines indicate payment levels. Hence, the consumption set consists of the four lines. For example, $z$ denotes the bundle consisting of object $b$ and the payment equal to the distance of $z$ from the vertical dotted line. A preference $R_i$ can be described by drawing (non-intersecting) indifference vectors through these consumption bundles (lines). One such indifference vector passing through $z$ is shown in Fig. 1. This indifference vector actually consists of four points: $(0, V^{R_i}(0, z))$, $(a, V^{R_i}(a, z))$, $z \equiv (b, t)$, and $(c, V^{R_i}(c, z))$ as shown. Parts of the indifference line in Fig. 1 which lie between the consumption bundle lines is useless and has no meaning, and it is only displayed for convenience. As we go to the right along the horizontal lines starting from any bundle, we get worse bundles (due to money monotonicity). Similarly, bundles to the left of a particular bundle are better than that bundle. This is shown in Fig. 1 with respect to the indifference vector.

Our modeling of preferences captures income effects even though we do not model income explicitly. We explain this point when we introduce positive income effect in Section 4.1.

### 2.2. Desirable mechanisms

Let $\mathcal{R}^C$ denote the set of all classical preferences and $\mathcal{R}^Q$ denote the set of all quasilinear preferences. We will consider an arbitrary subset of classical preferences $\mathcal{R} \subseteq \mathcal{R}^C$ - we will put specific restrictions on $\mathcal{R}$ later. A preference of agent $i$ is denoted by $R_i \in \mathcal{R}$. A preference profile is a list of preferences $R \equiv (R_1, \ldots, R_n)$. Given $i \in N$ and $N' \subseteq N$, let $R_{-i} \equiv (R_j)_{j \neq i}$ and $R_{-N'} \equiv (R_j)_{j \in N'}$, respectively.

An *object allocation* is an $n$-tuple $(a_1, \ldots, a_n) \in L^n$ such that no real (non-null) object is assigned to two agents, i.e., $a_i \neq a_j$ for all $i, j \in N$ with $a_i, a_j \neq 0$. The set of all object allocations is denoted by $A$. A (feasible) allocation is an $n$-tuple $((a_1, t_1), \ldots, (a_n, t_n)) \in (L \times \mathbb{R})^n$ such that $(a_1, \ldots, a_n) \in A$, where $(a_i, t_i)$ is the bundle of agent $i$. Let $Z$ denote the set of all feasible allocations. For every allocation $(z_1, \ldots, z_n) \in Z$, we will denote by $z_i$ the bundle of agent $i$.

An **mechanism** is a map $f : \mathcal{R}^n \to Z$. By definition, we restrict ourselves to **deterministic** mechanisms. Allowing for randomization will entail considering preferences over lotteries of allocations. This brings substantial difficulty in modeling and analysis. We do not know how our results will extend if we allow for randomization.

At a preference profile $R \in \mathcal{R}^n$, we denote the bundle of agent $i$ in mechanism $f$ as $f_i(R) \equiv (a_i(R), t_i(R))$, where $a_i(R)$ and $t_i(R)$ are respectively the object allocated to agent $i$ and $i$'s payment at preference profile $R$. We call $a(\cdot) \equiv (a_1(\cdot), \ldots, a_n(\cdot))$ and $t(\cdot) \equiv (t_1(\cdot), \ldots, t_n(\cdot))$ the object allocation mechanism and the payment mechanism, respectively of $f$.

**Definition 2.** A mechanism $f : \mathcal{R}^n \to Z$ is **desirable** if it satisfies the following properties:

1. **Strategy-proofness.** for every $i \in N$, for every $R_{-i} \in \mathcal{R}^{n-1}$, and for every $R_i, R'_i \in \mathcal{R}$, we have $f_i(R_i, R_{-i})\ R_i\ f_i(R'_i, R_{-i})$.
2. **(Ex-post) individual rationality (IR).** for every $i \in N$, for every $R \in \mathcal{R}^n$, we have $f_i(R)\ R_i\ (0, 0)$.
3. **Equal treatment of equals (ETE).** for every $i, j \in N$, for every $R \in \mathcal{R}^n$ with $R_i = R_j$, we have $f_i(R)\ I_i\ f_j(R)$.
4. **No wastage (NW).** for every $R \in \mathcal{R}^n$ and for every $a \in M$, there exists some $i \in N$ such that $a_i(R) = a$.

Besides desirability, for some of our results, we will require some form of restrictions on payments.

**Definition 3.** A mechanism $f : \mathcal{R}^n \to Z$ satisfies **no subsidy** if for every $R \in \mathcal{R}^n$ and for every $i \in N$, we have $t_i(R) \geq 0$.

No subsidy can be considered desirable to exclude "fake" agents, who participate in mechanisms just to take away available subsidy. It is an axiom satisfied by most standard mechanisms in practice. It is practical in settings where the seller may not have any means to finance any agent.

## 3. The minimum Walrasian equilibrium price mechanism

In this section, we define the notion of a Walrasian equilibrium, and use it to define a desirable mechanism. A price vector $p \in \mathbb{R}_+^{|L|}$ defines a price for every object with $p_0 = 0$. At any price vector $p \in \mathbb{R}_+^{|L|}$, let $D(R_i, p) := \{a \in L : (a, p_a)\ R_i\ (b, p_b)\ \forall\ b \in L\}$ denote the demand set of agent $i$ with preference $R_i$ at price vector $p$.

**Definition 4.** An object allocation $(a_1, \ldots, a_n) \in A$ and a price vector $p \in \mathbb{R}_+^{|L|}$ is a **Walrasian equilibrium** at a preference profile $R \in \mathcal{R}^n$ if

1. $a_i \in D(R_i, p)$ for all $i \in N$ and
2. $p_a = 0$ for all $a \in M \setminus \{a_1, \ldots, a_n\}$.

We refer to $p$ and $((a_1, p_{a_1}), \ldots, (a_n, p_{a_n}))$ defined above as a **Walrasian equilibrium price vector** and a **Walrasian equilibrium allocation** at $R$ respectively.

Since we assume $n > m$ and preferences satisfy desirability of objects, the conditions of Walrasian equilibrium imply that for all $a \in M$, we have $a_i = a$ for some $i \in N$.[9]

A Walrasian equilibrium price vector $p$ is a **minimum Walrasian equilibrium price vector** at preference profile $R$ if for every Walrasian equilibrium price vector $p'$ at $R$, we have $p_a \leq p'_a$ for all $a \in L$. At every $R \in (\mathcal{R}^C)^n$, a Walrasian equilibrium exists (Alkan and Gale, 1990), the set of Walrasian equilibrium price vectors forms a lattice with a unique minimum and a unique maximum Walrasian equilibrium price vector (Demange and Gale, 1985). We denote the minimum Walrasian equilibrium price vector at $R \in (\mathcal{R}^C)^n$ as $p^{min}(R)$. Notice that by desirability of objects, if $n > m$, then for every $a \in M$, we have $p_a^{min}(R) > 0$.[10]

We give an example to illustrate the notion of minimum Walrasian equilibrium price vector. Suppose $N = \{1, 2, 3\}$ and $M = \{a, b\}$. Fig. 2 shows some indifference vectors of a preference profile $R \equiv (R_1, R_2, R_3)$ and the corresponding minimum Walrasian equilibrium price vector $p^{min}(R) \equiv p^{min} \equiv (p_0^{min} = 0, p_a^{min}, p_b^{min})$.

First, note that

$$D(R_1, p^{min}) = \{a\}, D(R_2, p^{min}) = \{a, b\}, D(R_3, p^{min}) = \{0, b\}.$$

Hence, a Walrasian equilibrium is the allocation where agent 1 gets object $a$, agent 2 gets object $b$, and agent 3 gets the null object at the price vector $p^{min}$. Also, $p^{min}$ is the minimum Walrasian equilibrium price vector. To see this, let $p$ be any other Walrasian equilibrium price vector. If $p_a < p_a^{min}$ and $p_b < p_b^{min}$, then no agent demands the null object, contradicting Walrasian equilibrium. Thus, $p_a \geq p_a^{min}$ or $p_b \geq p_b^{min}$. If $p_b < p_b^{min}$, then by $p_a \geq p_a^{min}$, both agents

9 To see this, suppose that there is $a \in M$ such that $a_i \neq a$ for each $i \in N$. Then, by the second condition of Walrasian equilibrium, $p_a = 0$. By $n > m$, $a_i = 0$ for some $i \in N$. By desirability of objects, $(a, 0)\ P_i\ (a_i, 0)$, contradicting the first condition of Walrasian equilibrium.

10 To see this, suppose $p_a^{min}(R) = 0$ for some $a \in M$. Then any agent $i \in N$ who is not assigned in the Walrasian equilibrium will prefer $(a, 0)$ to $(0, 0)$ contradicting the fact that he is assigned a bundle from his demand set. Indeed, this argument holds for any Walrasian equilibrium price vector.

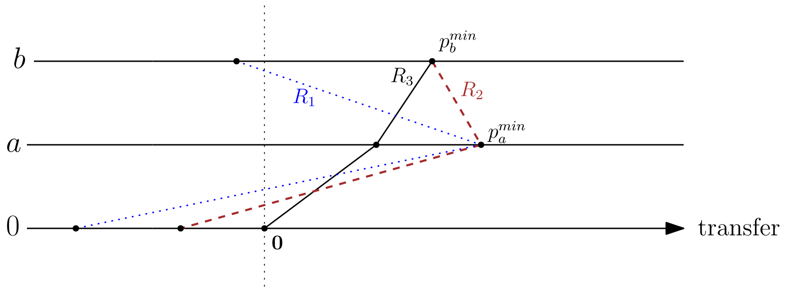


Fig. 2. The minimum Walrasian equilibrium price vector.

2 and 3 will demand only object $b$, contradicting Walrasian equilibrium. Thus, $p_b \geq p_b^{min}$. But, if $p_a < p_a^{min}$, both agents 1 and 2 will demand only object $a$, a contradiction to Walrasian equilibrium. Hence, $p \geq p^{min}$.

We now describe a desirable mechanism satisfying no subsidy. The mechanism picks a minimum Walrasian equilibrium allocation at every profile of preferences. Although the minimum Walrasian equilibrium price vector is unique at every preference profile, there may be multiple supporting object allocations – all these object allocations must be indifferent to all the agents. To handle this multiplicity problem, we introduce some notation. Let $Z^{min}(R)$ denote the set of all allocations at a minimum Walrasian equilibrium at preference profile $R$. Note that since $n > m$, if $((a_1, p_{a_1}), \ldots, (a_n, p_{a_n})) \in Z^{min}(R)$, then $p \equiv (p_a)_{a \in L} = p^{min}(R)$.

**Definition 5.** A mechanism $f^{min} : \mathcal{R}^n \to Z$ is a **minimum Walrasian equilibrium price (MWEP)** mechanism if $f^{min}(R) \in Z^{min}(R)\ \forall\ R \in \mathcal{R}^n$.

As discussed earlier, at any preference profile $R$, $Z^{min}(R)$ may contain multiple allocations but each agent is indifferent between its allocations in this set. Hence, we refer to $f^{min}$ as *the* MWEP mechanism, even though there can be more than one MWEP mechanism (depending on which allocation in $Z^{min}(R)$ is picked at every $R$).

Demange and Gale (1985) showed that the MWEP mechanism is strategy-proof. Clearly, it also satisfies IR, ETE, NW, and no subsidy. We document this fact below.

**Fact 1** *(Demange and Gale (1985)). The MWEP mechanism is desirable and satisfies no subsidy.*

Demange and Gale (1985) show that the MWEP mechanism satisfies a stronger incentive property called *(weak) group-strategy-proofness*, which means that coalitional incentive constraints hold. Further, the MWEP mechanism satisfies stronger fairness properties – it is *anonymous* (permuting preferences of agents does not change the outcome) and *envy-free*.

It is worth comparing the MWEP mechanism with the VCG mechanism for quasilinear preferences. Indeed, there is a naive way to generalize the VCG mechanism to any classical preference domain. Consider a preference profile $R$. For every agent $i \in N$ with preference $R_i$, let $v_i^a := V^{R_i}(a, (0, 0))$ for all $a \in M$. Let $v_i^0 = 0$ for all $i \in N$. Now, we compute the allocation and payments according to the VCG mechanism with respect to this profile of vectors $(v_1, \ldots, v_n)$. Such a generalized VCG mechanism coincides with the MWEP mechanism if the domain is the quasilinear domain (Leonard, 1983). Else, the generalized VCG mechanism is very different from the MWEP mechanism. Further, it is not strategy-proof if the domain is not the quasilinear domain (Morimoto and Serizawa, 2015).

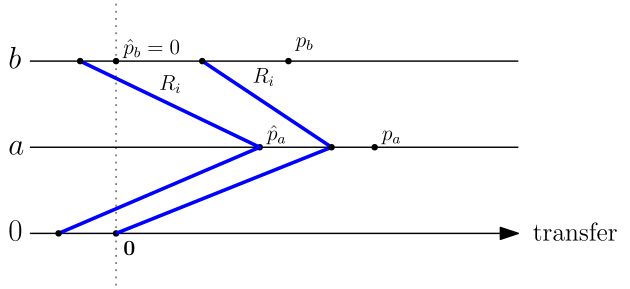


Fig. 3. Illustration of richness.

## 4. The results

In this section, we formally state our two results. The proofs of both the results are in Appendix A. Before we state the results, we explain the domain richness they use.

### 4.1. Rich domains

For each pair of price vectors $p, \hat{p} \in \mathbb{R}_+^{|L|}$, we write $p > \hat{p}$ if $p_a > \hat{p}_a$ for all $a \in M$. The domain of preferences that we consider for our first result requires the following richness.

**Definition 6.** A domain of preferences $\mathcal{R}$ is **rich** if for all $a \in M$ and for every $\hat{p}$ with $\hat{p}_a > 0$, $\hat{p}_b = 0$ for all $b \neq a$ and for every $p > \hat{p}$, there exists $R_i \in \mathcal{R}$ such that

$$D(R_i, \hat{p}) = \{a\} \text{ and } D(R_i, p) = \{0\}.$$

Fig. 3 illustrates this notion of richness with two objects $a$ and $b$. Two possible price vectors $p$ and $\hat{p}$ are shown and two indifference vectors of a preference $R_i$ are shown such that $D(R_i, p) = \{0\}$ and $D(R_i, \hat{p}) = \{a\}$.

The requirement of the richness condition is weak enough to be satisfied by many domains of interest. Obviously, if a domain of preferences is rich, then any superset of that domain is also rich. We give below some interesting examples of rich domains. Any superset of these domains is also rich.

*Quasilinear domain* Any domain of preferences containing $\mathcal{R}^Q$ satisfies richness. To see this, fix an object $a \in M$ and a price vector $\hat{p}$ with $\hat{p}_b = 0$ for all $b \neq a$ and $\hat{p}_a > 0$. Consider any other price vector $p > \hat{p}$. Now, consider the quasilinear preference $R_i$ given by the valuation vector $v$ such that

$$v^b = \begin{cases} \hat{p}_b + 2\epsilon & \text{if } b = a, \\ \epsilon & \text{if } b \neq a, \end{cases}$$

where $\epsilon > 0$ is small enough such that $v^a = \hat{p}_a + 2\epsilon < p_a$ and $\epsilon < p_b$ for all $b \in M \setminus \{b\}$. This means that $D(R_i, \hat{p}) = \{a\}$ but $D(R_i, p) = \{0\}$.

*Positive income effect domain* The set of all positive income effects preferences and the set of all non-negative income effect preferences satisfy richness.

**Definition 7.** A preference $R_i$ satisfies **positive income effect** if for every $a, b \in L$ and for every $t, t'$ with $t < t'$ and $(b, t')\ I_i\ (a, t)$, we have

$$(b, t' - \delta)\ P_i\ (a, t - \delta) \quad \forall\, \delta > 0.$$

A preference $R_i$ satisfies **non-negative income effect** if for every $a, b \in L$ and for every $t, t'$ with $t < t'$ and $(b, t')\ I_i\ (a, t)$, we have

$$(b, t' - \delta)\ R_i\ (a, t - \delta) \quad \forall\, \delta > 0.$$

Let $\mathcal{R}^{++}$ be the set of all positive income effect preferences and $\mathcal{R}^{+}$ be the set of all non-negative income effect preferences.

A standard definition of positive income effect will say that a preferred object is more preferred as income increases. We do not model income explicitly, but the zero payment corresponds to the endowed income. Thus, in our model, when income increases by $\delta > 0$, the origin of consumption space moves to right by $\delta$. This movement is equivalent to sliding indifference vectors to left. In other words, if the origin is fixed, the increase of income by $\delta$ is expressed as the decrease of payments of all bundles by $\delta$. In the above definition, $(b, t')\ I_i\ (a, t)$ and $t' > t$ imply that object $b$ is strictly preferred to object $a$ at any common payment levels $t'' \in [t, t']$. Then, positive income effect requires that when payments are decreased by $\delta$, $b$ will be preferred to $a$, i.e., $(b, t' - \delta)\ P_i\ (a, t - \delta)$. Hence, our modeling of preferences captures income effects even though we do not model income explicitly.

Both $\mathcal{R}^{+}$ and $\mathcal{R}^{++}$ are rich domains. The fact that $\mathcal{R}^{+}$ is rich follows from the observation that $\mathcal{R}^{Q} \subseteq \mathcal{R}^{+}$ and $\mathcal{R}^{Q}$ is rich. Even though $\mathcal{R}^{++} \cap \mathcal{R}^{Q} = \emptyset$, $\mathcal{R}^{++}$ is still a rich domain.

*Quasilinearity with borrowing cost* The set of all quasi-linear preferences with non-linear borrowing cost satisfies richness. Imagine a situation in which an agent has a quasilinear preference with valuation $v$, but has to borrow money from banks at interest rate $r > 0$ if his payment for an object exceeds his income $I > 0$. Then, given $t \in \mathbb{R}$, his *cost* of payment, which we denote by $c(t, I, r)$, is as follows.

$$c(t, I, r) = \begin{cases} t & \text{if } t \leq I, \\ I + (t - I)(1 + r) & \text{if } t > I. \end{cases}$$

Thus, for each pair $(a, t), (b, t') \in L \times \mathbb{R}$, the agent weakly prefers $(a, t)$ to $(b, t')$ if and only if $v(a) - c(t, I, r) \geq v(b) - c(t', I, r)$. Such preferences are obviously not quasilinear. Let $\mathcal{R}^{B}$ be the set of all such preferences. Then, $\mathcal{R}^{B}$ is rich.

*Single-peaked domain* The set of all single-peaked preferences satisfies richness. Imagine a condominium in which each floor has one room. Some agents prefer the highest floor because of good views, some prefer the lowest to avoid walking up stairs, and some prefer middle floors. Then, it is natural that each agent has a single-peaked preference – an ideal floor, and as we go away from the ideal floor, we go down our preference.

Formally, there is a strict order $\succ$ over $L$ such that for each $a \in M$, $a \succ 0$. A preference $R_i$ is **single-peaked** if there is a unique object $\tau(R_i)$ such that for all $t \in \mathbb{R}$

- $(\tau(R_i), t)\ P_i\ (a, t)$ for all $a \in M \setminus \{\tau(R_i)\}$ and
- if $\tau(R_i) \succ a \succ b$ or $b \succ a \succ \tau(R_i)$, then $(a, t)\ P_i\ (b, t)$.

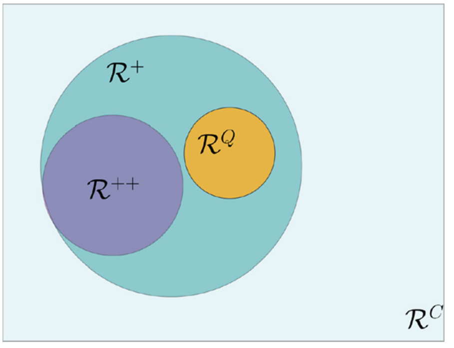


Fig. 4. Illustration of relationship between rich domains: $\mathcal{R}^C, \mathcal{R}^Q, \mathcal{R}^+, \mathcal{R}^{++}$.

In other words, an agent with preference $R_i$ has a "peak" floor, say $\tau(R_i)$, such that when the prices of all floors are the same, he prefers $\tau(R_i)$ to other floors, and for any two floors $a$ and $b$, if $b \succ a \succ \tau(R_i)$ or $\tau(R_i) \succ a \succ b$, he prefers $a$ to $b$. Let $\mathcal{R}^S$ be the set of all single-peaked preferences. Then, $\mathcal{R}^S$ is rich.

We can summarize the above discussions in this claim.

**Claim 1.** *The following domains are rich:* $\mathcal{R}^Q, \mathcal{R}^+, \mathcal{R}^{++}, \mathcal{R}^B, \mathcal{R}^S, \mathcal{R}^C$.

We omit a formal proof for the above claim. However, the intuition for its proof is similar to the quasilinear domain proof outlined above. Given $a \in M$ and two price vectors $p, \hat{p} \in \mathbb{R}_+^{|L|}$ with $\hat{p} < p$, in those domains, we can find a preference $R_i$ that has two indifference vectors satisfying the following: $V^{R_i}(a, (0, 0)) < p_a$ and for each $b \in M \setminus \{a\}$, $V^{R_i}(b, (0, 0))$ is close to zero and $V^{R_i}(b, (a, \hat{p}_a)) < 0$. Then, for preferences satisfying these conditions, the demand sets at $p$ and $\hat{p}$ contain only 0 and $a$, respectively. The relationship between some of the rich domains is shown in Fig. 4.

Richness is a condition which ensures a variety of preferences in the domain. For instance, if we just take a domain containing two (or any finite) quasilinear preferences, it will not satisfy richness. A concrete domain which violates richness is studied in Zhou and Serizawa (2018). They consider a domain where objects are *commonly ranked*. For instance, suppose there are two objects, $M = \{a, b\}$, and there is a common ranking of objects given by the ordering $\succ$: $b \succ a$. Further, assume that agents have quasilinear preferences over consumption bundles. A quasilinear preference, represented by a valuation vector $v_i \in \mathbb{R}^2_{++}$, must satisfy $v_i^b > v_i^a > v_i^0 = 0$. This means that the set of quasilinear preferences satisfying common object ranking is a smaller subset of $\mathcal{R}^Q$. Such a domain cannot satisfy richness. To see this, consider a price vector where $\hat{p}_a > 0$ and $\hat{p}_b = 0$ (as in Definition 6). By common object ranking, for any valuation vector $v_i$ of any agent $i$, we must have $v_i^b > v_i^a$ and this means that $v_i^b - \hat{p}_b > v_i^a - \hat{p}_a$. This implies that agent $i$ with this preference cannot demand object $a$ at price vector $\hat{p}$.[11]

[11] Google conducts auctions of advertisement slots on search pages – these auctions are known as the *sponsored search auction* (Edelman et al., 2007). Usually, each advertiser is assigned at most one advertisement slot in a sponsored search auction. It is plausible that advertisement slots higher up on the page have more value than those lower down. Thus, this is an example of a domain which satisfies the common ranking assumption of Zhou and Serizawa (2018), and hence, it violates richness. Our results cannot be applied to this model.

Another domain which violates richness is the *identical objects domain*. As the name suggests, in this domain, all the objects are identical. This means that if the prices are the same for all the objects, then the agent is indifferent between all the objects. Just as we argued about the common object ranking domain, it is not difficult to see that the identical objects domain violates richness – with identical objects $v_i^a = v_i^b$ and the arguments do not change in the previous paragraph. Adachi (2014) studies the domain of quasilinear preferences when objects are identical and provides an example of a desirable mechanism satisfying no subsidy, which is not the Vickrey auction (the MWEP mechanism in this case).

### 4.2. Ex-post revenue maximization of desirable mechanisms

We now formally state our first main result. For any mechanism $f : \mathcal{R}^n \to Z$, we define the **revenue** at preference profile $R \in \mathcal{R}^n$ as

$$\textsc{Rev}^f(R) := \sum_{i \in N} t_i(R).$$

**Definition 8.** A mechanism $f : \mathcal{R}^n \to Z$ **revenue dominates** another mechanism $g : \mathcal{R}^n \to Z$ if

$$\textsc{Rev}^f(R) \geq \textsc{Rev}^g(R) \quad \forall\, R \in \mathcal{R}^n.$$

A mechanism is **ex-post revenue optimal** among a class of mechanisms if it belongs to this class of mechanisms and revenue dominates every mechanism in this class.

Since revenue domination need not be a complete binary relation in a class of mechanisms, an ex-post revenue optimal mechanism may not exist. Our main result shows that there is a *unique* ex-post revenue optimal mechanism among the class of desirable mechanisms satisfying no subsidy, and it is the MWEP mechanism.

**Theorem 1.** *Suppose $\mathcal{R}$ is a rich domain of preferences. The MWEP mechanism is the unique ex-post revenue optimal mechanism among the class of desirable mechanisms satisfying no subsidy defined on $\mathcal{R}^n$.*

Theorem 1 clearly implies that the MWEP mechanism is *ex-ante* revenue optimal among the class of desirable mechanisms satisfying no subsidy. This is independent of the prior on the preferences of the agents.

We use Claim 1 to spell out our result in specific domains.

**Corollary 1.** *Suppose $\mathcal{R} \in \{\mathcal{R}^Q, \mathcal{R}^+, \mathcal{R}^{++}, \mathcal{R}^B, \mathcal{R}^S, \mathcal{R}^C\}$. The MWEP mechanism is the unique ex-post revenue optimal mechanism among the class of desirable mechanisms satisfying no subsidy defined on $\mathcal{R}^n$.*

In the quasilinear domain, the outcome of the MWEP mechanism coincides with the VCG mechanism. Hence, the VCG mechanism is ex-post revenue optimal in the quasilinear domain among the class of desirable mechanisms satisfying no subsidy. Note that Holmstrom's celebrated theorem (Holmstrom, 1979) does not imply this result since it uses Pareto efficiency but we do not. Similarly, Krishna and Perry (1998) show that among the class of *Pareto efficient*, BIC and IIR mechanisms, the VCG mechanism maximizes expected revenue in the quasilinear domain. This result works for multiple object auction problems even when agents can be allocated more than one object. Again, this result uses Pareto efficiency but we do not.

A closer inspection of the richness reveals that if $p$ is too small, then richness requires the existence of a preference where the valuations (with respect to $(0, 0)$) for real objects are very small. We can weaken this richness to a weaker condition which requires that valuations lie in an interval of the form $(v^{min}, v^{max})$, where $v^{min}$ and $v^{max}$ are any lower and upper bounds on the valuation of the objects such that $v^{\max} > v^{min} \geq 0$ and $v^{max} \in \mathbb{R}_+ \cup \{+\infty\}$. Theorem 1 continues to hold in such domains.

We now show how Theorem 1 can be strengthened in some specific rich domains. In particular, if the domain contains all the positive income effect preferences, then our result can be strengthened – we can replace no subsidy in Theorem 1 by the following no bankruptcy condition.

**Definition 9.** A mechanism $f : \mathcal{R}^n \to Z$ satisfies **no bankruptcy** if there exists $\ell \leq 0$ such that for every $R \in \mathcal{R}^n$, we have $\sum_{i \in N} t_i(R) \geq \ell$.

Obviously, no bankruptcy is a weaker property than no subsidy.[12] No bankruptcy is motivated by settings where the seller has limited means to finance the auction participants. Theorem 1 can now be strengthened in the positive income effect domain (which is a rich domain).

**Theorem 2.** *Suppose $\mathcal{R} \supseteq \mathcal{R}^{++}$. The MWEP mechanism is the unique ex-post revenue optimal mechanism among the class of desirable mechanisms satisfying no bankruptcy defined on $\mathcal{R}^n$.*

Analogous to Corollary 1, the following is a corollary of Theorem 2.

**Corollary 2.** *Suppose $\mathcal{R} \in \{\mathcal{R}^+, \mathcal{R}^{++}, \mathcal{R}^C\}$. The MWEP mechanism is the unique ex-post revenue optimal mechanism among the class of desirable mechanisms satisfying no bankruptcy defined on $\mathcal{R}^n$.*

We remark about a possible extension of our results to other models of combinatorial auctions. If we just consider the quasilinear preference domain, the VCG mechanism generalizes to other combinatorial auction models. However, a crucial feature of the VCG mechanism in our model (heterogeneous objects and unit demand buyers) is that it coincides with the MWEP mechanism in the quasilinear domain. This plays a crucial role in all our proofs. This equivalence is lost in other models of combinatorial auctions in the quasilinear domain (Gul and Stacchetti, 1999; Bikhchandani and Ostroy, 2006), and further, the Walrasian equilibrium price vector may fail to exist in other models (Bikhchandani and Ostroy, 2006). Hence, it is not clear how our result extends to other models of combinatorial auctions even in the quasilinear domain. We keep this as an agenda for future research.

On the other hand, when the set of preferences includes all or a very rich class of non-quasilinear preferences, strategy-proofness and Pareto efficiency (along with other axioms) have been shown to be incompatible if the unit demand assumption is violated – Kazumura and Serizawa (2016) show this for multi-object allocation problems where agents can be allocated more

[12] In the literature, the no-deficit condition is sometimes imposed instead of no subsidy. A mechanism $f : \mathcal{R}^n \to Z$ satisfies *no deficit* if for each $R \in \mathcal{R}^n$, $\sum_{i \in N} t_i(R) \geq 0$. It is clear that no bankruptcy is weaker than no deficit.

than one object; Baisa (2020) shows this for homogeneous object allocation problems where agents can be assigned any number of units. In other words, no canonical mechanism is known to exist once we relax the unit demand assumption, and it is not clear how our results will extend.

Finally, we discuss the connections of our results to Pareto efficient mechanism design.

**Definition 10.** A mechanism $f : \mathcal{R}^n \to Z$ is **Pareto efficient** if at every preference profile $R \in \mathcal{R}^n$, there exists no allocation $((\hat{a}_1, \hat{t}_1), \ldots, (\hat{a}_n, \hat{t}_n)) \in Z$ such that

$$(\hat{a}_i, \hat{t}_i)\ R_i\ f_i(R) \qquad \forall\, i \in N$$
$$\sum_{i \in N} \hat{t}_i \geq \textsc{Rev}^f(R),$$

with either the second inequality holding strictly or some agent $i$ strictly preferring $(\hat{a}_i, \hat{t}_i)$ to $f_i(R)$.

Notice that by distributing some money among all the agents, we can always make each agent better off than the allocation in any mechanism. Hence, the above definition requires that there should not exist another allocation where the auctioneer's revenue is not less and every agent is weakly better off.

The MWEP mechanism is Pareto efficient (Morimoto and Serizawa, 2015). Our results establish that even if a seller maximizes her revenue among desirable mechanisms satisfying no subsidy, the resulting mechanism will be Pareto efficient. We state this as corollaries below.

**Corollary 3.** *Let $\mathcal{R}$ be rich and $f : \mathcal{R}^n \to Z$ be ex-post revenue optimal among desirable mechanisms satisfying no subsidy. Then, $f$ is efficient.*

**Corollary 4.** *Let $\mathcal{R} \supseteq \mathcal{R}^{++}$ and $f : \mathcal{R}^n \to Z$ be ex-post revenue optimal among desirable mechanisms satisfying no bankruptcy. Then, $f$ is efficient.*

## 5. Desirable mechanisms satisfying no subsidy

How large is the class of desirable mechanisms satisfying no subsidy in a rich domain? The answer to this question will depend on the domain of the mechanism. In this section, we provide an example of a desirable mechanism satisfying no subsidy for the non-negative income effect domain. In Kazumura et al. (2020b), this example is extended to a family of such mechanisms. Kazumura et al. (2020b) also includes an example (which can be extended to a family of mechanisms), due to Tierney (2019), of a desirable mechanism satisfying no subsidy for the quasilinear domain. These mechanisms are different from the MWEP mechanism. Hence, at least in these two rich domains, we can conclude that the MWEP mechanism is not the unique desirable mechanism satisfying no subsidy and our ex-post revenue maximization requirement is necessary for Theorem 1.

Now, we describe a desirable mechanism satisfying no subsidy for the non-negative income effect domain $\mathcal{R}^+$. Our mechanism and the mechanism due to Tierney (2019) for the quasilinear domain is a variant of the MWEP mechanism. The variation resembles a slot machine in the sense that if for an agent $i \in N$, the preference profile of the other agents are aligned in a special way and constitutes a "discounting combination", then agent $i$ can get discounts from her payment in the MWEP mechanism. Thus, it differs from the MWEP mechanism at zero measure of preference profiles. To retain the properties of the MWEP mechanism, we need to give such discounts carefully. The discounting combination and discounts are constructed such that the new mechanism remains desirable and satisfies no subsidy.

Though the mechanism can be defined very generally, we define it for the simple case when $M = \{a, b\}$ and $N = \{1, 2, 3, 4\}$. First, we formalize the idea of a discounting combination in this case.

**Definition 11.** A **discounting combination** is a collection of three distinct preferences $T \equiv \{R^\alpha, R^\beta, R^\gamma\} \subset \mathcal{R}^+$ such that for each $R_i \in T$,

$$(a, 2)\ I_i\ (b, 2)\ I_i\ (0, 0) \text{ and } (a, 1)\ I_i\ (b, 1).$$

We say a preference profile $R$ is a **discounting combination for agent** $i$ if

$$\{R_j : j \neq i\} = T.$$

The discounting combination satisfies some properties at two price vectors: $\underline{p} = (0, 1, 1)$ and $\bar{p} = (0, 2, 2)$. If $R_i$ is in discounting combination, then $D(R_i, \underline{p}) = \{a, b\}$ and $D(R_i, \bar{p}) = \{0, a, b\}$. If the domain is the quasilinear domain, this is impossible to achieve since there can only be one quasilinear preference which can satisfy these properties (where values for both the objects is 2). On the other hand, a discounting combination requires three distinct preferences. Hence, a discounting combination (as defined in Definition 11) cannot be defined in the quasilinear domain. However, a discounting combination can be defined in the non-negative income effect domain. This makes our mechanism different from the mechanism in Tierney (2019) for the quasilinear domain. Consequently, the arguments to show desirability of both the mechanisms are different.

Our mechanism will be defined using a discounting combination. From now on, we fix a discounting combination $T := \{R^\alpha, R^\beta, R^\gamma\} \subset \mathcal{R}^+$ as in Definition 11. Before we define the mechanism, we prove two useful claims. The claims below relate the minimum Walrasian equilibrium allocation to the price vectors (0, 1, 1) and (0, 2, 2).

**Claim 2.** *If $R \equiv (R_1, R_2, R_3, R_4) \in (\mathcal{R}^+)^4$ is a discounting combination for agent $i$, then $p^{min}(R) = (0, 2, 2)$.*

**Proof.** Let $R \equiv (R_1, R_2, R_3, R_4) \in (\mathcal{R}^+)^4$ be a discounting combination for agent $i$ and $\{R_j : j \neq i\} = T$. Then by the definition of discounting combination, $D(R_j, (0, 2, 2)) = \{0, a, b\}$ for each $j \neq i$. Hence, (0, 2, 2) is a Walrasian equilibrium price vector.

Let $p' \leq (0, 2, 2)$ be such that $p'_a < 2$ or $p'_b < 2$. Then by the definition of discounting combination, $0 \notin D(R_j, p')$ for every $j \neq i$. Hence, only agent $i$ may demand 0. Thus, by $n = 4$ and $m = 2$, $p'$ cannot be a Walrasian equilibrium. □

Note that at any preference profile $R$, there can be a maximum of two agents for whom $R$ is a discounting combination. The next claim establishes an important property involving discounting combinations – we will use this property crucially to define our mechanism. The proof of this claim is given in Appendix B.

**Claim 3.** *For each $R \in (\mathcal{R}^+)^4$, there exists an object allocation $(a_1, \ldots, a_4)$ such that $\{a_1, a_2, a_3, a_4\} = \{0, a, b\}$ and for each $i \in N$,*

1. *if $R$ is a discounting combination for agent $i$, then $a_i \in D(R_i, (0, 1, 1))$,*
2. *if $R$ is not a discounting combination for agent $i$, then there exists a minimum Walrasian equilibrium price allocation $((b_1, p^{min}_{b_1}(R)), \ldots, (b_4, p^{min}_{b_4}(R))) \in Z^{min}(R)$ such that $b_i = a_i$.*

Claim 3 says that at every preference profile there is an object allocation such that (1) if an agent has a discounting combination it assigns her an object in the demand set at a *lower* price, i.e., (0, 1, 1); (2) if an agent does not have a discounting combination, it assigns her an object from *some* minimum Walrasian equilibrium, which has a higher price of (0, 2, 2). This property will be crucial in defining a desirable mechanism, which will be different from the MWEP mechanism. Notice that the object allocation in Claim 3 can be different from the object allocation in the MWEP mechanism. Consider a preference profile $R$ which is a discounting combination for some agents. Clearly, the number of such agents will be no more than two. Hence, there are at least two agents such that $R$ is not a discounting combination for them. According to Claim 3, the objects allocated to these agents may correspond to *different* minimum Walrasian equilibrium price allocations. Hence, the object allocation of the mechanism may not be the same as the object allocation in the MWEP mechanism.

**Definition 12.** The **MWEP mechanism with discounting combination**, denoted by $f^*$, is defined as follows: for every $R \in (\mathcal{R}^+)^4$, $(a^*_1(R), a^*_2(R), a^*_3(R), a^*_4(R))$ is an object allocation satisfying Claim 3 and for every $i \in N$

$$t^*_i(R) = \begin{cases} \underline{p}_{a^*_i(R)} & \text{if } R \text{ is a discounting combination for } i, \\ p^{min}_{a^*_i(R)}(R) & \text{otherwise,} \end{cases}$$

where $\underline{p} \equiv (0, 1, 1)$.

It is clear that $f^*$ satisfies individual rationality, equal treatment of equals, no wastage, and no subsidy. We show that it is also strategy-proof. As shown below in the proof, Claim 3 plays an important role in showing strategy-proofness.

**Proposition 1.** *The MWEP mechanism with discounting combination is strategy-proof.*

**Proof.** Fix $R \in (\mathcal{R}^+)^4$, and $i \in N$. If $R$ is not a discounting combination for $i$, then by changing his preference to $R'_i$, $(R'_i, R_{-i})$ is not a discounting combination for $i$. By Claim 3, in both the preference profiles, we can pick the respective minimum Walrasian equilibrium allocation, and by Demange and Gale (1985), $i$ cannot manipulate to $R'_i$.

If $R$ is a discounting combination for $i$, then by changing his preference to $R'_i$, $(R'_i, R_{-i})$ is also a discounting combination for $i$. As a result, we get that $a^*_i(R) \in D(R_i, (0, 1, 1))$ and $a^*_i(R'_i, R_{-i}) \in D(R'_i, (0, 1, 1))$. Clearly, agent $i$ cannot manipulate to $R'_i$. □

Note that if $R$ is a discounting combination for $i$, then she pays according to (0, 1, 1) which is lower than $(0, 2, 2) = p^{min}(R)$ (Claim 2). Hence, $f^*$ is different from the MWEP mechanism. This clarifies that we cannot afford to drop ex-post revenue maximization in Theorem 1. In that sense, our result is *not* entirely an axiomatic exercise, and revenue maximization is an essential part of our result.

## 6. Our axioms

What happens if we drop the following axioms from Theorem 1: no wastage, equal treatment of equals, and no subsidy? While a general answer to this question is difficult, we provide some answer in the domain of quasilinear preferences $\mathcal{R}^Q$. In this domain, the MWEP mechanism is the VCG mechanism. We also assume a *symmetric* prior setting: each agent's value for each of the objects is drawn using the same distribution and this distribution satisfies the monotone hazard rate property. If the number of agents is at least twice the number of objects, Roughgarden et al. (2012) show that the **expected** revenue of the VCG mechanism in this model is at least half the expected revenue from the ex-ante revenue optimal mechanism among the class of strategy-proof and ex-post individually rational mechanisms.[13] As an immediate corollary of Theorem 1, we see that the ex-post revenue optimal mechanism among the class of desirable mechanisms satisfying no subsidy in the quasilinear domain can guarantee at least half of the expected revenue of the ex-ante revenue optimal mechanism among the class of strategy-proof and ex-post individually rational mechanisms.

We now show that each of the axioms used in Theorem 1 is necessary.

NOTION OF INCENTIVE COMPATIBILITY AND IR. Consider a mechanism that chooses the maximum Walrasian equilibrium allocation at every profile. Such a mechanism will satisfy no subsidy and all the properties of desirability except strategy-proofness. Similarly, the MWEP mechanism supplemented by a *participation fee* satisfies no subsidy and all the properties of desirability except ex-post IR. Both these mechanisms generate more revenue than the MWEP mechanism. Hence, strategy-proofness and ex-post IR are necessary for our results to hold.

What is less clear is if we can relax the notion of incentive compatibility to Bayesian incentive compatibility in our results. In general, the set of Bayesian incentive compatible mechanisms is larger in our model even in the quasilinear domain and they will have different ex-post revenue properties than the MWEP mechanism. To see this, consider the single object auction model in the quasilinear domain. We know that the Vickrey auction and the first-price auction generates the same ex-ante revenue if values of buyers are independently and identically drawn. So, there are profiles of preferences where the first-price auction must generate more revenue than the Vickrey auction. Our notion of incentive compatibility eliminates such mechanisms. We also do not allow randomized mechanisms.

NO WASTAGE. No wastage gets rid of all mechanisms with reserve price. Hence, it is easy to see that no wastage is required for our result – in the quasilinear domain of preferences with one object, Myerson (1981) shows that the Vickrey auction with an *optimally* chosen reserve price maximizes expected revenue for independent and identically distributed values of agents. Such a mechanism wastes the object and generates more revenue than the Vickrey auction at some profiles of preferences.

No wastage is also necessary in a more indirect manner. Consider the domain of quasilinear preferences with two objects $M \equiv \{a, b\}$ and $N = \{1, 2, 3\}$. We show that the seller may increase her revenue by *not* selling all the objects. Consider a profile of valuations as follows:

[13] This result is quite striking because the ratio of expected revenue of the VCG mechanism and the ex-ante revenue optimal mechanism is bounded by a *constant*, i.e., it does not depend on $m$ or $n$. Roughgarden et al. (2012) have similar constant approximation bounds for $n < 2m$ also.

$$v_1^a = v_1^b = 5$$
$$v_2^a = v_2^b = 4$$
$$v_3^a = v_3^b = 1.$$

The MWEP price at this profile is $p_a^{min} = p_b^{min} = 1$, which generates a revenue of 2 to the seller. On the other hand, suppose the seller conducts a Vickrey auction of object $a$ only. Then, he generates a revenue of 4. Hence, the seller can increase her ex-post revenue at some profiles of valuations by withholding objects.

EQUAL TREATMENT OF EQUALS. There are various fairness notions in the mechanism design literature. A typical notion of *ex-post* fairness is envy freeness (Varian, 1974; Sprumont, 2013). A typical notion of *ex-ante* fairness is anonymity (Sprumont, 1991; Moulin and Shenker, 1992; Barbera and Jackson, 1995).[14] Equal treatment of equals is the weakest fairness notion in the sense that it is weaker than each of envy-freeness and anonymity.

There are many examples of mechanisms violating equal treatment of equals in the mechanism design literature. In the single object auction model in quasilinear domain, if values of agents are drawn from different distributions, then revenue is maximized by an *asymmetric* mechanism (Myerson, 1981). Hence, for some profiles of preferences, such mechanisms must generate more revenue than the Vickrey auction. Equal treatment of equals rules out such mechanisms.

Another example that shows the necessity of equal treatment equals in our result is the following. Suppose that there are two agents and one object, and the preferences of the agents are quasilinear. Hence, the preference of each agent $i \in \{1, 2\}$ can be described by his *valuation* for the object $v_i$.

We define the following mechanism: the object is first offered to agent 1 at price $p > 0$; if agent 1 accepts the offer, then he gets the object at price $p$ and agent 2 does not get anything and does not pay anything; else, agent 2 is given the object for free.

This mechanism generates a revenue of $p$ whenever $v_1 > p$ (but generates zero revenue otherwise). However, note that the Vickrey auction generates a revenue of $v_2$ when $v_1 > v_2$. Hence, if $v_1 > p > v_2$, then this mechanism generates more revenue than the Vickrey auction. Also, this mechanism satisfies no subsidy and all the properties of desirability except equal treatment of equals.

NO SUBSIDY. It is tempting to conjecture that no subsidy can be relaxed in quasilinear domain of preferences. The following example shows that this need not be true.

Consider an example with one object and two agents in the quasilinear domain - hence, preferences of agents can be represented by their valuations $v_1$ and $v_2$. Further, assume that valuations lie in $\mathbb{R}_{++}$. Choose $k \in (0, 1)$ and define the mechanism $f \equiv (a, t)$ as follows: for every $(v_1, v_2)$

$$a(v_1, v_2) = \begin{cases} (1, 0) & \text{if } kv_1 > v_2, \\ (0, 1) & \text{otherwise,} \end{cases}$$
$$t_1(v_1, v_2) = \begin{cases} -(v_2 - kv_2) & \text{if } a_1(v_1, v_2) = 0, \\ \frac{v_2}{k} - (v_2 - kv_2) & \text{if } a_1(v_1, v_2) = 1, \end{cases}$$
$$t_2(v_1, v_2) = \begin{cases} 0 & \text{if } a_2(v_1, v_2) = 0, \\ kv_1 & \text{if } a_2(v_1, v_2) = 1. \end{cases}$$

[14] Anonymity is sometime called *symmetry* in the literature (Manelli and Vincent, 2010; Deb and Pai, 2016). Though it is stronger than equal treatment of equals in our model, it is often used when random mechanisms are allowed.

It is straightforward to check that the mechanism is strategy-proof. It is also not difficult to see that utilities of the agents are always non-negative, and hence, individual rationality holds. Finally, if $v_1 = v_2$, we have

$$a_1(v_1, v_2) = 0, a_2(v_1, v_2) = 1, \quad t_1(v_1, v_2) = -(v_2 - kv_2), t_2(v_1, v_2) = kv_1.$$

Hence, net utility of agent 1 is $v_2 - kv_2$ and that of agent 2 is $v_1 - kv_1$, which are equal since $v_1 = v_2$. This shows that the mechanism satisfies equal treatment of equals.

However, the mechanism pays agent 1 when he does not get the object. Thus, it violates no subsidy. The revenue from this mechanism when $kv_1 > v_2$ is

$$v_2\left(\frac{1}{k} + k - 1\right) \geq v_2.$$

The Vickrey auction generates a revenue of $v_2$ when $kv_1 > v_2$. Hence, this mechanism generates more revenue than the Vickrey auction when $kv_1 > v_2$. This shows that we cannot drop no subsidy from Theorem 1.[15]

## 7. Relation to the literature

If there is only one object, Myerson (1981) shows that the Vickrey auction with an appropriately chosen reserve price is *ex-ante* revenue optimal (maximize expected revenue) if preferences of agents are quasilinear and independently and identically drawn. As an immediate corollary to this result, one sees that the Vickrey auction is the ex-ante revenue optimal mechanism among all mechanisms satisfying no wastage in his model. Our Theorem 1 generalizes this corollary of Myerson (1981): it works for multiple objects if agents demand at most one object; it works without quasilinearity; it works without any restriction on prior. Of course, our generalization requires extra axioms (no subsidy and equal treatment of equals) and requires stronger notions of incentive (strategy-proofness) and participation (ex-post IR) constraints. We also focus exclusively on deterministic mechanisms.

Ever since the work of Myerson (1981), various extensions of his work to multi-object auctions have been studied in quasilinear domain. Most of these extensions focus on the single agent (or, screening problem of a monopolist) with additive valuations (value for a bundle of objects is the sum of values of objects). Armstrong (1996, 2000) are early papers that show the difficulty in extending Myerson's optimal mechanisms to multiple objects case. These difficulties are precisely formulated in Rochet and Choné (1998); Thanassoulis (2004); Manelli and Vincent (2006, 2007); Hart and Nisan (2019); Hart and Reny (2015); Daskalakis et al. (2017). Thirumulanathan et al. (2019) is the closest paper to ours, where he considers a single unit-demand buyer buying from a seller selling two heterogeneous objects. He characterizes the menu of the optimal mechanism for a large class of priors.

In our model with multiple unit demand buyers, even with quasilinearity, the multiple dimensions of private information will be valuations for each object. As illustrated in Armstrong (1996, 2000), the multiple dimensions of private information implies that the incentive constraints become complicated to handle. Whether agents can be allocated at most one object or multiple objects, the multidimensional nature of private information makes the optimization problem extremely difficult to handle. Because we work in a model without quasilinearity, we are essentially operating in an "infinite" dimensional mechanism design problem. Hence, we should expect the

[15] Further inspection reveals that the revenue from this mechanism when $v_1 = v_2 = v$ is $kv - v(1-k) = v(2k-1)$. So, if $k < \frac{1}{2}$, this revenue approaches $-\infty$ as $v \to \infty$. Hence, this mechanism even violates no bankruptcy.

problems discussed in quasilinear environment to appear in an even more complex way in our model. In Kazumura et al. (2020a), we show how the Myersonian approach may not work in mechanism design problems without quasilinearity. Hence, solving for full optimality without imposing the additional axioms that we put seems to be even more challenging in our model. In that sense, our results provide a useful resolution to this complex problem.

To circumvent the difficulties from the multidimensional private information and multiple agents, a literature in computer science has developed approximately optimal mechanisms for our model - multiple objects and multiple unit demand agents (but with quasilinearity). Contributions in this direction include Chawla et al. (2010a,b); Briest et al. (2010); Cai et al. (2012). Many of these approximate mechanisms allow for randomization. Further, these approximately optimal mechanisms involve reserve prices and violate no wastage axiom. It is unlikely that these results extend to environments without quasilinearity.

Our work can be connected to a result by Bulow and Klemperer (1996) and its extension by Roughgarden et al. (2015). Bulow and Klemperer (1996) show that (under standard independent and identical agent assumption with *regular* distribution) a single object optimal mechanism (with quasilinear preferences) for $n$ agents generates less expected revenue than a single object Vickrey auction for $(n+1)$ agents. This result has been extended to our multi-object unit-demand agent setting with quasilinear preferences: the expected revenue maximizing mechanism for $n$ agents generates less expected revenue than the VCG mechanism for $(m+n)$ agents, where $m$ is the number of objects (Roughgarden et al., 2015). Our results complement these results by establishing an axiomatic revenue maximizing foundation of the MWEP mechanism (even when preferences are not quasilinear).

We motivated our no wastage axiom by saying that the seller may not be able to commit to a no sale in future if the objects are not sold. If the seller can commit to a mechanism after a no-sale, then we can invoke a revelation principle and our results will follow. However, in many realistic settings, the seller is not able to commit to a future mechanism. Skreta (2015) analyzes a single object auction model in quasilinear domain and models the non-commitment of the seller explicitly. In a finite-period model, she finds that the expected revenue maximizing mechanism takes the same form as in the case of commitment. Her optimal mechanism does not satisfy no wastage, i.e., it is still optimal for the seller to not trade the object at the last period.

Ausubel and Cramton (1999) consider a model of a seller selling *identical* objects to a set of buyers who can consume at most one unit. They assume quasilinear preferences and explore the consequences of ex-post resale. They show that the Vickrey auction with reserve price stands out as the optimal (expected revenue maximizing) mechanism with resale (in a subclass of allocation rules called the *monotonic aggregate* allocation rules). They also offer other results to show that an inefficient allocation is suboptimal if there is perfect resale. While they do not consider non-quasilinear preferences and the heterogeneous objects model, their results also hint that some form of revenue maximization and perfect resale leads to a restricted Pareto efficient mechanism (i.e., whenever there is sale, the object is allocated efficiently).

There is a short but important literature on object allocation problem with non-quasilinear preferences. Baisa (2016) considers the single object model and allows for randomization with non-quasilinear preferences. He introduces a novel mechanism in his setting and studies its optimality properties (in terms of revenue maximization). We do not consider randomization and our solution concept is different from his. Further, ours is a model with multiple objects.

The literature with non-quasilinear preferences and multiple objects have traditionally looked at Pareto efficient mechanisms. As discussed earlier, the closest paper is Morimoto and Serizawa (2015) who consider the same model as ours. They characterize the MWEP mechanism using

Pareto efficiency, individual rationality, incentive compatibility, and no subsidy when the domain includes *all* classical preferences - see an extension of this characterization in a smaller domain in Zhou and Serizawa (2018). Similar characterizations are also available for other settings: Sakai (2008, 2013b,a) provide such characterizations in the single object auction model; Saitoh and Serizawa (2008); Ashlagi and Serizawa (2012); Adachi (2014) in the homogeneous object auction model with unit demand preferences. Pareto efficiency and the *complete* class of classical preferences play a critical role in pinning down the MWEP mechanism in these papers. As we point out in Section 5, even in the quasilinear domain of preferences, there are desirable mechanisms satisfying no subsidy which are different from the MWEP mechanism. By imposing revenue maximization as an objective instead of Pareto efficiency, we get the MWEP mechanism in our model. Pareto efficiency is obtained as an implication (Corollaries 3 and 4). Finally, our results work for not only the complete class of classical preferences, but for a large variety of domains, such as the class of all quasilinear preferences, one including all non-quasilinear preferences, one including all preferences exhibiting positive income effects, etc.

Tierney (2019) considers axioms like *no discrimination*, *welfare continuity*, and some stronger form of strategy-proofness to give various characterizations of the MWEP mechanism with reserve prices in the quasilinear domain. Using our result, he shows that in the *quasilinear domain*, the MWEP mechanism is the unique mechanism satisfying strategy-proofness, *no-discrimination*, individual rationality, no wastage, and *welfare continuity*.

## Appendix A. Proofs of Theorems 1 and 2

In this section, we present all the proofs. The proofs use the following fact very crucially: the MWEP mechanism chooses a Walrasian equilibrium outcome. Before diving into the proofs, we want to stress here that a greedy approach of proving our results would be to first prove that any desirable mechanism satisfying no subsidy and maximizing revenue must be Pareto efficient. In the quasilinear domain, using revenue equivalence will then pin down the MWEP (VCG) mechanism. This approach will fail in our setting because our results work even without quasilinearity and revenue equivalence does not hold in such domains (Kazumura et al., 2020a). Further, it is not obvious even in quasilinear domain that the desirability, no subsidy, and the revenue optimality implies Pareto efficiency. Our proofs work by showing various implications of desirability and no subsidy on consumption bundles of agents. It uses richness of the domain to derive these implications. In that sense, it departs from traditional Myersonian techniques, where revenue maximization is a programming problem with object allocation mechanisms as decision variables.

It is worth discussing how our proofs are different from Morimoto and Serizawa (2015), who characterize the MWEP mechanism. Their focus is on Pareto efficiency and their proofs depend on this. Since we use only no wastage as a efficiency desideratum, which is much weaker than Pareto-efficiency, we need to develop our own proof techniques to establish our results.

We start off by showing an elementary lemma which shows that at every preference profile, if a mechanism gives every agent weakly better consumption bundles than the MWEP mechanism, then its revenue is no more than any MWEP mechanism. This lemma will be used to prove both our results.

**Lemma 1.** *For every mechanism $f : \mathcal{R}^n \to Z$ and for every $R \in \mathcal{R}^n$, the following holds:*

$$\big[f_i(R)\ R_i\ f_i^{min}(R)\ \forall\, i \in N\big] \Rightarrow \big[\textsc{Rev}^{f^{min}}(R) \geq \textsc{Rev}^{f}(R)\big],$$

*where $f^{min}$ is the MWEP mechanism.*

**Proof.** Fix a profile of preferences $R \in \mathcal{R}^n$ and denote $f_i^{min}(R) = (a_i, p_{a_i}^{min}(R))$ for each $i \in N$. Now, for every $i \in N$, we have $f_i(R) \equiv (a_i(R), t_i(R))\ R_i\ (a_i, p_{a_i}^{min}(R))$ and by the Walrasian equilibrium property, $(a_i, p_{a_i}^{min}(R))\ R_i\ (a_i(R), p_{a_i(R)}^{min}(R))$. This gives us $t_i(R) \leq p_{a_i(R)}^{min}(R)$ for each $i \in N$. Hence,

$$\text{Rev}^f(R) = \sum_{i \in N} t_i(R) \leq \sum_{i \in N} p_{a_i(R)}^{min}(R) \leq \text{Rev}^{f^{min}}(R),$$

where the last inequality follows from $p^{min}(R) \in \mathbb{R}_+^{|L|}$. □

*A.1. Proof of Theorem 1*

We start with a series of Lemmas before providing the main proof. Throughout, we assume that $\mathcal{R}$ is a rich domain of preferences and $f$ is a desirable mechanism satisfying no subsidy on $\mathcal{R}^n$. For the proofs, we need the following definition.

**Definition 13.** A preference $R_i$ is $(a, t)$-**favoring** for $t \geq 0$ and $a \in M$ if for price vector $p$ with $p_a = t$, $p_b = 0$ for all $b \neq a$, we have $D(R_i, p) = \{a\}$.

An equivalent way to state this is that $R_i$ is $(a, t)$-favoring for $t > 0$ and $a \in M$ if $V^{R_i}(b, (a, t)) < 0$ for all $b \neq a$. A slightly stronger version of $(a, t)$-favoring preference is the following.

**Definition 14.** A preference $R_i$ is $(a, t)^\epsilon$-favoring for $t \geq 0$, $a \in M$, and $\epsilon > 0$ if it is $(a, t)$-favoring and

$$V^{R_i}(a, (0, 0)) < t + \epsilon$$
$$V^{R_i}(b, (0, 0)) < \epsilon\ \forall\, b \in M \setminus \{a\}.$$

The following lemma shows that if $\mathcal{R}$ is rich, then $(a, t)^\epsilon$-favoring preferences exist for every $(a, t) \in M \times \mathbb{R}_+$ and $\epsilon > 0$.

**Lemma 2.** *Suppose $\mathcal{R}$ is rich. Then, for every bundle $(a, t) \in M \times \mathbb{R}_+$ and for every $\epsilon > 0$, there exists a preference $R_i \in \mathcal{R}$ such that it is $(a, t)^\epsilon$-favoring.*

**Proof.** Define $\hat{p}$ as follows: $\hat{p}_a = t, \quad \hat{p}_b = 0\ \ \forall\, b \neq a$.
Define $p$ as follows: $p_a = t + \epsilon, \quad p_0 = 0, \quad p_b = \epsilon\ \ \forall\, b \in M \setminus \{a\}$.
By richness, there exists $R_i$ such that $D(R_i, \hat{p}) = \{a\}$ and $D(R_i, p) = \{0\}$. But this implies that $R_i$ is $(a, t)$-favoring. Further, $V^{R_i}(a, (0, 0)) < t + \epsilon$ and $V^{R_i}(b, (0, 0)) < \epsilon\ \forall\, b \in M \setminus \{a\}$. Hence, $R_i$ is $(a, t)^\epsilon$-favoring. □

Using this, we prove the following lemma which will be used in the proof.

**Lemma 3.** *For every preference profile $R \in \mathcal{R}^n$, for every $i \in N$, for every $t \in \mathbb{R}_+$, if there exists $j \neq i$ such that $R_j$ is $(a_i(R), t)$-favoring, then $t_i(R) > t$.*

**Proof.** Suppose $t_i(R) \leq t$. Since $R_j$ is $(a_i(R), t)$-favoring, $t_i(R) \leq t$ implies that $R_j$ is also $f_i(R) \equiv (a_i(R), t_i(R))$-favoring. Consider a preference profile $R' \equiv (R'_i = R_j, R'_{-i} = R_{-i})$. By equal treatment of equals (since $R'_i = R'_j = R_j$),

$$f_i(R')\ I_j\ f_j(R'). \tag{A.1}$$

We argue that $f_i(R') = f_i(R)$. If $a_i(R') = a_i(R)$, then strategy-proofness implies that $t_i(R') = t_i(R)$ and we are done. Assume for contradiction that $a_i(R) = a \neq b = a_i(R')$. By strategy-proofness, $(b, t_i(R'))\ R'_i\ (a, t_i(R))$, which implies that $t_i(R') \leq V^{R'_i}(b, (a, t_i(R)))$. Since $R'_i = R_j$ is $(a, t_i(R))$-favoring, we have $V^{R'_i}(b, (a, t_i(R))) < 0$. This implies that $t_i(R') < 0$, which is a contradiction to no subsidy. Hence, we have

$$f_i(R') = f_i(R). \tag{A.2}$$

Combining Inequality (A.1) and Equation (A.2), we get that $f_i(R)\ I_j\ f_j(R')$. Hence, $t_j(R) = V^{R_j}(a_j(R'), f_i(R)) < 0$, where the strict inequality followed from the fact $R_j$ is $f_i(R)$-favoring and $a_i(R) = a_i(R') \neq a_j(R')$. This is a contradiction to no subsidy. □

We will now prove Theorem 1 using these lemmas.

PROOF OF THEOREM 1

**Proof.** The proof is completed by proving the following proposition.

**Proposition 2.** *Let $f : \mathcal{R}^n \to Z$ be a desirable mechanism satisfying no subsidy, where $\mathcal{R}$ is a rich domain of preferences. Then, for every $R \in \mathcal{R}^n$ and every $i \in N$,*

$$f_i(R)\ R_i\ f_i^{min}(R).$$

**Proof.** Fix a preference profile $R \in \mathcal{R}^n$. Let $(z_1, \ldots, z_n) \equiv f^{min}(R)$ be the allocation chosen by the MWEP mechanism $f^{min}$ at $R$. Let $\underline{p} \equiv \min_{a \in M} p_a^{min}(R)$. Clearly, $\underline{p} > 0$. For simplicity of notation, we will denote $z_i \equiv (a_i, p_i)$, where $p_i \equiv p_{a_i}^{min}(R)$ for all $i \in N$.

Assume for contradiction that there is some agent $i \in N$ such that $z_i\ P_i\ f_i(R)$. We first construct a finite sequence of *distinct* agents and preferences, without loss of generality $(1, R'_1), \ldots, (n, R'_n)$, satisfying certain properties. Let $N_0 \equiv \emptyset$, $N_k \equiv \{1, \ldots, k\}$ for each $k \geq 1$, and $(R'_{N_0}, R_{-N_0}) \equiv R$. This sequence satisfies the properties that for every $k \in \{1, \ldots, n\}$,

1. $z_k\ P_k\ f_k(R'_{N_{k-1}}, R_{-N_{k-1}})$ for each $k \geq 1$,
2. $a_k \neq 0$,
3. $R'_k$ is $(z_k)^{\epsilon_k}$-favoring for some $\epsilon_k > 0$ with $\epsilon_k < \min\{V^{R_k}(a_k, f_k(R'_{N_{k-1}}, R_{-N_{k-1}})) - p_k, \underline{p}\}$.

Now, we construct this sequence inductively.

**Step 1 - Constructing** $(1, R'_1)$. Let $i = 1$. By our assumption, $z_1\ P_1\ f_1(R)$. This implies $p_1 - V^{R_1}(a_1, f_1(R)) < 0$. Thus, there is $\epsilon_1 > 0$ such that $\epsilon_1 < \min\{V^{R_1}(a_1, f_1(R)) - p_1, \underline{p}\}$. By Lemma 2, there is a $(z_1)^{\epsilon_1}$-favoring preference $R'_1$. Suppose $a_1 = 0$. Then, $(0, 0) = z_1\ P_1\ f_1(R)$, which contradicts individual rationality. Hence, $a_1 \neq 0$.

**Step 2 - Constructing** $(k, R'_k)$ for $k > 1$. We proceed inductively - suppose, we have already constructed $(1, R'_1), \ldots, (k-1, R'_{k-1})$ satisfying Properties 1, 2, and 3. By no wastage and the fact that $a_{k-1} \neq 0$, there is agent $j \in N$ such that $a_j(R'_{N_{k-1}}, R_{-N_{k-1}}) = a_{k-1}$.

If $j = k-1$, then individual rationality of $f$ and $a_j(R'_{N_{k-1}}, R_{-N_{k-1}}) = a_{k-1}$ imply that

$$t_{k-1}(R'_{N_{k-1}}, R_{-N_{k-1}}) \leq V^{R'_{k-1}}(a_{k-1}, (0,0)) < p_{k-1} + \epsilon_{k-1}$$
$$< V^{R_{k-1}}(a_{k-1}, f_{k-1}(R'_{N_{k-2}}, R_{-N_{k-2}})),$$

where the second inequality followed from the fact that $R'_{k-1}$ is $(z_{k-1})^{\epsilon_{k-1}}$-favoring, and the last inequality followed from the definition of $\epsilon_{k-1}$. Thus, by $a_j(R'_{N_{k-1}}, R_{-N_{k-1}}) = a_{k-1}$, we have

$$f_{k-1}(R'_{N_{k-1}}, R_{-N_{k-1}}) \; P_{k-1} \; f_{k-1}(R'_{N_{k-2}}, R_{-N_{k-2}}),$$

which contradicts strategy-proofness. Hence, $j \neq k-1$.

If $j \in N_{k-2}$, then by individual rationality of $f$, we get

$$t_j(R'_{N_{k-1}}, R_{-N_{k-1}}) \leq V^{R'_j}(a_{k-1}, (0,0)) < \epsilon_j < p_{k-1}, \tag{A.3}$$

where the second inequality followed from the fact that $R'_j$ is $(z_j)^{\epsilon_j}$-favoring and $j \neq (k-1)$, and the last inequality followed from the definition of $\epsilon_j$. But, notice that agent $(k-1) \neq j$ and $R'_{k-1}$ is $z_{k-1}$-favoring (since it is $(z_{k-1})^{\epsilon_{k-1}}$-favoring). Further $a_j(R'_{N_{k-1}}, R_{-N_{k-1}}) = a_{k-1}$. Then, Lemma 3 implies that $t_j(R'_{N_{k-1}}, R_{-N_{k-1}}) > p_{k-1}$, which contradicts Inequality (A.3).

Thus, we have established $j \notin N_{k-1}$, i.e., $j$ is a distinct agent not in $N_{k-1}$. Hence, we denote $j \equiv k$, and note that

$$z_k \; R_k \; z_{k-1} \; P_k \; f_k(R'_{N_{k-1}}, R_{-N_{k-1}}),$$

where the first preference relation follows from the Walrasian equilibrium property and the second follows from the fact that $a_k(R'_{N_{k-1}}, R_{-N_{k-1}}) = a_{k-1}$ and $p_{k-1} < t_k(R'_{N_{k-1}}, R_{-N_{k-1}})$ (Lemma 3). Hence Property 1 is satisfied for agent $k$. Next, if $a_k = 0$, then $(0,0) = z_k \; P_k \; f_k(R'_{N_{k-1}}, R_{-N_{k-1}})$ contradicts individual rationality. Hence, Property 2 also holds. By $z_k \; P_k \; f_k(R'_{N_{k-1}}, R_{-N_{k-1}})$, $p_k - V^{R_k}(a_k, f_k(R'_{n_{k-1}}, R_{-N_{k-1}})) > 0$. Thus, there is $\epsilon_k > 0$ such that $\epsilon_k < \min\{V^{R_k}(a_k, f_k(R'_{N_{k-1}}, R_{-N_{k-1}})) - p_k, \underline{p}\}$. Hence, by Lemma 2, there is a $z_k^{\epsilon_k}$-favoring $R'_k$.

Thus, we have constructed a sequence $(1, R'_1), \ldots, (n, R'_n)$ such that $a_k \neq 0$ for all $k \in N$. This is impossible since $n > m$, giving us the required contradiction. $\square$

By Lemma 1 and Proposition 2, the MWEP mechanism is an ex-post revenue optimal mechanism among the class of desirable mechanisms satisfying no subsidy defined on a rich domain.

Finally, we show that the MWEP mechanism is the unique ex-post revenue optimal mechanism among the class of desirable mechanisms satisfying no subsidy defined on a rich domain. Suppose $\hat{f} \equiv (\hat{a}, \hat{t})$ is another (not the MWEP) desirable mechanism satisfying no subsidy that is ex-post revenue optimal among the class of desirable mechanisms satisfying no subsidy. Then, there is some preference profile $R$ and an agent $i$ such that the object $\hat{a}_i(R)$ assigned to agent $i$ by the mechanism $\hat{f}$ is not in her demand set at $p^{min}(R)$. Let $(a_j, p^{min}_{a_j}(R))$ denote the consumption bundle assigned to each agent $j \in N$ at preference profile $R$ by the MWEP mechanism $f^{min}$. Hence,

$$(a_i, p^{min}_{a_i}(R)) \; P_i \; (\hat{a}_i(R), p^{min}_{\hat{a}_i(R)}(R)).$$

For all $j \neq i$, by the definition of Walrasian equilibrium, we have

$$(a_j, p^{min}_{a_j}(R))\ R_j\ (\hat{a}_j(R), p^{min}_{\hat{a}_j(R)}(R)).$$

Proposition 2 implies that for all $j \in N$,

$$(\hat{a}_j(R), \hat{t}_j(R))\ R_j\ (a_j, p^{min}_{a_j}(R)).$$

Combining the above relations, for all $j \in N$, we have $(\hat{a}_j(R), \hat{t}_j(R))\ R_j\ (\hat{a}_j(R), p^{min}_{\hat{a}_j(R)}(R))$ with strict relation holding for agent $i$. This implies that $\hat{t}_j(R) \le p^{min}_{\hat{a}_j(R)}(R)$ for all $j \in N$ with strict inequality holding for agent $i$. Adding it over all the agents, we get

$$\text{REV}^{\hat{f}}(R) = \sum_{j\in N} \hat{t}_j(R) < \sum_{j \in N} p^{min}_{\hat{a}_j(R)}(R) \le \text{REV}^{f^{min}}(R),$$

which is a contradiction to the ex-post revenue optimality of $\hat{f}$. □

*A.2. Proof of Theorem 2*

We now fix a desirable mechanism $f : \mathcal{R}^n \to Z$, where $\mathcal{R} \supseteq \mathcal{R}^{++}$. Further, we assume that $f$ satisfies no bankruptcy, where the corresponding bound as $\ell \le 0$. We start by proving an analogue of Lemma 3.

**Lemma 4.** *For every preference profile $R \in \mathcal{R}^n$, for every $i \in N$, and every $(a, t) \in M \times \mathbb{R}_+$ with $a = a_i(R)$, if there exists $j \neq i$ such that for each $b \in L \setminus \{a\}$,*

$$V^{R_j}(b, (a, t)) < -n\big(\max_{k\in N}\max_{c\in M} V^{R_k}(c, (0, 0))\big) + \ell,$$

*then $t_i(R) > t$.*

**Proof.** Assume for contradiction $t_i(R) \le t$. Consider $R'_i = R_j$. By strategy-proofness, $f_i(R'_i, R_{-i})\ R'_i\ f_i(R) = (a, t_i(R))$. By equal treatment of equals,

$$f_j(R'_i, R_{-i})\ I_j\ f_i(R'_i, R_{-i})\ R_j\ (a, t_i(R)).$$

Note that either $a_i(R'_i, R_{-i}) \neq a$ or $a_j(R'_i, R_{-i}) \neq a$. Without loss of generality, assume that $a_j(R'_i, R_{-i}) = b \neq a$. Then, using the fact that $(b, t_j(R'_i, R_{-i}))\ R_j\ (a, t_i(R))$ and $t_i(R) \le t$, we get

$$\begin{aligned} t_j(R'_i, R_{-i}) &\le V^{R_j}(b, (a, t_i(R))) \\ &\le V^{R_j}(b, (a, t)) \\ &< -n\big(\max_{k\in N}\max_{c\in M} V^{R_k}(c, (0, 0))\big) + \ell. \end{aligned}$$

By individual rationality

$$t_i(R'_i, R_{-i}) \le V^{R'_i}(a_i(R'_i, R_{-i}), (0, 0)) \le \max_{c\in M} V^{R'_i}(c, (0, 0)).$$

Further, individual rationality also implies that for all $k \notin \{i, j\}$,

$$t_k(R'_i, R_{-i}) \le V^{R_k}(a_k(R'_i, R_{-i}), (0, 0)) \le \max_{c\in M} V^{R_k}(c, (0, 0)).$$

Adding these three sets of inequalities above, we get

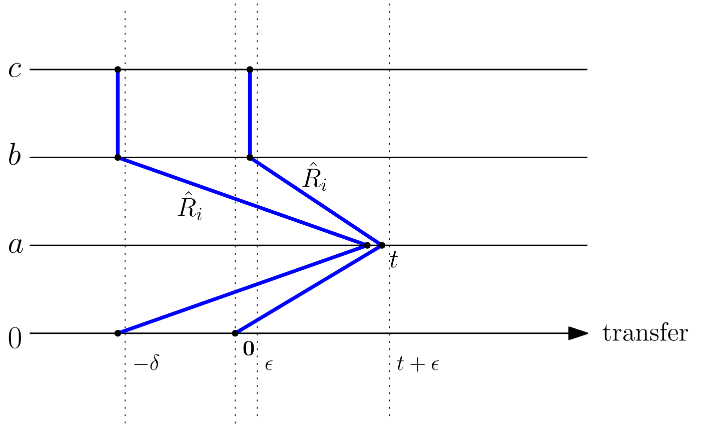


Fig. 5. Illustration of $\hat{R}_i$.

$$\begin{aligned}
&\sum_{k\in N} t_k(R'_i, R_{-i}) \\
&< -n\big(\max_{k\in N}\max_{c\in M} V^{R_k}(c,(0,0))\big) + \ell + \max_{c\in M} V^{R'_i}(c,(0,0)) + \sum_{k\in N\setminus\{i,j\}} \max_{c\in M} V^{R_k}(c,(0,0)) \\
&= -n\big(\max_{k\in N}\max_{c\in M} V^{R_k}(c,(0,0))\big) + \ell + \max_{c\in M} V^{R_j}(c,(0,0)) + \sum_{k\in N\setminus\{i,j\}} \max_{c\in M} V^{R_k}(c,(0,0)) \\
&\le -n\big(\max_{k\in N}\max_{c\in M} V^{R_k}(c,(0,0))\big) + (n-1)\big(\max_{k\in N\setminus\{i\}}\max_{c\in M} V^{R_k}(c,(0,0))\big) + \ell \\
&\le \ell.
\end{aligned}$$

This contradicts no bankruptcy. □

Using Lemma 4, we can mimic the proof of Theorem 1 to complete the proof of Theorem 2. We start by defining a class of positive income effect preferences by strengthening the notion of $(a,t)^\epsilon$-favoring preference. For every $(a,t)\in M\times\mathbb{R}_+$, for each $\epsilon>0$, and for each $\delta>0$, let $\mathcal{R}((a,t),\epsilon,\delta)$ be the set of preferences such that for each $\hat{R}_i\in\mathcal{R}((a,t),\epsilon,\delta)$, the following holds:

1. $\hat{R}_i$ is $(a,t)^\epsilon$-favoring and
2. $V^{\hat{R}_i}(b,(a,t)) < -\delta$ for all $b\neq a$.

A graphical illustration of $\hat{R}_i$ is provided in Fig. 5. Since $\delta>0$, it is clear that a $\hat{R}_i$ can be constructed in $\mathcal{R}((a,t),\epsilon,\delta)$ such that it exhibits positive income effect. Hence, $\mathcal{R}^{++}\cap\mathcal{R}((a,t),\epsilon,\delta)\neq\emptyset$.

PROOF OF THEOREM 2

**Proof.** Now, we can mimic the proof of Theorem 1. We only show parts of the proof that requires some change. As in the proof of Theorem 1, by Lemma 1, we only need to show that for every profile of preferences $R\in\mathcal{R}^n$ and for every $i\in N$, $f_i(R)\ R_i\ f_i^{min}(R)$, where $f^{min}$ is the MWEP mechanism. Assume for contradiction that there is some profile of preferences $R \in \mathcal{R}^n$ and some agent $i \in N$ such that $z_i \; P_i \; f_i(R)$, where $(z_1, \ldots, z_n) \equiv f^{min}(R)$ be the allocation chosen by the MWEP mechanism at $R$. Let $\underline{p} \equiv \min_{a \in M} p_a^{min}(R)$. For simplicity of notation, we will denote $z_j \equiv (a_j, p_j)$, where $p_j \equiv p_{a_j}^{min}(R)$, for all $j \in N$.

Define $\bar{\delta} > 0$ as follows:

$$\bar{\delta} := n\big(\max_{k \in N} \max_{c \in M} V^{R_k}(c, (0, 0))\big) - \ell.$$

We first construct a finite sequence of agents and preferences, without loss of generality $(1, R_1'), \ldots, (n, R_n')$, satisfying certain properties. Let $N_0 \equiv \emptyset$, $N_k \equiv \{1, \ldots, k\}$ for each $k \geq 1$, and $(R'_{N_0}, R_{-N_0}) \equiv R$. This sequence satisfies the properties that for every $k \in \{1, \ldots, n\}$,

1. $z_k \; P_k \; f_k(R'_{N_{k-1}}, R_{-N_{k-1}})$ for each $k \geq 1$,
2. $a_k \neq 0$,
3. $R_k' \in \mathcal{R}^+ \cap \mathcal{R}(z_k, \epsilon, \bar{\delta})$ for some $\epsilon_k > 0$ with $\epsilon_k < \min\{V^{R_k}(a_k, f_k(R'_{N_{k-1}}, R_{-N_{k-1}})) - p_k, \underline{p}\}$.

Now, we can complete the construction of this sequence inductively as in the proof of Theorem 1 (using Lemma 4 instead of Lemma 3), giving us the desired contradiction.

The uniqueness proof is identical to the proof of uniqueness given in Theorem 1. □

## Appendix B. Proof of Claim 3

Before we start the proof of Claim 3, we point out a technical property of non-negative income effect preferences. The claim below shows a form of monotonicity of demand sets with non-negative income effect preferences.

**Claim 4.** *Let $p, p' \in \mathbb{R}^3_+$ be price vectors such that $p'_a = p'_b < p_a = p_b$. For each $R_i \in \mathcal{R}^+$, if $D(R_i, p) \cap M \neq \emptyset$, then $D(R_i, p') \subseteq D(R_i, p)$.*

**Proof.** Let $D(R_i, p) \cap M \neq \emptyset$. Since $p'_a = p'_b < p_a = p_b$, it must be that $0 \notin D(R_i, p')$. Assume for contradiction that $D(R_i, p') \setminus D(R_i, p) \neq \emptyset$. Then, by $0 \notin D(R_i, p')$, without loss of generality, let $a \in D(R_i, p')$ and $a \notin D(R_i, p)$.

By $D(R_i, p) \cap M \neq \emptyset$ and $a \notin D(R_i, p)$, we have $b \in D(R_i, p)$ and $(b, p_b) \; P_i \; (a, p_a)$. The latter implies $V^{R_i}(a, (b, p_b)) < p_a = p_b$.

Let $\delta := p_a - p'_a = p_b - p'_b > 0$. By $R_i \in \mathcal{R}^+$, $V^{R_i}(a, (b, p_b)) < p_b$, $(b, p_b) \; I_i \; (a, V^{R_i}(a, (b, p_b)))$ and $\delta > 0$, we have $(b, p_b - \delta) \; R_i \; (a, V^{R_i}(a, (b, p_b)) - \delta)$. By $V^{R_i}(a, (b, p_b)) < p_a$, $V^{R_i}(a, (b, p_b)) - \delta < p'_a$, and so, $(a, V^{R_i}(a, (b, p_b)) - \delta) \; P_i \; (a, p'_a)$. Thus, $(b, p'_b) = (b, p_b - \delta) \; P_i \; (a, p'_a)$. This contradicts $a \in D(R_i, p')$. □

Proof of Claim 3

**Proof.** Let $R \in (\mathcal{R}^+)^4$ and $S(R) := \{i \in N : R$ is discounting combination for $i\}$. If $S(R)$ is empty, then the claim follows because $Z^{min}(R)$ is non-empty. As we discussed just above Claim 3, if $S(R)$ is non-empty, then $|S(R)| \leq 2$. So, we consider two cases.

Case 1. $S(R) = \{i, j\}$. By Claim 2, we have $p^{min}(R) = (0, 2, 2)$. Since $R$ is a discounting combination for two agents, for every $k \in N = \{1, 2, 3, 4\}$, we must have $R_k \in T$ with

$R_i = R_j$. Thus, by Definition 11, for every $k \in N = \{1, 2, 3, 4\}$, $D(R_k, (0, 2, 2)) = \{0, a, b\}$, and $D(R_i, (0, 1, 1)) = D(R_j, (0, 1, 1)) = \{a, b\}$.

Consider an object allocation $(a_1, a_2, a_3, a_4)$ such that $a_i, a_j \in \{a, b\}$ and for each $k \in N \setminus \{i, j\}$, $a_k = 0$. Then, $(a_1, a_2, a_3, a_4)$ satisfies Conditions (1) and (2) of the claim for this case.

CASE 2. $S(R) = \{i\}$. By Claim 2, we have $p^{min}(R) = (0, 2, 2)$. Also, by (b) of Definition 11, we have

$$D_k(R_k, (0, 2, 2)) = \{0, a, b\} \qquad k \in N \setminus \{i\}. \tag{B.1}$$

Consider an object allocation $(a_1, a_2, a_3, a_4)$ such that $a_i \in D(R_i, (0, 1, 1))$ and $\{a_1, a_2, a_3, a_4\} = \{0, a, b\}$. Then, since $S(R) = \{i\}$, the object allocation $(a_1, a_2, a_3, a_4)$ satisfies Condition (1) of the claim. To show Condition (2) of the claim, we consider two subcases.

CASE 2A. Suppose $D(R_i, (0, 2, 2)) \cap M \neq \emptyset$. Then, by Claim 4 we get $a_i \in D(R_i, (0, 1, 1)) \subseteq D(R_i, (0, 2, 2))$. Thus, by Equation (B.1), we have $((a_1, \overline{p}_{a_1}), \ldots, (a_4, \overline{p}_{a_4})) \in Z^{min}(R)$, where $\overline{p} \equiv (0, 2, 2)$. Thus, $(a_1, a_2, a_3, a_4)$ also satisfies Condition 2.

CASE 2B. Suppose $D(R_i, (0, 2, 2)) \cap M = \emptyset$, i.e., $D(R_i, (0, 2, 2)) = \{0\}$. Choose any $k \notin S(R)$. Then, $k \neq i$. Consider any object allocation $(b_1, b_2, b_3, b_4)$ satisfying $\{b_1, b_2, b_3, b_4\} = \{0, a, b\}$ such that $b_k = a_k$ and $b_i = 0$. By $D(R_i, (0, 2, 2)) = \{0\}$ Equation (B.1), and the fact that $p^{min}(R) = (0, 2, 2)$, we get $((b_1, \overline{p}_{b_1}), \cdots, (b_4, \overline{p}_{b_4})) \in Z^{min}(R)$, where $\overline{p} \equiv (0, 2, 2)$. Thus, $(a_1, a_2, a_3, a_4)$ also satisfies Condition 2.

This exhausts all the cases and completes the proof. □

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