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Abstract: We study the quantum (C^*) convexity structure of normalized positive operator valued measures (POVMs) on measurable spaces. In particular, it is seen that unlike extreme points under classical convexity, C^*-extreme points of normalized POVMs on countable spaces (in particular for finite sets) are always spectral measures (normalized projection valued measures). More generally it is shown that atomic C^*-extreme points are spectral. A Krein–Milman type theorem for POVMs has also been proved. As an application it is shown that a map on any commutative unital C^*-algebra with countable spectrum (in particular $\mathbb{C}^n$) is C^*-extreme in the set of unital completely positive maps if and only if it is a unital $*$-homomorphism.

1. Introduction

The classical notion of convexity plays an important role in analysis in understanding various mathematical structures. Often the problem is to identify extreme points of a convex set. Once that is done, subsequently one may try to show that all points of the set are convex combinations of extreme points or their limits. There have been several approaches to generalize the notion of convexity to have a non-commutative (or quantum) variant, for example CP-convexity [18], matrix convexity [14], nc-convexity [9] and C^*-convexity ([15,25]).

One prominent and useful idea is to replace positive scalars in the interval [0, 1] as coefficients for convexity by positive, contractive and invertible elements in a C^*-algebra. This is the notion of quantum or C^*-convexity. The study of C^*-convexity and C^*-extreme points seems to have been started by Loebl and Paulsen [25] for subsets of C^*-algebras and subsequently many researchers have explored it. Farenick and Morenz [15] defined and initiated a study of C^*-convexity and C^*-extreme points (see Definition 7.4) for unital completely positive (UCP) maps on C^*-algebras taking values in the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$. They call these maps as *generalized states* following an earlier convention, as UCP maps taking values in

$\mathcal{B}(\mathcal{H})$ with $\mathcal{H}$ one dimensional are just states. In particular they show that for $n \in \mathbb{N}$, C^*-extreme UCP maps on the C^*-algebra $\mathbb{C}^n$, taking values in matrices (that is, $\mathcal{B}(\mathcal{H})$ with finite dimensional $\mathcal{H}$) are $*$-homomorphisms. Whether the same conclusion can be arrived at when the space $\mathcal{H}$ is infinite dimensional and separable was left open. We settle it here affirmatively in Theorem 7.7.

It should be mentioned here that there are several papers analyzing C^*-convexity of UCP maps: [15,17,19,26,38], to name a few. In [17,38], one can see some abstract characterizations of C^*-extreme points of UCP maps. There is a well-known relationship (see [20,33]) between UCP maps on the C^*-algebra $C(X)$ of continuous functions on a compact Hausdorff space X and positive operator valued measures (POVMs) on the Borel σ-algebra $\mathcal{O}(X)$ on X. Many authors while studying UCP maps on commutative C^*-algebras exploit this relationship. We follow the same approach and for the purpose first study POVMs.

Positive operator valued measures (POVMs) are called generalized measurements in quantum mechanics and are basic mathematical tools in quantum information theory. There is extensive literature on POVMs and we do not attempt a survey. Some standard references are [10,11,23,36] and [21]. The notions of C^*-convexity and C^*-extreme points have natural extensions to POVMs (see Definition 3.1 and 3.2). Here we study C^*-convexity of POVMs on a measurable space $(X, \mathcal{O}(X))$, where $\mathcal{O}(X)$ is a σ-algebra of subsets of a set X. The problem of identifying C^*-extreme points of POVMs has been open for several decades even for finite sets. The result from 1997 of Farenick and Morenz [15] translates to saying that C^*-extreme positive *matrix* valued measures on a finite set X are spectral measures (normalized projection valued measures). We generalize the result of [15] considerably, as we allow general POVMs on all countable spaces and still all the C^*-extreme points are spectral (Theorem 3.11). This is important because it is in stark contrast with classical (linear) convexity. Extreme points of POVMs under classical convexity are not necessarily spectral measures and are hard to describe even for finite sets, though abstract characterizations are available. C^*-extreme points being spectral measures have physical significance as they relate to classical measurements. Our result reinforces the idea that C^*-convexity is perhaps the suitable notion of convexity in the quantum setting.

Our main goal is to explore the C^*-convexity structure and identify the C^*-extreme points of POVMs taking values for arbitrary separable Hilbert spaces. We shall also present some results on usual (classical) extreme points of POVMs for comparison. We investigate POVMs via decomposing them into a sum of atomic and non-atomic POVMs. Some of these results on POVMs could be folklore in the literature, but we present them here for clarity of presentation and for completeness.

This paper is organized as follows. We start with the definition of POVMs on measurable spaces in Sect. 2 and state some known basic results such as Naimark's dilation theorem, Radon–Nikodym type theorem and so on. A brief description of atomic and non-atomic POVMs is given. In Sect. 3, we present some of our main results on C^*-extreme points. The most crucial technical step is in the proof of Theorem 3.8. Heinosaari and Pellonpää [22] have shown that extreme points of POVMs with commutative ranges are spectral. The same conclusion holds under C^*-convexity (Theorem 3.9) as well. Most importantly all atomic C^*-extreme points are also seen to be spectral (Theorem 3.11). This also helps us in proving that C^*-extreme points are spectral for finite dimensional Hilbert spaces, which we prove in full generality.

In Sect. 4, a notion of disjointness for spectral measures is introduced and we see that it is equivalent to mutual singularity. We study the behaviour of C^*-extreme points

on taking direct sums of mutually singular POVMs. In particular, we show that every C^*-extreme point decomposes into a direct sum of an atomic POVM and a non-atomic POVM, mutually singular to each other. Next in Sect. 5, we explore basic properties like C^*-convexity, atomicity etc under a notion of measure isomorphism of POVMs. In Sect. 6, we analyze POVMs on topological spaces. In this case, we consider the notion of regularity of POVMs and obtain some results analogous to classical measure theory. We also consider a topology on the collection of all POVMs and prove a Krein–Milman type theorem (Theorem 6.15). Lastly in Sect. 7, we describe a well-known correspondence between regular POVMs on a compact Hausdorff space X and completely positive maps on the space $C(X)$ of all continuous functions on X. Using the results got earlier for POVMs and this correspondence, we obtain a number of results for UCP maps on $C(X)$. In particular we show that C^*-extreme maps on commutative unital C^*-algebras with countable spectrum are $*$-homomorphisms (Theorem 7.7). Then making use of the theory of measure isomorphism of POVMs, we show that separable commutative unital C^*-algebras with uncountable spectrum always admit non $*$-homomorphic UCP maps as C^*-extreme points (Theorem 7.10). We also show a Krein–Milman type theorem for the collection of all UCP maps on $C(X)$ equipped with bounded-weak topology. In the concluding section we remark that the study of C^*-convexity can easily be extended to the setting of locally compact Hausdorff spaces by taking one point compactifications. We end with a question on identifying C^*-extreme points of unital completely positive maps on the C^*-algebra l^∞.

It may be remarked here that, although we have relegated a detailed description of the relationship between POVMs and completely positive maps to Sect. 7, occasionally even in earlier sections we would be making references to some known results from the theory of completely positive maps.

Convention. All Hilbert spaces on which POVMs and UCP maps act will be complex and separable as that is where our interest lies. However, when we consider Naimark's dilation of POVMs or Stinespring representations of UCP maps we may end up with non-separable Hilbert spaces and this has to be kept in mind. We follow the convention of the inner product being linear in the second variable. Throughout $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded operators on a complex separable Hilbert space $\mathcal{H}$. If $\mathcal{H}, \mathcal{K}$ are two Hilbert spaces, $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denotes the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. For a subset M of a Hilbert space, $[M]$ denotes the closed subspace generated by M. For any map f, $\mathrm{ran}(f)$ denotes its range. Usually A, B, C etc. will denote measurable subsets of general measurable spaces. Terms like μ, ν etc will denote arbitrary POVMs while π, ρ will be used specifically for spectral measures. The Hilbert space on which a spectral measure π acts will usually be denoted (and taken without mention) by $\mathcal{H}_\pi$. Terms like ϕ, ψ etc will denote completely positive maps on a C^*-algebra. By a positive measure, we mean a (not necessarily finite) usual scalar valued measure taking value in $[0, \infty]$. For our convenience, we always assume that singleton sets are measurable.

2. Basic Properties of POVMs

2.1. Positive operator valued measures. In this section, we recall the definition and some basics of positive operator valued measures. This would also help us in fixing the notation. See [10,23,33,36] and [21] for general references.

Unless stated otherwise, X is a non-empty set and $\mathcal{O}(X)$ denotes a σ-algebra of subsets of X. The pair $(X, \mathcal{O}(X))$ is called a *measurable space* and the elements of $\mathcal{O}(X)$ are called *measurable subsets*. We shall simply call X a measurable space without

mentioning the underlying σ-algebra $\mathcal{O}(X)$. To avoid some unnecessary complications in presentation, we assume that all singleton subsets of X are measurable. When X is a topological space, we shall assume $\mathcal{O}(X)$ to be the Borel σ-algebra on X. All topological spaces under consideration would be Hausdorff.

Definition 2.1. Let X be a measurable space and let $\mathcal{H}$ be a Hilbert space. A *positive operator valued measure(POVM)* on X with values in $\mathcal{B}(\mathcal{H})$ is a map $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ satisfying the following:

- $\mu(A) \geq 0$ in $\mathcal{B}(\mathcal{H})$ for all $A \in \mathcal{O}(X)$ and
- for every $h, k \in \mathcal{H}$, the map $\mu_{h,k} : \mathcal{O}(X) \to \mathbb{C}$ defined by

$$\mu_{h,k}(A) = \langle h, \mu(A)k \rangle \quad \text{for all } A \in \mathcal{O}(X), \tag{2.1}$$

is a complex measure.

Moreover, a POVM μ is called

(1) *normalized* if $\mu(X) = I_{\mathcal{H}}$, the identity operator on $\mathcal{H}$.
(2) *projection valued measure (PVM)* if $\mu(A)$ is a projection for each $A \in \mathcal{O}(X)$.
(3) *spectral measure* if μ is a PVM and is normalized.

It follows from the definition of POVM that, for any increasing (or decreasing) sequence $\{A_n\}$ of measurable subsets converging to A i.e. $A_n \subseteq A_{n+1}$ and $\cup_n A_n = A$ (or $A_n \supseteq A_{n+1}$ and $\cap_n A_n = A$), $\mu(A_n) \to \mu(A)$ in weak operator topology (WOT) in $\mathcal{B}(\mathcal{H})$. Since convergence of an increasing (or decreasing) sequence of bounded operators is equivalent for both weak operator topology and strong operator topology (SOT), it follows that $\mu(A_n) \to \mu(A)$ in SOT. Also, since on bounded subsets of $\mathcal{B}(\mathcal{H})$, WOT and σ-weak topology agree, we infer that $\mu(A_n) \to \mu(A)$ in σ-weak topology. Therefore, in the countable additivity of POVM:

$$\mu\left(\bigcup_{n=1}^{\infty} B_n \right) = \sum_{n=1}^{\infty} \mu(B_n), \quad B_n \in \mathcal{O}(X), B_n \cap B_m = \emptyset \text{ for } n \neq m,$$

the convergence of the series holds in WOT, SOT and σ-weak topologies. So for POVMs such sums can be considered in any of the three topologies.

For any POVM μ, by $\mu_{h,k}$ we would mean the complex measure defined in (2.1). It is clear that a POVM μ is determined by its associated family of complex measures $\{\mu_{h,k} : h, k \in \mathcal{H}\}$.

Notation. Let $POVM_{\mathcal{H}}(X)$ denote the collection of all POVMs on $\mathcal{O}(X)$ with values in $\mathcal{B}(\mathcal{H})$ and let $\mathcal{P}_{\mathcal{H}}(X)$ denote the collection of all normalized elements in $POVM_{\mathcal{H}}(X)$.

We frequently make use of the following remarks in subsequent results without always explicitly referring to them.

Remark 2.2. It is well-known that for a POVM μ, that $\mu(A)$ is a projection for all $A \in \mathcal{O}(X)$ (i.e. μ is a PVM) is equivalent to the fact that $\mu(B \cap C) = \mu(B)\mu(C)$ for all $B, C \in \mathcal{O}(X)$ (see pg 34, [36]).

Remark 2.3. Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM and let $\{B_i\}_{i \in I}$ be a collection of mutually disjoint measurable subsets such that $\mu(B_i) \neq 0$ for each $i \in I$. Then by using separability of $\mathcal{H}$, one can show that I is countable (Lemma 3.1, [12]) as follows: consider any strictly positive density operator S on $\mathcal{H}$ such that the map $T \mapsto \text{tr}(ST)$ (tr denotes trace) is a normal faithful state on $\mathcal{B}(\mathcal{H})$. Define the positive measure $\mu_S : \mathcal{O}(X) \to [0, \infty)$ by $\mu_S(A) = \text{tr}(S\mu(A))$ for all $A \in \mathcal{O}(X)$. Note that $\mu_S(B_i) \neq 0$ for all $i \in I$ and since $\sum_{i \in I} \mu_S(B_i) \leq \mu_S(\cup_{i \in I} B_i) < \infty$, we conclude that I is countable.

2.2. Naimark's dilation theorem. The classical dilation theorem of Naimark [28] shows that POVMs can be dilated to spectral measures: Let X be a measurable space and $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM. Then there exists a triple $(\pi, V, \mathcal{H}_\pi)$ where $\mathcal{H}_\pi$ is a Hilbert space, $\pi : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_\pi)$ is a spectral measure and $V \in \mathcal{B}(\mathcal{H}, \mathcal{H}_\pi)$ such that

$$\mu(A) = V^*\pi(A)V \quad \text{for all } A \in \mathcal{O}(X) \tag{2.2}$$

and the minimality condition: $\mathcal{H}_\pi = [\pi(\mathcal{O}(X))V\mathcal{H}]$ is satisfied. Moreover such a dilation is unique up to unitary equivalence. The triple $(\pi, V, \mathcal{H}_\pi)$ is called a *Naimark dilation triple* for μ. Since π is spectral, note from (2.2) that V is an isometry if and only if μ is a normalized POVM.

Naimark's theorem is text book material. The proof generally uses the usual GNS construction method. Some possible references are (Theorem II.11.F, [36]) and (Theorem 2.1.1, [23]). A proof using Stinespring's theorem for completely positive maps is also well-known (Theorem 4.6, [33]), but then POVMs under consideration will have to be assumed to be regular on the Borel σ−algebra of some locally compact Hausdorff space. As an immediate application of Naimark's dilation theorem we have the following result. Here and elsewhere, $\mathcal{M}'$ denotes the commutant of a subset $\mathcal{M}$ in $\mathcal{B}(\mathcal{H})$.

Proposition 2.4. *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a normalized POVM and $\mu(E)$ a projection for some $E \in \mathcal{O}(X)$. Then $\mu(E \cap A) = \mu(E)\mu(A) = \mu(A)\mu(E)$ for every $A \in \mathcal{O}(X)$. In particular, $\mu(E) \in \mu(\mathcal{O}(X))'$ and hence $\mathrm{ran}(\mu(E))$, the range of $\mu(E)$ is a reducing subspace for all $\mu(A)$, $A \in \mathcal{O}(X)$.*

Proof. Let $(\pi, V, \mathcal{H}_\pi)$ be the minimal Naimark dilation for μ. As noticed earlier, since μ is normalized and π is spectral, it follows that V is an isometry. Now for any $A \in \mathcal{O}(X)$, as $\mu(A) = V^*\pi(A)V$ and $V^*V = I_\mathcal{H}$, we get

$$[V\mu(A) - \pi(A)V]^* \cdot [V\mu(A) - \pi(A)V] = [\mu(A)V^* - V^*\pi(A)] \cdot [V\mu(A) - \pi(A)V]$$
$$= \mu(A)^2 - \mu(A)^2 - \mu(A)^2 + \mu(A)$$
$$= \mu(A)^2 - \mu(A).$$

In particular, since $\mu(E)$ is a projection, we get $V\mu(E) = \pi(E)V$. For any $A \in \mathcal{O}(X)$, therefore

$$\mu(A)\mu(E) = V^*\pi(A)V\mu(E) = V^*\pi(A)\pi(E)V = V^*\pi(A \cap E)V = \mu(A \cap E).$$

Similarly or by taking adjoint of the last equation we get $\mu(E)\mu(A) = \mu(E \cap A)$.   □

Definition 2.5. A POVM μ is *concentrated* on a measurable subset E if $\mu(A) = \mu(A \cap E)$ for all $A \in \mathcal{O}(X)$.

Note that a POVM μ being concentrated on a subset E just means that $\mu(X \backslash E) = 0$. This is not same as saying that E is the support of μ. In fact when X is a topological space, the *support* of μ is defined as the smallest closed subset C such that $\mu(C) = \mu(X)$.

Proposition 2.6. *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM with the minimal Naimark dilation $(\pi, V, \mathcal{H}_\pi)$. Then for any $A \in \mathcal{O}(X)$, $\mu(A) = 0$ if and only if $\pi(A) = 0$. In particular, μ is concentrated on $E \in \mathcal{O}(X)$ if and only if π is concentrated on E.*

Proof. Let $\mu(A) = 0$. Then for any $B \in \mathcal{O}(X)$ and $h \in \mathcal{H}$, we get

$$\langle \pi(A)\pi(B)Vh, \pi(B)Vh \rangle = \langle V^*\pi(B \cap A)Vh, h \rangle = \langle \mu(B \cap A)h, h \rangle \leq \langle \mu(A)h, h \rangle = 0.$$

Since $\{\pi(B)Vh; h \in \mathcal{H}, B \in \mathcal{O}(X)\}$ is total in $\mathcal{H}_\pi$ by the minimality condition, we conclude that $\pi(A) = 0$. The converse is obvious. The second assertion follows from the first. $\qquad\square$

Remark 2.7. As we have already mentioned in Convention, all Hilbert spaces on which POVMs act are assumed to be separable. But note that the Hilbert space $\mathcal{H}_\pi$ in the minimal Naimark dilation $(\pi, V, \mathcal{H}_\pi)$ of a POVM need not always be separable. Nevertheless, notions like atoms and atomic/non-atomic POVMs (Definition 2.11), mutual singularity (Definition 4.1) of POVMs, regularity (Definition 6.1) of a POVM etc. do not need the assumption of separability of the Hilbert space and hence will naturally be considered for the spectral measure π.

2.3. Radon–Nikodym type theorem. In classical measure theory, the Radon–Nikodym derivative of a (σ-finite) positive measure absolutely continuous with respect to another (σ-finite) positive measure is a well-established fact. There have been several attempts to generalize it to the case of absolutely continuous POVMs (which is defined in a similar way as usual positive measures), especially for finite dimensional Hilbert spaces, see for example [16,27]. In this paper however we consider a different notion of comparison of POVMs. We say ν is *dominated* by μ (denoted by $\nu \leq \mu$) if $\mu - \nu$ is a POVM. Here also a Radon–Nikodym type of theorem is known and is well studied. It is analogous to a Radon–Nikodym theorem for completely positive maps by Arveson (Theorem 1.4.2, [1]). See [35] for a more recent account of this result of Arveson and its implications to quantum information theory.

For readers convenience we present an outline of the proof. Here the operator D can be thought of as the Radon–Nikodym derivative of ν with respect to μ.

Theorem 2.8 (Radon–Nikodym type theorem). *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM with the minimal Naimark dilation $(\pi, V, \mathcal{H}_\pi)$. Then for a POVM $\nu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$, $\nu \leq \mu$ (i.e. $\mu - \nu$ is a POVM) if and only if there exists a positive contraction $D \in \pi(\mathcal{O}(X))'$ such that $\nu(A) = V^*D\pi(A)V$ for all $A \in \mathcal{O}(X)$.*

Proof. The proof of 'if' part is obvious. For the converse, assume that $\mu - \nu$ is a POVM. Let $(\mathcal{H}_\rho, \rho, W)$ be the minimal Naimark dilation for ν and define an operator $T : \mathcal{H}_\pi \to \mathcal{H}_\rho$ as follows: first define T on the subspace $\text{span}\{\pi(A)Vh; A \in \mathcal{O}(X), h \in \mathcal{H}\}$ of $\mathcal{H}_\pi$ by $T(\pi(A)Vh) = \rho(A)Wh$, for all $A \in \mathcal{O}(X), h \in \mathcal{H}$ and extend it linearly. One can easily show that T is a well-defined contraction by using the fact that $\sum_{i,j=1}^{n} \langle h_i, (\mu - \nu)(A_i \cap A_j)h_j \rangle \geq 0$ for any $A_i \in \mathcal{O}(X), h_i \in \mathcal{H}, 1 \leq i \leq n$. So it extends as a contraction to its closure $\mathcal{H}_\pi$, which we still denote by T. Set $D = T^*T$. Then D is a positive contraction and it is immediate to verify that $D \in \pi(\mathcal{O}(X))'$ and $\nu(A) = V^*D\pi(A)V$ for all $A \in \mathcal{O}(X)$. $\qquad\square$

2.4. Extreme POVMs. The set $\mathcal{P}_\mathcal{H}(X)$, which is the collection of all normalized POVMs on X with values in $\mathcal{B}(\mathcal{H})$ is clearly a convex set. Extreme points of this set are well studied, especially when X is a finite set or a compact Hausdorff space and $\mathcal{H}$ is a finite dimensional Hilbert space (see [6,16,32] and [22]). In this paper, we are not focusing

much on extreme points of $\mathcal{P}_{\mathcal{H}}(X)$. Nevertheless, we provide some results for the sake of comparison with C^*-extreme points. It is to be noted that even when X is finite with more than two points, the set of extreme points is difficult to describe. This is true even when $\mathcal{H}$ is finite dimensional. The following abstract characterization of extreme points of $\mathcal{P}_{\mathcal{H}}(X)$ is again inspired by Arveson's result (Theorem 1.4.6, [1]) which characterizes the extreme points of unital completely positive maps on a C^*-algebra. This must have been noted by several researchers for the case of POVMs and so we just outline the proof.

Theorem 2.9 (Extreme point criterion). *Suppose that $\mu \in \mathcal{P}_{\mathcal{H}}(X)$ has the minimal Naimark dilation $(\pi, V, \mathcal{H}_\pi)$. Then a necessary and sufficient criterion for μ to be extreme in $\mathcal{P}_{\mathcal{H}}(X)$ is that the map $D \mapsto V^*DV$ from $\pi(\mathcal{O}(X))'$ to $\mathcal{B}(\mathcal{H})$ is injective.*

Proof. First assume that μ is extreme in $\mathcal{P}_{\mathcal{H}}(X)$. Let $V^*DV = 0$ for some $D \in \pi(\mathcal{O}(X))'$. Without loss of generality, we can assume that $-I_{\mathcal{H}_\pi} \leq D \leq I_{\mathcal{H}_\pi}$. Write $\mu = (\mu^+ + \mu^-)/2$ where $\mu^\pm(\cdot) = V^*(I_{\mathcal{H}_\pi} \pm D)\pi(\cdot)V$. Then as μ is extreme in $\mathcal{P}_{\mathcal{H}}(X)$, we must have $\mu = \mu^+$. Hence $V^*D\pi(\cdot)V = 0$, which implies $D = 0$. For the converse, assume the injectivity of the map $D \mapsto V^*DV$, and let $\mu = (\mu_1 + \mu_2)/2$ for $\mu_1, \mu_2 \in \mathcal{P}_{\mathcal{H}}(X)$. By Radon–Nikodym type theorem, there are positive contractions $D_i \in \pi(\mathcal{O}(X))'$, $i = 1, 2$ such that $\frac{\mu_i(\cdot)}{2} = V^*D_i\pi(\cdot)V$. But then as μ_i is normalized, we have $V^*(2D_i - I_{\mathcal{H}_\pi})V = 0$ and hence the hypothesis implies $2D_i = I_{\mathcal{H}_\pi}$. Thus we get $\mu_i(\cdot) = V^*\pi(\cdot)V = \mu(\cdot)$ for $i = 1, 2$, which proves that μ is extreme in $\mathcal{P}_{\mathcal{H}}(X)$. $\square$

The following is an immediate corollary of this theorem. It can also be seen directly, as projections are extremal in the set of positive contractions.

Corollary 2.10. *Every spectral measure is extreme in $\mathcal{P}_{\mathcal{H}}(X)$.*

We briefly discuss here a result in Holevo [23] (see Theorem 2.1.2 therein), which describes some significant differences that can arise when dimension of the Hilbert space changes from finite to infinite. Let $\mathcal{P}_{\mathcal{H}}^0(X)$ denote the set of all spectral measures, and let $\mathcal{P}_{\mathcal{H}}^1(X)$ denote the set of POVMs with commuting ranges. Note that $\mathcal{P}_{\mathcal{H}}^0(X) \subseteq \mathcal{P}_{\mathcal{H}}^1(X)$. Let $\text{Ext}(\mathcal{P}_{\mathcal{H}}(X))$ denote the set of extreme points of $\mathcal{P}_{\mathcal{H}}(X)$, and let $\text{co}(S)$ denote the convex hull of a subset S of $\mathcal{P}_{\mathcal{H}}(X)$. Holevo considers the following topology on $\mathcal{P}_{\mathcal{H}}(X)$ given by the convergence: a net μ_i converges to μ in $\mathcal{P}_{\mathcal{H}}(X)$ if $\text{tr}(T\mu_i(A)) \rightarrow \text{tr}(T\mu(A))$, for all $A \in \mathcal{O}(X)$ and trace class operators T on $\mathcal{H}$ i.e. $\mu_i(A) \rightarrow \mu(A)$ in σ-weak topology (this is equivalent to saying that $\mu_i(A) \rightarrow \mu(A)$ in WOT for all $A \in \mathcal{O}(X)$). In Sect. 6, we also consider a strictly weaker topology that we define for POVMs on topological spaces (see Definition 6.11).

Let n denote the cardinality of the set X. If $n = 2$, then the extreme points of $\mathcal{P}_{\mathcal{H}}(X)$ are exactly the spectral measures (this case relates closely to the classical probability theory), as well as we have $\mathcal{P}_{\mathcal{H}}^1(X) = \overline{\text{co}}(\mathcal{P}_{\mathcal{H}}^0(X))$. Note that this happens regardless of the dimension of the Hilbert space $\mathcal{H}$. On the other hand, when $n > 2$ the situation becomes more complicated. To be precise, if $n > 2$ then there are always some extreme points of $\mathcal{P}_{\mathcal{H}}(X)$ which are not spectral measures, and we have $\mathcal{P}_{\mathcal{H}}^1(X) \subsetneq \overline{\text{co}}(\mathcal{P}_{\mathcal{H}}^0(X)) \subseteq \mathcal{P}_{\mathcal{H}}(X)$. Moreover, the latter inclusion is strict when $\dim \mathcal{H} < \infty$, while they are equal when $\dim \mathcal{H} = \infty$.

The scenario in the case of C^*-extreme points of $\mathcal{P}_{\mathcal{H}}(X)$ (see Definition 3.2) is less complex. Indeed if X is countable, or if $\dim \mathcal{H} < \infty$ and X is arbitrary, then C^*-extreme points of $\mathcal{P}_{\mathcal{H}}(X)$ are always spectral measures (see Theorem 3.11 and Theorem 3.13 below). This is in stark contrast with extreme points case. Since spectral measures are more tractable objects and have classical significance, it seems very natural to study the theory of C^*-convexity of POVMs.

2.5. Atomic and non-atomic POVMs. One of the approaches that we take in this paper for exploring C^*-extreme points is via the decomposition of POVMs into atomic and non-atomic POVMs and analysing them separately. So we recall here the definitions and give some of their properties. These notions have been widely studied in classical measure theory. See [24] for a very general exposition.

Definition 2.11. Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM. A subset $A \in \mathcal{O}(X)$ is called an *atom* for μ if $\mu(A) \neq 0$ and whenever $B \subseteq A$ in $\mathcal{O}(X)$,

$$\text{either} \quad \mu(B) = 0 \quad \text{or} \quad \mu(B) = \mu(A).$$

A POVM μ is called *atomic* if every $A \in \mathcal{O}(X)$ with $\mu(A) \neq 0$ contains an atom. A POVM μ is called *non-atomic* if it has no atom.

We shall frequently make use of the following remark, which is easy to verify.

Remark 2.12. If A is an atom for a POVM μ then for any $B \subseteq A$ in $\mathcal{O}(X)$, either $\mu(B) = 0$ or $A \cap B$ is an atom for μ.

It is a well-known fact that every finite (more generally σ-finite) positive measure decomposes uniquely as a sum of an atomic positive measure and a non-atomic positive measure. In a similar fashion, every POVM decomposes uniquely as a sum of an atomic POVM and a non-atomic POVM ([10,27]). Although the proof in [27] (which itself is inspired from the classical case) is for POVMs on locally compact Hausdorff spaces, the same proof will work for general measurable spaces (see the proof of Theorem 4.9 below). We state it here.

Theorem 2.13 (Theorem 3.10, [27]). *Every POVM decomposes uniquely as a sum of an atomic POVM and a non-atomic POVM.*

We end this section by making a useful observation on atoms of POVMs which shall be frequently used in the paper.

Proposition 2.14. *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM with the minimal Naimark dilation $(\pi, V, \mathcal{H}_\pi)$. Then a subset $A \in \mathcal{O}(X)$ is an atom for μ if and only if A is an atom for π. In particular, μ is atomic (non-atomic) if and only if π is atomic (non-atomic).*

Proof. For any subset $A \in \mathcal{O}(X)$, A is an atom for μ if and only if $\mu(A) \neq 0$ and for each $A' \subseteq A$ in $\mathcal{O}(X)$, we have either $\mu(A') = 0$ or $\mu(A \backslash A') = 0$. Equivalently $\pi(A) \neq 0$ and we have either $\pi(A') = 0$ or $\pi(A \backslash A') = 0$ from Proposition 2.6, which in turn is same as saying that A is an atom for π. The second assertion easily follows from the first.   □

3. Main Results on C^*-Extreme Points

As mentioned earlier, $\mathcal{P}_\mathcal{H}(X)$ denotes the collection of all normalized POVMs from $\mathcal{O}(X)$ to $\mathcal{B}(\mathcal{H})$. We already saw that $\mathcal{P}_\mathcal{H}(X)$ is a convex set and Theorem 2.9 gives an abstract characterization of extreme points of $\mathcal{P}_\mathcal{H}(X)$. In the rest of the paper, we look into a non-commutative convexity structure of $\mathcal{P}_\mathcal{H}(X)$, called quantum convexity or C^*-convexity. As said earlier, the notion of C^*-convexity was introduced in [25] for a subset of $\mathcal{B}(\mathcal{H})$. In [15], it is generalized to the collection of unital completely positive maps. Further one can see the definition of C^*-convexity being modified and studied by [26] in different settings. The notion has also been studied by Farenick et al. [16] for positive operator valued measures, which is our main interest in this paper. Some general references on this topic are [15,17,19,25,26,38] and [16].

Definition 3.1. For any $\mu_i \in \mathcal{P}_{\mathcal{H}}(X)$ and $T_i \in \mathcal{B}(\mathcal{H}), 1 \leq i \leq n$ with $\sum_{i=1}^{n} T_i^* T_i = I_{\mathcal{H}}$, a sum of the form

$$\mu(\cdot) = \sum_{i=1}^{n} T_i^* \mu_i(\cdot) T_i \tag{3.1}$$

is called a *C*-convex combination* for μ. The operators T_i's here are called *C*-coefficients*. When T_i's are invertible, the sum in (3.1) is called a *proper C*-convex combination* for μ.

Observe that $\mathcal{P}_{\mathcal{H}}(X)$ is a *C*-convex set* in the sense that it is closed under C^*-convex combinations. Now the following definition of C^*-extreme points is the POVM analogue of the definition in [15] for unital completely positive maps.

Definition 3.2. A normalized POVM μ is called a *C*-extreme point* in $\mathcal{P}_{\mathcal{H}}(X)$ if, whenever $\sum_{i=1}^{n} T_i^* \mu_i(\cdot) T_i$ is a proper C^*-convex combination for μ, then each μ_i is unitarily equivalent to μ i.e. there are unitary operators $U_i \in \mathcal{B}(\mathcal{H})$ such that $\mu_i(\cdot) = U_i^* \mu(\cdot) U_i$ for $1 \leq i \leq n$.

3.1. Abstract characterizations of C*-extreme points. Farenick and Zhou [17] obtained a characterization of C^*-extreme points for unital completely positive maps. The same can be translated into the language of POVMs and one obtains a characterization for C^*-extreme points of $\mathcal{P}_{\mathcal{H}}(X)$.

As we are dealing with the more general case of arbitrary measurable spaces and also because we are deliberately making slight changes in the statements, we are providing the proof here for completeness.

Theorem 3.3 (Theorem 3.1, [17]). *Let* $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ *be a normalized POVM with the minimal Naimark dilation* $(\pi, V, \mathcal{H}_\pi)$. *Then* μ *is a C*-extreme point in* $\mathcal{P}_{\mathcal{H}}(X)$ *if and only if for any positive operator* $D \in \pi(\mathcal{O}(X))'$ *with* $V^* D V$ *being invertible, there exists a co-isometry* $U \in \pi(\mathcal{O}(X))'$ *(i.e.* $UU^* = I_{\mathcal{H}_\pi}$*) satisfying* $U^* U D^{1/2} = D^{1/2}$ *and an invertible operator* $S \in \mathcal{B}(\mathcal{H})$ *such that* $U D^{1/2} V = V S$.

Proof. First assume that μ is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X)$. Let $D \in \pi(\mathcal{O}(X))'$ be positive with $V^* D V$ invertible. Choose $\alpha > 0$ such that $\|\alpha D\| < 1$. This ensures that $I_{\mathcal{H}_\pi} - \alpha D$ is positive and invertible. Also $\|\alpha V^* D V\| < 1$ and hence $I_{\mathcal{H}} - \alpha V^* D V$ is positive and invertible. Set

$$T_1 = (\alpha V^* D V)^{1/2} \ \text{ and } \ T_2 = (I_{\mathcal{H}} - \alpha V^* D V)^{1/2}.$$

Then both T_1 and T_2 are invertible and $T_1^* T_1 + T_2^* T_2 = I_{\mathcal{H}}$. Now we define POVMs $\mu_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}), i = 1, 2$ by

$$\mu_1(A) = T_1^{-1} \left(\alpha V^* D \pi(A) V \right) T_1^{-1} \text{ and } \mu_2(A) = T_2^{-1} \left(V^* (I_{\mathcal{H}_\pi} - \alpha D) \pi(A) V \right) T_2^{-1}, \tag{3.2}$$

for all $A \in \mathcal{O}(X)$. It is clear that μ_i is a POVM and $\mu_i(X) = I_{\mathcal{H}}$. Also,

$$T_1^* \mu_1(A) T_1 + T_2^* \mu_2(A) T_2 = V^* \pi(A) V = \mu(A) \quad \text{for all } A \in \mathcal{O}(X).$$

Therefore since μ is C^*-extreme, there exists a unitary $W \in \mathcal{B}(\mathcal{H})$ such that $\mu(\cdot) = W^*\mu_1(\cdot)W$. This implies

$$\mu(\cdot) = W^*T_1^{-1}(\alpha V^*D\pi(\cdot)V)T_1^{-1}W = \left(\sqrt{\alpha}W^*T_1^{-1}V^*D^{1/2}\right)\pi(\cdot)\left(\sqrt{\alpha}D^{1/2}VT_1^{-1}W\right)$$
$$= V_1^*\pi(\cdot)V_1,$$

where $V_1 = \sqrt{\alpha}D^{1/2}VT_1^{-1}W \in \mathcal{B}(\mathcal{H}, \mathcal{H}_\pi)$. Note that $V_1^*V_1 = V_1^*\pi(X)V_1 = \mu(X) = I_{\mathcal{H}}$, and so V_1 is an isometry. Now if we set

$$\mathcal{K} = [\pi(\mathcal{O}(X))V_1\mathcal{H}] \subseteq \mathcal{H}_\pi,$$

then $\mathcal{K}$ is a reducing subspace for all $\pi(A)$, $A \in \mathcal{O}(X)$. Also $\mathrm{ran}(V_1) = \pi(X)V_1\mathcal{H} \subseteq \mathcal{K}$, and if we think V_1 as an operator from $\mathcal{H}$ into $\mathcal{K}$, then $(\pi(\cdot)_{|\mathcal{K}}, V_1, \mathcal{K})$ is the minimal Naimark dilation for μ. Therefore, by the uniqueness of minimal dilation, there exists a unitary $U : \mathcal{K} \to \mathcal{H}_\pi$ satisfying

$$UV_1 = V \quad \text{and} \quad \pi(A)U = U\pi(A)_{|\mathcal{K}} \quad \text{for all } A \in \mathcal{O}(X).$$

Extend U to the whole of $\mathcal{H}_\pi$ by assigning it to be 0 on $\mathcal{H}_\pi \ominus \mathcal{K}$, which we still denote by U. Here $\mathcal{H}_\pi \ominus \mathcal{K}$ denotes the orthogonal complement of $\mathcal{K}$ in $\mathcal{H}_\pi$. It is immediate that

$$U^*U = P_\mathcal{K} \quad \text{and} \quad UU^* = I_{\mathcal{H}_\pi}$$

where $P_\mathcal{K}$ is the projection of $\mathcal{H}_\pi$ onto $\mathcal{K}$, which is to say that U is a co-isometry. We also have

$$\pi(A)U = U\pi(A) \quad \text{for all } A \in \mathcal{O}(X),$$

and hence $U \in \pi(\mathcal{O}(X))'$. Set

$$S = \sqrt{\alpha}^{-1}W^*T_1 \in \mathcal{B}(\mathcal{H}).$$

Then S is invertible and, since $UV_1 = V$ and W is a unitary, we get

$$VS = UV_1S = \sqrt{\alpha}\sqrt{\alpha}^{-1}UD^{1/2}VT_1^{-1}WW^*T_1 = UD^{1/2}V.$$

The only remaining thing is to show that $U^*UD^{1/2} = D^{1/2}$. Since U^*U is a projection onto $\mathcal{K}$, this will follow once we show that $\mathcal{K} = \overline{\mathrm{ran}}(D^{1/2})$. Now using $\mathcal{H}_\pi = [\pi(\mathcal{O}(X))V\mathcal{H}]$ and invertibility of T_1 and W, we obtain

$$\mathcal{K} = [\pi(\mathcal{O}(X))V_1\mathcal{H}] = [\pi(\mathcal{O}(X))D^{1/2}V\left(\sqrt{\alpha}T_1^{-1}W\mathcal{H}\right)]$$
$$= [D^{1/2}\pi(\mathcal{O}(X))V\mathcal{H}] = \overline{\mathrm{ran}}(D^{1/2}).$$

For the converse, assume that the given statement in 'only if' part is true. Let $\mu = \sum_{i=1}^n T_i^*\mu_i(\cdot)T_i$ be a proper C^*-convex combination. Fix any $i \in \{1, \ldots, n\}$. Since $T_i^*\mu_i(\cdot)T_i \leq \mu$, it follows from Radon–Nikodym type Theorem (Theorem 2.8) that there exists a positive operator $D_i \in \pi(\mathcal{O}(X))'$ satisfying

$$T_i^*\mu_i(A)T_i = V^*D_i\pi(A)V \quad \text{for all } A \in \mathcal{O}(X).$$

Then $V^*D_iV = T_i^*T_i$ and since T_i is invertible, it follows that V^*D_iV is invertible. Hence the hypothesis ensures the existence of an operator $U_i \in \pi(\mathcal{O}(X))'$ satisfying

$U_i^* U_i D_i^{1/2} = D_i^{1/2}$ and an invertible operator $S_i \in \mathcal{B}(\mathcal{H})$ such that $U_i D_i^{1/2} V = V S_i$. Thus,

$$
\begin{aligned}
T_i^* \mu_i(\cdot) T_i &= V^* D_i \pi(\cdot) V = V^* D_i^{1/2} \pi(\cdot) D_i^{1/2} V \\
&= V^* D_i^{1/2} \pi(\cdot) U_i^* U_i D_i^{1/2} V = V^* D_i^{1/2} U_i^* \pi(\cdot) U_i D_i^{1/2} V \\
&= \left(U_i D_i^{1/2} V \right)^* \pi(\cdot) \left(U_i D_i^{1/2} V \right) = (V S_i)^* \pi(\cdot) (V S_i) \\
&= S_i^* \left(V^* \pi(\cdot) V \right) S_i = S_i^* \mu(\cdot) S_i,
\end{aligned}
$$

which implies $\mu_i = T_i^{*-1} S_i^* \mu(\cdot) S_i T_i^{-1} = R_i^* \mu(\cdot) R_i$, where $R_i = S_i T_i^{-1}$. It is clear that R_i is invertible and since, $R_i^* R_i = \mu_i(X) = I_{\mathcal{H}}$, it follows that R_i is a unitary. This shows that μ_i is unitarily equivalent to μ, as required to conclude that μ is a C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$. $\square$

Remark 3.4. In the statement of the theorem above, U is a co-isometry. It is not clear at this point as to whether U can be chosen to be a unitary, as claimed in (Theorem 3.1, [17]). Of course this is automatic if $\mathcal{H}_\pi$ is finite dimensional.

The following is an immediate corollary of Theorem 3.3. This can also be deduced from a result of Loebl and Paulsen (Proposition 26, [25]), which says that projections are C^*-extreme points in the set of all positive contractions of $\mathcal{B}(\mathcal{H})$, although it needs a bit of effort.

Corollary 3.5. *Every spectral measure is a C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$.*

Proof. If μ is a spectral measure then the minimal dilation for μ can be taken to be $(\mu, I_{\mathcal{H}}, \mathcal{H})$. For positive $D \in \mu(X)'$ with $D(= I_{\mathcal{H}}^* D I_{\mathcal{H}})$ invertible, we can take $U = I_{\mathcal{H}}$ and $S = D^{1/2}$ to satisfy the criterion. $\square$

Zhou in his thesis [38] gave another characterization of C^*-extreme points for unital completely positive maps. The result translates to POVM case as follows. Again as there is a slight change in the statement, we provide the proof.

Corollary 3.6 (Theorem 3.1.5, [38]). *Let $\mu \in \mathcal{P}_{\mathcal{H}}(X)$. Then μ is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X)$ if and only if for any POVM $\nu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ with $\nu \leq \mu$ and $\nu(X)$ invertible, there exists an invertible operator $S \in \mathcal{B}(\mathcal{H})$ such that $\nu(A) = S^* \mu(A) S$ for all $A \in \mathcal{O}(X)$.*

Proof. First assume that μ is a C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$. Let $\nu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM such that $\nu \leq \mu$ and $\nu(X)$ is invertible. Let $(\pi, V, \mathcal{H}_\pi)$ be the minimal Naimark dilation for μ. By Theorem 2.8, there exists a positive operator $D \in \pi(\mathcal{O}(X))'$ such that

$$
\nu(A) = V^* D \pi(A) V \quad \text{for all } A \in \mathcal{O}(X).
$$

Since $V^* D V = \nu(X)$ and $\nu(X)$ is invertible, it follows that $V^* D V$ is invertible. Therefore, by Theorem 3.3 there exists a co-isometry $U \in \pi(\mathcal{O}(X))'$ satisfying $U^* U D^{1/2} = D^{1/2}$ and an invertible operator $S \in \mathcal{B}(\mathcal{H})$ such that $U D^{1/2} V = V S$. So for any $A \in \mathcal{O}(X)$, we get

$$
\begin{aligned}
\nu(A) &= V^* D \pi(A) V = V^* D^{1/2} \pi(A) D^{1/2} V = V^* D^{1/2} \pi(A) U^* U D^{1/2} V \\
&= V^* D^{1/2} U^* \pi(A) U D^{1/2} V
\end{aligned}
$$

$$= \left(UD^{1/2}V\right)^* \pi(A)\left(UD^{1/2}V\right) = (VS)^*\pi(A)(VS)$$
$$= S^*\left(V^*\pi(A)V\right)S = S^*\mu(A)S.$$

Conversely, assume the given statement in the 'only if' part is true. Let $\mu = \sum_{i=1}^n T_i^*\mu_i$ $(\cdot)T_i$ be a proper C^*-convex combination. Then $T_i^*\mu_i(\cdot)T_i \leq \mu$ for each i. Also, since $T_i^*\mu_i(X)T_i = T_i^*T_i$ and T_i is invertible, it follows that $T_i^*\mu_i(X)T_i$ is invertible. Hence by hypothesis, there exists an invertible operator $S_i \in \mathcal{B}(\mathcal{H})$ such that for all $A \in \mathcal{O}(X)$, we have

$$T_i^*\mu_i(A)T_i = S_i^*\mu(A)S_i$$

which when put differently yields

$$\mu_i(A) = U_i^*\mu(A)U_i,$$

where $U_i = S_i T_i^{-1}$. But, since $U_i^*U_i = U_i^*\mu(X)U_i = \mu_i(X) = I_\mathcal{H}$ and U_i is invertible, it follows that U_i is a unitary. This shows that μ_i is unitarily equivalent to μ, as was required.   □

We wish to mention that the condition of $\nu(X)$ being invertible in the corollary above cannot be dropped. The original statement (Theorem 3.1.5, [38]) is somewhat ambiguous about the invertibility requirement in the characterization. But it is crucial as the following example shows. Here $\mathbb{T}$ is the unit circle and $\mathcal{O}(\mathbb{T})$ is the Borel σ-algebra of $\mathbb{T}$.

Example 3.7. Consider the normalized POVM $\mu : \mathcal{O}(\mathbb{T}) \to \mathcal{B}(\mathcal{H}^2)$ defined by

$$\mu(A) = P_{\mathcal{H}^2}M_{\chi_A}|_{\mathcal{H}^2} = T_{\chi_A} \quad \text{for all } A \in \mathcal{O}(\mathbb{T}),$$

where $\mathcal{H}^2$ is the Hardy space on $\mathbb{T}$ and $T_f = P_{\mathcal{H}^2}M_f|_{\mathcal{H}^2}$ denotes the Toeplitz operator for any $f \in L^\infty(= L^\infty(\mathbb{T}, l))$, where l denotes the one-dimensional Lebesgue measure on $\mathbb{T}$. Here χ_A denotes the characteristic function for the subset A. It is known that μ is C^*-extreme in $\mathcal{P}_{\mathcal{H}^2}(\mathbb{T})$ (see Example 7.9). Let $C \subseteq \mathbb{T}$ be a Borel subset such that $l(C) \neq 0$ and $l(\mathbb{T}\backslash C) \neq 0$. Consider $\nu : \mathcal{O}(\mathbb{T}) \to \mathcal{B}(\mathcal{H}^2)$ defined by

$$\nu(A) = \mu(A \cap C) = P_{\mathcal{H}^2}M_{\chi_{(A\cap C)}}|_{\mathcal{H}^2} = T_{\chi_{(A\cap C)}} \quad \text{for all } A \in \mathcal{O}(\mathbb{T}).$$

It is clear that ν is a POVM and $\nu \leq \mu$. Also $\nu(\mathbb{T}) = T_{\chi_C}$ is not invertible. We claim that there is no operator $S \in \mathcal{B}(\mathcal{H})$ such that $\nu(\cdot) = S^*\mu(\cdot)S$. Suppose this is not the case and S is one such operator. Note that $S^*S = \nu(\mathbb{T}) = T_{\chi_C}$. Since $l(C)$ and $l(\mathbb{T}\backslash C)$ are non-zero, we have $\chi_C \neq 0$ and $\chi_{(\mathbb{T}\backslash C)} \neq 0$ in L^∞. It is then a fact due to Coburn that T_{χ_C} and $T_{\chi_{(\mathbb{T}\backslash C)}}$ are one-one operators (see Proposition 7.24, [13]) and hence S^*S is one-one, which further implies that S is one-one. Therefore, again as $T_{\chi_{(\mathbb{T}\backslash C)}}$ is one-one, it follows the operator $T_{\chi_{(\mathbb{T}\backslash C)}}S$ is one-one. But on the other hand, we have

$$\left(T_{\chi_{(\mathbb{T}\backslash C)}}^{1/2}S\right)^* \left(T_{\chi_{(\mathbb{T}\backslash C)}}^{1/2}S\right) = S^*T_{\chi_{(\mathbb{T}\backslash C)}}S = S^*\mu(\mathbb{T}\backslash C)S = \nu(\mathbb{T}\backslash C) = 0$$

which implies

$$T_{\chi_{(\mathbb{T}\backslash C)}}^{1/2}S = 0$$

and hence $T_{\chi_{(\mathbb{T}\backslash C)}}S = 0$, leading us to a contradiction.

3.2. C*-extreme points with commutative ranges. With these two characterizations of C*-extreme points at our disposal, we are now ready to present the main results of this paper. Gregg [19] shows that if a POVM μ is C*-extreme in $\mathcal{P}_{\mathcal{H}}(X)$ (for a compact Hausdorff space X) then for any A in $\mathcal{O}(X)$, the spectrum of $\mu(A)$ is either contained in $\{0, 1\}$ (so that $\mu(A)$ is a projection) or it is whole of the interval $[0, 1]$. Our main observation is that the second situation can be avoided in a variety of cases. The proof uses straightforward Borel functional calculus, with a carefully chosen family of functions. These functions are necessarily discontinuous and so C*-algebra setting and continuous functional calculus will not suffice.

Theorem 3.8. *Let μ be a C*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$. If $E \in \mathcal{O}(X)$ is such that $\mu(A)\mu(E) = \mu(E)\mu(A)$ for all $A \subseteq E$ in $\mathcal{O}(X)$, then $\mu(E)$ is a projection. In particular if $\mu(E)$ commutes with all $\mu(B)$ for $B \in \mathcal{O}(X)$, then $\mu(E)$ is a projection.*

Proof. The second assertion is immediate from the first. So assume the hypothesis in the first statement. We claim that $\sigma(\mu(E)) \cap (r, s) = \emptyset$ for all $0 < r < s < 1$, where $\sigma(\mu(E))$ denotes the spectrum of the operator $\mu(E)$. As $\mu(E)$ is a positive contraction, it will follow that $\sigma(\mu(E)) \subseteq \{0, 1\}$, which in turn will imply that $\mu(E)$ is a projection. So fix $0 < r < s < 1$, and define the map $f := f_{r,s} : [0, 1] \to [0, 1]$ by

$$f_{r,s}(t) = \begin{cases} 1 & \text{if } t \notin [r, s], \\ \frac{r}{1-r}\left(\frac{1}{t} - 1\right) & \text{if } t \in [r, s]. \end{cases} \tag{3.3}$$

Clearly f is continuous except at one point namely s, and hence it is a Borel measurable function. So for any operator $0 \le T \le I_{\mathcal{H}}$, it follows from spectral theory that $f(T)$ is a well defined bounded operator. Further we note for each $t \in [0, 1]$, that

$$0 < \alpha := \left(\frac{r}{1-r}\right)\left(\frac{1-s}{s}\right) \le f(t) \le 1$$

and consequently,

$$\alpha I_{\mathcal{H}} \le f(T) \le I_{\mathcal{H}}. \tag{3.4}$$

Now consider the map $\nu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ defined by

$$\nu(B) = \mu(B \cap E)f(\mu(E)) + \mu(B \backslash E) \tag{3.5}$$

for any $B \in \mathcal{O}(X)$.

We show that ν is a POVM by observing the following:

- For each $B \in \mathcal{O}(X)$, our hypothesis says that $\mu(B \cap E)$ and $\mu(E)$ commute and it then implies from spectral theory that $\mu(B \cap E)$ commutes with $f(\mu(E))$. Therefore, as both $\mu(B \cap E)$ and $f(\mu(E))$ are positive operators, it follows that their product $\mu(E \cap B)f(\mu(E))$ is a positive operator, which amounts to saying that $\nu(B) \ge 0$ in $\mathcal{B}(\mathcal{H})$.
- If $B_1, B_2, \ldots$ is a countable collection of mutually disjoint measurable subsets of X and $B = \cup_n B_n$, then since μ is a POVM, we have in WOT convergence,

$$\begin{aligned} \nu(\cup_n B_n) &= \mu((\cup_n B_n) \cap E)f(\mu(E)) + \mu((\cup_n B_n)\backslash E) \\ &= \mu(\cup_n(B_n \cap E))f(\mu(E)) + \mu(\cup_n(B_n \backslash E)) \\ &= \sum_n [\mu(B_n \cap E)f(\mu(E))] + \sum_n \mu(B_n \backslash E) \end{aligned}$$

$$= \sum_n [\mu(B_n \cap E) f(\mu(E)) + \mu(B_n \backslash E)]$$

$$= \sum_n \nu(B_n).$$

This shows that μ is countably additive, which in particular implies that the function $B \mapsto \langle h, \nu(B)k \rangle$ is a complex measure on X for all $h, k \in \mathcal{H}$.

The observations above imply that ν is a POVM. Further since $f(\mu(E)) \leq I_{\mathcal{H}}$ from (3.4), it follows for each $B \in \mathcal{O}(X)$, that

$$\nu(B) = \mu(B \cap E) f(\mu(E)) + \mu(B \backslash E) \leq \mu(B \cap E) + \mu(B \backslash E) = \mu(B)$$

which is to say $\nu \leq \mu$. Also since $f(\mu(E)) \geq \alpha I_{\mathcal{H}}$ from (3.4), and $\mu(E) \leq I_{\mathcal{H}}$, we note that

$$\begin{aligned}
\nu(X) &= \mu(E) f(\mu(E)) + \mu(X \backslash E) \\
&\geq \alpha \mu(E) + \mu(X \backslash E) \\
&= \alpha \mu(E) + I_{\mathcal{H}} - \mu(E) \\
&= I_{\mathcal{H}} - (1 - \alpha) \mu(E) \\
&\geq I_{\mathcal{H}} - (1 - \alpha) I_{\mathcal{H}} \\
&= \alpha I_{\mathcal{H}},
\end{aligned}$$

which is equivalent to saying that $\nu(X)$ is invertible. Therefore, as μ is a C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$, it follows from Corollary 3.6 that there exists an invertible operator $T \in \mathcal{B}(\mathcal{H})$ satisfying the condition

$$\nu(B) = T^* \mu(B) T \quad \text{for all } B \in \mathcal{O}(X). \tag{3.6}$$

We note that $\nu(X) = T^*T = |T|^2$ and hence,

$$|T| = \nu(X)^{1/2} = [\mu(E) f(\mu(E)) + I_{\mathcal{H}} - \mu(E)]^{1/2} \tag{3.7}$$

where $|T|$ denotes the square root of the positive operator T^*T. Set $S = \mu(E)$. By taking $B = E$ in (3.6), we have

$$T^* S T = T^* \mu(E) T = \nu(E) = \mu(E) f(\mu(E)) = S f(S).$$

Let $T = U|T|$ be the polar decomposition of T. Then U is a unitary and $|T|$ is invertible, as T is invertible. Consequently,

$$U^* S U = |T|^{-1} S f(S) |T|^{-1}. \tag{3.8}$$

Now let $g : [0, 1] \to [0, 1]$ be the map defined by

$$g(t) = \frac{t f(t)}{1 - t + t f(t)} = \begin{cases} t & \text{if } t \notin [r, s], \\ r & \text{if } t \in [r, s]. \end{cases}$$

Then $g(S)$ is a well-defined bounded operator and we get

$$g(S) = S f(S) [I_{\mathcal{H}} - S + S f(S)]^{-1}.$$

Hence (3.7) and (3.8) yield

$$U^*SU = g(S).$$

Therefore by spectral mapping theorem (Theorem IX.8.11, [8]), spectrum of S satisfies the following:

$$\sigma(S) = \sigma(U^*SU) = \sigma(g(S)) \subseteq \mathrm{essran}(g),$$

where $\mathrm{essran}(g)$ is the essential range of g with respect to the spectral measure corresponding to the operator S. But,

$$\mathrm{essran}(g) \subseteq \overline{\mathrm{ran}}(g) \subseteq [0, r] \cup [s, 1],$$

which implies that $\sigma(S) \subseteq [0, r] \cup [s, 1]$. This is same as saying $\sigma(S) \cap (r, s) = \emptyset$, which is what we wanted to show. □

A direct application of Theorem 3.8 is possible for C^*-extreme points with commutative range. We say a POVM μ to be *commutative* if its range is commutative. It has been shown [22] that a commutative normalized POVM is an extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ if and only if it is spectral. A similar kind of result for C^*-extreme points holds true following the theorem above; if a C^*-extreme point μ in $\mathcal{P}_{\mathcal{H}}(X)$ is commutative, then it follows from Theorem 3.8 that $\mu(A)$ is projection for all $A \in \mathcal{O}(X)$ and hence μ is spectral. Thus we have arrived at the following theorem.

Theorem 3.9. *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a commutative normalized POVM. Then μ is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X)$ if and only if it is a spectral measure.*

3.3. Atomic C^*-extreme points. Theorem 3.8 is quite powerful. Here we have more applications of it. We examine atomic C^*-extreme points. First consider the following lemma. Recall our assumption that singletons are measurable subsets.

Lemma 3.10. *Let μ be a C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$. Then $\mu(E)$ is a projection for every atom E for μ. In particular $\mu(\{x\})$ is a projection for all $x \in X$ and consequently $\mu(A)$ is a projection for every countable subset A of X.*

Proof. If E is an atom for μ then for each $B \subseteq E$ in $\mathcal{O}(X)$, either $\mu(B) = 0$ or $\mu(B) = \mu(E)$ and hence, $\mu(B)$ commutes with $\mu(E)$. Therefore Theorem 3.8 is applicable and it follows that $\mu(E)$ is a projection. This further implies that for each $x \in X$, since either $\mu(\{x\}) = 0$ or $\{x\}$ is an atom for μ, $\mu(\{x\})$ is a projection.

Finally let $x, y \in X$ be two distinct points and set $P = \mu(\{x\})$ and $Q = \mu(\{y\})$. Note that

$$P + Q = \mu(\{x\}) + \mu(\{y\}) = \mu(\{x, y\}) \leq I_{\mathcal{H}}$$

and hence $P \leq I_{\mathcal{H}} - Q$. Because P and Q are projections as proved above, it follows that $P(I_{\mathcal{H}} - Q) = P$, which in turn yields

$$PQ = 0.$$

In other words, $\mu(\{x\})$ and $\mu(\{y\})$ are mutually orthogonal projections for any two distinct points x and y. Therefore, for any at most countable subset $A = \{x_1, x_2, \ldots\}$ of

X, the collection $\{\mu(\{x_n\})\}$ consists of projections mutually orthogonal to one another and since

$$\mu(A) = \sum_n \mu(\{x_n\}) \quad \text{(in WOT)},$$

we conclude that $\mu(A)$ is a projection.   □

The POVMs on finite sets have been natural setting for many applications in quantum theory. Several researchers have looked into the convexity structure in this set up and the structure of extreme points is very well studied. They are not always spectral measures. When it comes to C^*-convexity, it is shown in [16] that only spectral measures are C^*-extreme when $\mathcal{H}$ is finite dimensional. Here we show that it is true in full generality.

Following the results above, we now give a characterization of all atomic C^*-extreme points in $\mathcal{P}_{\mathcal{H}}(X)$. This in particular characterizes all C^*-extreme points in $\mathcal{P}_{\mathcal{H}}(X)$ whenever X is finite.

Theorem 3.11. *An atomic normalized POVM μ on a measurable space X is a C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ if and only if μ is spectral. In particular, if X is a countable measurable space then any C^*-extreme point of $\mathcal{P}_{\mathcal{H}}(X)$ is spectral.*

Proof. We have seen that spectral measures are always C^*-extreme. Conversely, assume that μ is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X)$. Let $\{B_i\}_{i \in I}$ be a maximal collection of mutually disjoint atoms for μ, which exists thanks to Zorn's lemma. Then Lemma 3.10 says that $\mu(B_i)$ is a projection for each $i \in I$. Also, since B_i's are mutually disjoint, it follows from Proposition 2.4 that $\{\mu(B_i) : i \in I\}$ is a collection of mutually orthogonal projections. Hence, as $\mathcal{H}$ is separable, we conclude that I is countable. Further for any $A \in \mathcal{O}(X)$, we have

$$\mu(A) = \sum_{i \in I} \mu(A \cap B_i), \qquad (3.9)$$

otherwise, we would get $\mu(A \backslash (\cup_i (A \cap B_i))) \neq 0$ and since μ is atomic, there is an atom, say $A_1 \subseteq A \backslash (\cup_i (A \cap B_i))$ for μ. But then $\{B_i\}_{i \in I} \cup \{A_1\}$ is a collection of mutually disjoint atoms for μ, which violates the maximality of the collection $\{B_i\}_{i \in I}$. Similarly note from Lemma 3.10 that for any $A \in \mathcal{O}(X)$, since either $\mu(A \cap B_i) = 0$ or $A \cap B_i$ is an atom for μ, the collection $\{\mu(A \cap B_i)\}_{i \in I}$ consists of mutually orthogonal projections. Consequently it follows from equation (3.9), that $\mu(A)$ is a projection. This proves that μ is spectral. Since any POVM on a countable measurable space is atomic, the second assertion follows.   □

Since all spectral measures are also extreme (in the usual sense), we have the following corollary. Note that the same is always true for general measurable spaces, whenever $\mathcal{H}$ is a finite dimensional Hilbert space ([16]).

Corollary 3.12. *If X is a countable (in particular, finite) measurable space, then every C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ is extreme.*

3.4. The case of finite dimensional Hilbert space. We end this section by recording the case of finite dimensional Hilbert spaces and general measurable spaces. This set up has been widely studied by several researchers. We recall that it is proved in [16] for a compact Hausdorff space X and a finite dimensional $\mathcal{H}$, that every C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ is spectral. We extend this result to full generality using Theorem 3.11.

Theorem 3.13. *Let $\mathcal{H}$ be a finite dimensional Hilbert space and X a measurable space. Then any C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ is spectral.*

Proof. Firstly, finite dimensionality of $\mathcal{H}$ ensures that every C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ is also extreme (Proposition 2.1, [16]). Now we show that every extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ is atomic (see Lemma 2, [6] for topological spaces) as follows: if μ is extreme in $\mathcal{P}_{\mathcal{H}}(X)$ and $(\pi, V, \mathcal{H}_\pi)$ is the minimal Naimark dilation for μ, then the map

$$D \mapsto V^*DV$$

from $\pi(\mathcal{O}(X))'$ to $\mathcal{B}(\mathcal{H})$ is one-to-one by Theorem 2.9. Since $\mathcal{H}$ is finite dimensional, $\mathcal{B}(\mathcal{H})$ is a finite dimensional algebra and hence $\pi(\mathcal{O}(X))'$ is a finite-dimensional algebra. Therefore, since $\pi(\mathcal{O}(X)) \subseteq \pi(\mathcal{O}(X))'$ and $\mathcal{H}_\pi = [\pi(\mathcal{O}(X)) V \mathcal{H}]$, it follows that $\mathcal{H}_\pi$ is also finite-dimensional.

Consequently $\{\pi(A) : A \in \mathcal{O}(X)\}$ is a commuting family of projections on a finite dimensional Hilbert space $\mathcal{H}_\pi$ and hence it is a finite set. This implies that π is atomic. Then by Proposition 2.14, μ is also atomic. Thus we have shown that every C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ is atomic. The proof is complete in view of Theorem 3.11. $\qquad\square$

Remark 3.14. In the theorem above, we noticed that any spectral measure acting on a finite dimensional Hilbert space is atomic.

4. Mutually Singular POVMs

4.1. Mutual singularity. The notion of mutual singularity of positive measures is very familiar from classical measure theory. We consider the similar notion of mutually singular POVMs. Our main aim here is to discuss the behaviour of C^*-extremity for direct sums of mutually singular POVMs. This helps us in characterization of C^*-extreme points, as we show that every C^*-extreme POVM can be decomposed into a direct sum of an atomic and a non-atomic normalized POVM.

Definition 4.1. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces and X a measurable space. Two POVMs $\mu_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$, are called *mutually singular*, denoted $\mu_1 \perp \mu_2$, if there exist disjoint measurable subsets X_1 and X_2 of X such that $\mu_i(A) = \mu_i(A \cap X_i)$ for all $A \in \mathcal{O}(X)$.

The following proposition is a direct consequence of the classical case. It is a well-known fact that an atomic finite positive measure is always mutually singular to a non-atomic positive measure. We use it below.

Proposition 4.2. *Let $\mu_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$ be two POVMs such that μ_1 is atomic and μ_2 is non-atomic. Then they are mutually singular.*

Proof. Consider strictly positive density operators S_i on $\mathcal{H}_i$ such that $T \mapsto \text{tr}(S_i T)$ (tr denotes trace) are faithful normal states on $\mathcal{B}(\mathcal{H}_i)$ for $i = 1, 2$. Then $\lambda_i : \mathcal{O}(X) \to [0, \infty)$ defined by

$$\lambda_i(A) = \text{tr}(\mu_i(A) S_i) \quad \text{for all } A \in \mathcal{O}(X),$$

are positive measures which, for any $A \in \mathcal{O}(X)$ satisfy

$$\mu_i(A) = 0 \text{ if and only if } \lambda_i(A) = 0. \tag{4.1}$$

This in particular implies that λ_1 is atomic and λ_2 is non-atomic. Therefore, as mentioned above, λ_1 is mutually singular to λ_2 (Theorem 2.5, [24]). This in turn implies due to (4.1) that μ_1 is mutually singular to μ_2. $\qquad\square$

4.2. Disjoint spectral measures. Inspired by the notion of disjointness for representations of C^*-algebras (see [1,2]), we introduce a similar notion for spectral measures. We do not know whether this concept has been studied before. We establish here that singularity and disjointness of spectral measures are in fact same.

Let $\pi : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_\pi)$ be a spectral measure and let $\mathcal{H}$ be a closed subspace of $\mathcal{H}_\pi$ such that $\mathcal{H}$ is invariant (and hence reducing) under $\pi(A)$ for all $A \in \mathcal{O}(X)$. Then the mapping $A \mapsto \pi(A)_{|\mathcal{H}}$ gives rise to another spectral measure from $\mathcal{O}(X)$ to $\mathcal{B}(\mathcal{H})$, and is called a *sub-spectral measure* of π.

Definition 4.3. Two spectral measures $\pi_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_{\pi_i})$, $i = 1, 2$ are called *disjoint* if no non-zero sub-spectral measure of π_1 is unitarily equivalent to any sub-spectral measure of π_2.

Let $\lambda : \mathcal{O}(X) \to [0, \infty]$ be a σ-finite measure such that $L^2(\lambda)$ is a separable Hilbert space. Consider the map $\pi^\lambda : \mathcal{O}(X) \to \mathcal{B}(L^2(\lambda))$ defined by

$$\pi^\lambda(A) = M_{\chi_A} \quad \text{for all } A \in \mathcal{O}(X), \tag{4.2}$$

where M_{χ_A} is the multiplication operator by the characteristic function χ_A. It is straightforward to verify that π^λ is a spectral measure. Also $\pi^\lambda(A) = 0$ if and only if $\lambda(A) = 0$ for any $A \in \mathcal{O}(X)$. Such spectral measures are known as canonical spectral measures.

We first prove that the notion of singularity and disjointness are same in the case of canonical spectral measures. The proof here follows the same technique which is usually employed for representations (see Theorem 2.2.2, [2]).

Lemma 4.4. *Let λ_1 and λ_2 be two σ-finite positive measures on X. Then λ_1 is mutually singular to λ_2 if and only if π^{λ_1} and π^{λ_2} are disjoint.*

Proof. Let π^{λ_1} and π^{λ_2} be disjoint spectral measures. Assume to the contrary that λ_1 and λ_2 are not mutually singular. Then by Lebesgue decomposition theorem, there is a non-zero σ-finite positive measure, say λ, such that λ is absolutely continuous with respect to both λ_1 and λ_2. Using Radon–Nikodym derivative $\frac{d\lambda}{d\lambda_i}$ of λ with respect to λ_i, it is not hard to see that π^λ is unitarily equivalent to $\pi^{\lambda_i}(\cdot)_{|\mathcal{K}_i}$, where $\mathcal{K}_i = \operatorname{ran}(\pi^{\lambda_i}(C_i))$ and $C_i = \{x \in X; \frac{d\lambda}{d\lambda_i}(x) > 0\}$. It is clear that since $\lambda_i(C_i) \neq 0$, we have $\mathcal{K}_i \neq 0$ which contradicts disjointness of π^{λ_1} and π^{λ_2}. The proof of the converse is contained in the next theorem. $\square$

We use the familiar notion of direct sum in the next theorem and in subsequent results. The *direct sum* of a collection $\{\mu_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_i)\}_{i \in I}$ of POVMs is the map $\oplus_i \mu_i : \mathcal{O}(X) \to \mathcal{B}(\oplus_i \mathcal{H}_i)$ defined by

$$(\oplus_i \mu_i)(A) = \oplus_i \mu_i(A) \quad \text{for all } A \in \mathcal{O}(X). \tag{4.3}$$

It is immediate that $\oplus_i \mu_i$ is a POVM. Further it is normalized if and only if each μ_i is normalized. Also $\oplus_i \mu_i$ is a spectral measure if and only if each μ_i is a spectral measure. Similar to an equivalent criterion for disjointness of representations (Proposition 2.1.4, [2]), we have the following result for spectral measures. This also shows that the notions of singularity and disjointness are same.

Theorem 4.5. *Let $\pi_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_{\pi_i})$, $i = 1, 2$ be two spectral measures. Then the following are equivalent:*

(1) π_1 and π_2 are mutually singular.

(2) π_1 is disjoint to π_2.
(3) If for $T \in \mathcal{B}(\mathcal{H}_{\pi_1}, \mathcal{H}_{\pi_2})$, $T\pi_1(A) = \pi_2(A)T$ for all $A \in \mathcal{O}(X)$, then $T = 0$.

Proof. (1) $\implies$ (3): Let C_1 and C_2 be disjoint measurable subsets such that $\pi_i(A) = \pi_i(A \cap C_i)$ for all $A \in \mathcal{O}(X)$ and $i = 1, 2$. Let $T \in \mathcal{B}(\mathcal{H}_{\pi_1}, \mathcal{H}_{\pi_2})$ be such that $T\pi_1(A) = \pi_2(A)T$ for all $A \in \mathcal{O}(X)$. Then, since $\pi_1(C_1) = I_{\mathcal{H}_{\pi_1}}$ and $\pi_2(C_1) = 0$, it follows that

$$T = T\pi_1(C_1) = \pi_2(C_1)T = 0.$$

(3) $\implies$ (2): if π_1 and π_2 are not disjoint, then there are non-zero closed subspaces $\mathcal{K}_i$ of $\mathcal{H}_{\pi_i}$ invariant under $\pi_i(A)$ for all $A \in \mathcal{O}(X)$, and a unitary $U : \mathcal{K}_1 \to \mathcal{K}_2$ such that

$$U\pi_1(A)_{|\mathcal{K}_1} = \pi_2(A)_{|\mathcal{K}_2} U \quad \text{for all } A \in \mathcal{O}(X).$$

Extend U to $\mathcal{H}_{\pi_1}$ by assigning 0 on $\mathcal{H}_{\pi_1} \ominus \mathcal{K}_1$, and call it $\tilde{U}$. Then it is immediate that $\tilde{U} \neq 0$ and $\tilde{U}\pi_1(A) = \pi_2(A)\tilde{U}$ for all $A \in \mathcal{O}(X)$, violating the condition in part (3).

(2) $\implies$ (1): Let π_1 and π_2 be disjoint. By Hahn-Hellinger Theorem (Theorem 7.6, [30]), there exists a collection, say $\{\lambda_n^i\}_{n\in\mathbb{N}\cup\{\infty\}}$, of σ-finite positive measures (possibly zero measures) mutually singular to one another such that, upto unitary equivalence, we have

$$\pi_i = \bigoplus_{n\in\mathbb{N}\cup\{\infty\}} n \cdot \pi^{\lambda_n^i}$$

for $i = 1, 2$. Here $n \cdot \pi^{\lambda_n^i}$ denotes the direct sum of n copies of $\pi^{\lambda_n^i}$ (when $n = \infty$, the direct sum is countably infinite). Because π_1 and π_2 are disjoint, each $\pi^{\lambda_n^1}$ must be disjoint to $\pi^{\lambda_m^2}$ for $m, n \in \mathbb{N} \cup \{\infty\}$. It then follows from Lemma 4.4 that λ_n^1 is mutually singular to λ_m^2 as positive measures. Therefore for each n, m, there exist measurable subsets X_{nm}^1 and X_{nm}^2 satisfying $X_{nm}^1 \cap X_{nm}^2 = \emptyset$ and

$$\lambda_n^1(A) = \lambda_n^1(A \cap X_{nm}^1) \quad \text{and} \quad \lambda_m^2(A) = \lambda_m^2(A \cap X_{nm}^2),$$

for all $A \in \mathcal{O}(X)$. Set

$$X^1 = \cup_n \cap_m X_{nm}^1 \text{ and } X^2 = \cup_m \cap_n X_{nm}^2.$$

Then by usual set theory rules:

$$X^1 \cap X^2 = \left(\cup_n \cap_m X_{nm}^1\right) \cap \left(\cup_k \cap_l X_{lk}^2\right) = \cup_n \cup_k \left[\left(\cap_m X_{nm}^1\right) \cap \left(\cap_l X_{lk}^2\right)\right]$$
$$\subseteq \cup_n \cup_k \left(X_{nk}^1 \cap X_{nk}^2\right) = \emptyset,$$

by using $X_{nk}^1 \cap X_{nk}^2 = \emptyset$. Further for any $A \in \mathcal{O}(X)$ and fixed n, since $\lambda_n^1(A \cap X_{nm}^1) = \lambda_n^1(A)$ for all m, we have

$$\lambda_n^1(A) \geq \lambda_n^1(A \cap X^1) \geq \lambda_n^1(\cap_m \left(A \cap X_{nm}^1\right)) = \lim_l \lambda_n^1 \left(\cap_{m=1}^l (A \cap X_{nm}^1)\right) = \lambda_n^1(A),$$

where limit is taken in WOT. This implies $\lambda_n^1(A \cap X^1) = \lambda_n^1(A)$. Similarly, we get $\lambda_m^2(A \cap X^2) = \lambda_m^2(A)$ for each m. Put differently, we obtain $\pi^{\lambda_n^i}(A \cap X^i) = \pi^{\lambda_n^i}(A)$, which further implies that

$$\pi_i(A \cap X^i) = \bigoplus_{n \in \mathbb{N} \cup \{\infty\}} n \cdot \pi^{\lambda_n^i}(A \cap X^i) = \bigoplus_{n \in \mathbb{N} \cup \{\infty\}} n \cdot \pi^{\lambda_n^i}(A) = \pi_i(A),$$

for each $A \in \mathcal{O}(X)$ and $i = 1, 2$. Since X^1 and X^2 are disjoint, we conclude that π_1 is mutually singular to π_2. $\qquad\square$

Remark 4.6. In Theorem 4.5, we assumed that the spectral measures act on separable Hilbert spaces. But the implication (1) $\implies$ (3) is true even for non-separable Hilbert spaces and the proof is similar. To see this, let $\pi_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{K}_i)$, $i = 1, 2$ be two mutually singular spectral measures concentrated on measurable subsets X_i with $X_1 \cap X_2 = \emptyset$. Here $\mathcal{K}_i$ need not be separable. Let $T \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ be such that $T \pi_1(A) = \pi_2(A)T$ for all $A \in \mathcal{O}(X)$. Then, since $\pi_1(X_1) = \pi_1(X) = I_{\mathcal{K}_1}$ and $\pi_2(X_1) = 0$, we have $T = T \pi_1(X_1) = \pi_2(X_1)T = 0$. We use this fact in the next theorem.

4.3. Direct sums and C^*-extreme points. We explore the properties of being C^*-extreme or extreme for direct sums of mutually singular POVMs. Generally it is enough to look at individual components to obtain the same property for direct sums.

Theorem 4.7. *Let $\{\mu_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_i)\}_{i \in I}$ be a countable collection of normalized POVMs for some indexing set I such that μ_i and μ_j are mutually singular for $i \neq j$ in I. Then $\mu = \oplus_i \mu_i$ is C^*-extreme (extreme) in $\mathcal{P}_{\oplus_i \mathcal{H}_i}(X)$ if and only if each μ_i is C^*-extreme (extreme) in $\mathcal{P}_{\mathcal{H}_i}(X)$.*

Proof. For each $i \in I$, let $(\pi_i, V_i, \mathcal{H}_{\pi_i})$ be the minimal Naimark dilation for μ_i. Set $\mathcal{H} = \oplus_i \mathcal{H}_i$ and $\mathcal{H}_\pi = \oplus_i \mathcal{H}_{\pi_i}$ and let $\pi = \oplus_i \pi_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_\pi)$ and $V = \oplus_i V_i : \mathcal{H} \to \mathcal{H}_\pi$. Clearly π is a spectral measure and V is an isometry. It is straightforward to check that

$$[\pi(\mathcal{O}(X))V\mathcal{H}] = \mathcal{H}_\pi \text{ and } \mu(A) = V^*\pi(A)V \quad \text{for all } A \in \mathcal{O}(X).$$

This implies that $(\pi, V, \mathcal{H}_\pi)$ is the minimal Naimark dilation for μ. Also for $i \neq j$ in I, since μ_i is mutually singular to μ_j, it follows from Proposition 2.6 that π_i is mutually singular to π_j. Now we claim that

$$\pi(\mathcal{O}(X))' = \oplus_i \pi_i(\mathcal{O}(X))' = \left\{ \oplus_i S_i; \ S_i \in \pi_i(\mathcal{O}(X))' \right\}.$$

Let $S \in \pi(\mathcal{O}(X))' \subseteq \mathcal{B}(\oplus_i \mathcal{H}_{\pi_i})$. Then $S = [S_{ij}]$ for some $S_{ij} \in \mathcal{B}(\mathcal{H}_{\pi_j}, \mathcal{H}_{\pi_i})$. For any $A \in \mathcal{O}(X)$, therefore

$$[S_{ij}] (\oplus_i \pi_i(A)) = (\oplus_i \pi_i(A)[S_{ij}]$$

that is

$$[S_{ij}\pi_j(A)] = [\pi_i(A)S_{ij}]$$

and hence

$$S_{ij}\pi_j(A) = \pi_i(A)S_{ij} \quad \text{for all } i, j \in I.$$

In particular, this says that $S_{ii} \in \pi_i(\mathcal{O}(X))'$ for all $i \in I$. Also since π_i and π_j are mutually singular for $i \neq j$, it follows from Remark 4.6 that

$$S_{ij} = 0 \quad \text{for } i \neq j.$$

Thus

$$S = [S_{ij}] = \oplus_i S_{ii} \in \oplus_i \pi_i(\mathcal{O}(X))'.$$

This proves that $\pi(\mathcal{O}(X))' \subseteq \oplus_i \pi_i(\mathcal{O}(X))'$. The other inclusion of our claim is obvious.

In order to prove the equivalent assertions of C^*-extremity, we use the claim above and the necessary and sufficient criterion of Theorem 3.3 throughout the proof without always mentioning them. First assume that μ is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X)$. Fix $j \in I$ and let $D_j \in \pi_j(\mathcal{O}(X))'$ be positive such that $V_j^* D_j V_j$ is invertible. Define

$$D = \oplus_i D_i$$

by assigning $D_i = I_{\mathcal{H}_{\pi_i}}$ for $i \neq j$. It is clear that D is positive and $D \in \pi(\mathcal{O}(X))'$. Since

$$V^* D V = \oplus_i V_i^* D_i V_i$$

and $V_i^* D_i V_i$ is invertible for all i, it follows that $V^* D V$ is invertible. Therefore, as μ is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X)$, we get a co-isometry $U \in \pi(\mathcal{O}(X))'$ with $U^* U D^{1/2} = D^{1/2}$ and an invertible operator $T \in \mathcal{B}(\mathcal{H})$ such that $U D^{1/2} V = V T$. Then $T = [T_{ij}]$ for some $T_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$ and $U = \oplus_i U_i$ for $U_i \in \pi_i(\mathcal{O}(X))'$. Since U is a co-isometry, each U_i is a co-isometry. Also, since

$$\oplus_i U_i^* U_i D_i^{1/2} = \left(\oplus_i U_i^*\right)\left(\oplus_i U_i\right)\left(\oplus_i D_i^{1/2}\right) = U^* U D^{1/2} = D^{1/2} = \oplus_i D_i^{1/2},$$

it follows that

$$U_i^* U_i D_i^{1/2} = D_i^{1/2} \quad \text{for all } i.$$

Further, since

$$\oplus_i U_i D_i^{1/2} V_i = U D^{1/2} V = V T = (\oplus_i V_i)[T_{ij}] = [V_i T_{ij}], \qquad (4.4)$$

it follows for $i \neq j$ that, $V_i T_{ij} = 0$ and hence $T_{ij} = V_i^* V_i T_{ij} = 0$. This amounts to saying that $T = \oplus_i T_{ii}$ and its invertibility, in particular, implies that T_{jj} is invertible in $\mathcal{B}(\mathcal{H}_j)$. Also (4.4) yields

$$U_j D_j^{1/2} V_j = V_j T_{jj}.$$

As U_j is a co-isometry in $\pi_j(\mathcal{O}(X))'$ satisfying $U_j^* U_j D_j^{1/2} = D_j^{1/2}$ and T_{jj} is invertible in $\mathcal{B}(\mathcal{H}_j)$ such that $U_j D_j^{1/2} V_j = V_j T_{jj}$, we conclude that μ_j is C^*-extreme in $\mathcal{P}_{\mathcal{H}_j}(X)$.

Conversely, assume that each μ_i is C^*-extreme in $\mathcal{P}_{\mathcal{H}_i}(X)$. Let $D \in \pi(\mathcal{O}(X))'$ be positive such that $V^* D V$ is invertible. Then D is of the form $\oplus_i D_i$ for some $D_i \in \pi_i(\mathcal{O}(X))'$. Clearly each D_i is positive. Since $V^* D V$ is invertible and $V^* D V = \oplus_i V_i^* D_i V_i$, it follows that $V_i^* D_i V_i$ is invertible for all $i \in I$. Again, as μ_i is C^*-extreme in $\mathcal{P}_{\mathcal{H}_i}(X)$,

we obtain a co-isometry $U_i \in \pi_i(\mathcal{O}(X))'$ with $U_i^* U_i D_i^{1/2} = D_i^{1/2}$ and an invertible operator $T_i \in \mathcal{B}(\mathcal{H}_i)$ such that $U_i D_i^{1/2} V_i = V_i T_i$. Set

$$U = \oplus_i U_i \quad \text{and} \quad T = \oplus_i T_i.$$

Then $U \in \pi(\mathcal{O}(X))'$ and U is a co-isometry, as each U_i is a co-isometry. Likewise T is invertible in $\mathcal{B}(\mathcal{H})$, since each T_i is invertible. Further we note that

$$U^* U D^{1/2} = \oplus_i U_i^* U_i D_i^{1/2} = \oplus_i D_i^{1/2} = D^{1/2}.$$

Similarly we get

$$U D^{1/2} V = \oplus_i U_i D_i^{1/2} V_i = \oplus_i V_i T_i = V T.$$

Thus we conclude that μ is C^*-extreme in $\mathcal{P}_\mathcal{H}(X)$.

The case of equivalent assertions of extremity can be proved similarly, using the claim above and Theorem 2.9. Assume that μ is extreme in $\mathcal{P}_\mathcal{H}(X)$. Fix $j \in I$ and let $D_j \in \pi_j(\mathcal{O}(X))'$ be such that $V_j^* D_j V_j = 0$. Define

$$D = \oplus_i D_i$$

by assigning $D_i = 0$ for $i \neq j$. Clearly then $V^* D V = 0$. Hence, as μ is extreme in $\mathcal{P}_\mathcal{H}(X)$, it follows that $D = 0$, which in particular implies $D_j = 0$. This proves that μ_j is extreme in $\mathcal{P}_{\mathcal{H}_j}(X)$.

For the converse, assume that each μ_i is extreme in $\mathcal{P}_{\mathcal{H}_i}(X)$. Let $D \in \pi(\mathcal{O}(X))'$ be such that $V^* D V = 0$. Again by the claim above, we have $D = \oplus_i D_i$ for some $D_i \in \pi_i(\mathcal{O}(X))'$. Also the expression $V^* D V = 0$ implies

$$V_i^* D_i V_i = 0 \quad \text{for each } i.$$

Hence as μ_i is extreme, it follows that $D_i = 0$ for each i, which in turn shows $D = 0$. Thus we conclude that μ is extreme in $\mathcal{P}_\mathcal{H}(X)$. The proof is now complete. $\square$

The following corollary is just an explicit description of the theorem above.

Corollary 4.8. *Let $\mu \in \mathcal{P}_\mathcal{H}(X)$ and let $\{B_i\}$ be a collection of disjoint measurable subsets such that $X = \cup_i B_i$ and $\mu(B_i)$ is a projection for each i. Let $\mathcal{H}_i = \text{ran}(\mu(B_i))$ and define $\mu_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_i)$ by $\mu_i(A) = \mu(B_i \cap A)_{|\mathcal{H}_i}$ for all $A \in \mathcal{O}(X)$. Then μ is C^*-extreme (extreme) in $\mathcal{P}_\mathcal{H}(X)$ if and only if each μ_i is C^*-extreme (extreme) in $\mathcal{P}_{\mathcal{H}_i}(X)$.*

Proof. Let $(\pi, V, \mathcal{H}_\pi)$ be the minimal Naimark dilation for μ. Since $\mu(B_i)$ is a projection for each i and B_i's are mutually disjoint, it follows from Proposition 2.4 that $\mu(B_i)$'s are mutually orthogonal projections. Also each $\mathcal{H}_i$ is a reducing subspace for all $\mu(A)$, $A \in \mathcal{O}(X)$ by Proposition 2.4 and hence μ_i is a well-defined normalized POVM. Further, since $X = \cup_i B_i$, we have $\mathcal{H} = \oplus_i \mathcal{H}_i$ and $\mu = \oplus_i \mu_i$. The assertions now are direct consequence of Theorem 4.7. $\square$

As we mentioned earlier in Theorem 2.13, every POVM decomposes uniquely as a sum of atomic and non-atomic POVMs. Additionally if μ is C^*-extreme then we show that this decomposition can be made into a direct sum of atomic and non-atomic POVMs such that each of the components is C^*-extreme. The following theorem effectively provides a proof of Theorem 2.13 and then discusses its role in identifying C^*-extreme POVMs. The proof here follows almost the same procedure which can be found in [24,27].

Theorem 4.9. *Let μ be a C^*-extreme point in $\mathcal{P}_\mathcal{H}(X)$. Then $\mu = \mu_1 \oplus \mu_2$ where μ_1 is an atomic normalized POVM and μ_2 is a non-atomic normalized POVM and they are mutually singular. Such a decomposition is unique. Furthermore μ_1 and μ_2 are C^*-extreme and in particular μ_1 is spectral.*

Proof. Let $\{B_j\}_{j \in J}$ be a maximal collection of mutually disjoint atoms for μ, which exists due to Zorn's lemma. As in the proof of Theorem 3.11, since μ is C^*-extreme, we note using Lemma 3.10 that $\mu(B_j)$ is a projection for each j. Also $\{\mu(B_j)\}_{j \in J}$ are mutually orthogonal by Proposition 2.4. Since $\mathcal{H}$ is separable, it follows that J is countable. This further implies that if we set $X_1 = \cup_{j \in J} B_j$, then since

$$\mu(X_1) = \sum_{j \in J} \mu(B_j), \tag{4.5}$$

$\mu(X_1)$ is a projection. Now set $X_2 = X \backslash X_1$. For $i = 1, 2$, let $\mathcal{H}_i = \text{ran}(\mu(X_i))$, and define the operator valued measures $\mu_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_i)$ by

$$\mu_i(A) = \mu(A \cap X_i)_{|\mathcal{H}_i} = \mu(A)_{|\mathcal{H}_i} \quad \text{for all } A \in \mathcal{O}(X).$$

It is clear that each μ_i is a normalized POVM. Also $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and

$$\mu = \mu_1 \oplus \mu_2.$$

Now we show that μ_1 is atomic. Assume that $\mu_1(A) \neq 0$ for some $A \in \mathcal{O}(X)$. Then $\mu(A \cap X_1) \neq 0$ and, since

$$\mu(A \cap X_1) = \sum_{j \in J} \mu(A \cap B_j),$$

it follows that $\mu(A \cap B_j) \neq 0$ for some j and hence $\mu_1(A \cap B_j) \neq 0$. Therefore, as B_j is an atom for μ, $A \cap B_j$ is an atom for μ. Consequently, as $\mu_1(A \cap B_j) \neq 0$, it follows that $A \cap B_j$ is an atom for μ_1. Thus we have got an atom contained in the subset A with $\mu_1(A) \neq 0$, which shows that μ_1 is atomic. To prove that μ_2 is non-atomic, let if possible, A be an atom for μ_2. Since μ_2 is concentrated on X_2, $A \cap X_2$ is an atom for μ_2 and hence $A \cap X_2$ is an atom for μ. But then $\{B_j\}_{j \in J} \cup \{A \cap X_2\}$ is a collection of mutually disjoint atoms for μ, violating the maximality of the collection $\{B_j\}_{j \in J}$. Thus we conclude that μ_2 is non-atomic. It is clear that μ_1 and μ_2 are mutually singular.

To show the uniqueness, let $\nu_1 \oplus \nu_2$ be another such decomposition with atomic $\nu_1 \in \mathcal{P}_{\mathcal{K}_1}(X)$ and non-atomic $\nu_2 \in \mathcal{P}_{\mathcal{K}_2}(X)$ where $\mathcal{H} = \mathcal{K}_1 \oplus \mathcal{K}_2$. We shall show that $\mathcal{K}_i = \mathcal{H}_i$ and $\nu_i = \mu_i$ for $i = 1, 2$. Let Y_1 and Y_2 be disjoint measurable subsets such that $\nu_i(A) = \nu_i(A \cap Y_i)$ for all $A \in \mathcal{O}(X)$. We know from Proposition 4.2 that $\mu_1 \perp \nu_2$ and $\mu_2 \perp \nu_1$ and so Y_1 and Y_2 can be chosen so that $Y_1 \cap X_2 = Y_2 \cap X_1 = \emptyset$. Therefore for each $i = 1, 2$, since both μ_i and ν_i are concentrated on $X_i \cup Y_i$ and $(X_1 \cup Y_1) \cap (X_2 \cup Y_2) = \emptyset$, we can assume without loss of generality, that $X_i = Y_i$ (just replace X_i, Y_i by $X_i \cup Y_i$). Further note that

$$I_{\mathcal{K}_i} = \nu_i(Y_i) = \mu(Y_i)_{|\mathcal{K}_i} = \mu(X_i)_{|\mathcal{K}_i} = P_{\mathcal{H}_i|_{\mathcal{K}_i}},$$

where $P_{\mathcal{H}_i}$ denotes the projection of $\mathcal{H}$ onto $\mathcal{H}_i$. This implies $\mathcal{K}_i \subseteq \mathcal{H}_i$. By symmetry, we have $\mathcal{H}_i \subseteq \mathcal{K}_i$. Hence $\mathcal{K}_i = \mathcal{H}_i$. Similarly for all $A \in \mathcal{O}(X)$, we get

$$\nu_i(A) = \nu_i(A \cap Y_i) = \mu(A \cap Y_i)_{|\mathcal{K}_i} = \mu(A \cap X_i)_{|\mathcal{H}_i} = \mu_i(A \cap X_i) = \mu_i(A)$$

showing that $\nu_i = \mu_i$. The second statement follows from Theorem 4.7 and Theorem 3.11. $\qquad \square$

Remark 4.10. In the theorem above, we cannot expect a similar kind of direct sum decomposition for a normalized POVM which is not C^*-extreme. To see an example, let λ_1 and λ_2 be two probability measures on some measurable space X such that λ_1 is atomic while λ_2 is non-atomic. Let $T \in \mathcal{B}(\mathcal{H})$ be a positive contraction which is not a projection. Consider the POVM $\mu \in \mathcal{P}_{\mathcal{H}}(X)$ defined by $\mu(\cdot) = \lambda_1(\cdot)T + \lambda_2(\cdot)(I_{\mathcal{H}} - T)$. One can easily verify that no decomposition of μ into a direct sum of atomic and non-atomic normalized POVMs exists.

One reason for us to study the notion of mutually singular POVMs is the following result. Its proof follows from Theorem 4.9 and Theorem 4.7. Since we have already characterized all atomic C^*-extreme points (Theorem 3.11), it says in particular that it is sufficient to look for the characterization of non-atomic C^*-extreme points to understand the general situation.

Corollary 4.11. *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a normalized POVM and let $X_1 = \cup_{i \in I} B_i$ be the union of a maximal collection $\{B_i\}_{i \in I}$ of mutually disjoint atoms for μ. Let $X_2 = X \backslash X_1$. Then μ is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X)$ if and only if*

(1) the operators $\mu(X_1)$ and $\mu(X_2)$ are projections and,
(2) $\mu = \mu_1 \oplus \mu_2$ such that μ_i is C^*-extreme in $\mathcal{P}_{\mathcal{H}_i}(X)$,
  where $\mathcal{H}_i = \mathrm{ran}(\mu(X_i))$ and $\mu_i = \mu(\cdot)_{|\mathcal{H}_i}$ for $i = 1, 2$.

5. Measure Isomorphic POVMs

We digress a bit from the earlier developments and explore C^*-extreme properties from the perspective of measure isomorphism. In classical measure theory, this notion has been examined extensively. The idea is to neglect measure zero subsets in considering isomorphisms. One consequence is that most questions about abstract measure spaces get reduced to questions about sub σ-algebras of the Borel σ-algebra of the unit interval $[0, 1]$. In a sense this space is universal.

Measure isomorphism for POVMs seems to have been first studied in [12]. Our aim here is quite limited to investigate preservation of some natural properties of POVMs, especially C^*-extremity, under this isomorphism. Here too we see the role of the unit interval.

Let X be a measurable space and $\mathcal{H}$ a Hilbert space. Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM. For each $A \in \mathcal{O}(X)$, let $[A]_\mu$ denote the set

$$[A]_\mu := \{B \in \mathcal{O}(X); \mu(A \backslash B) = 0 = \mu(B \backslash A)\}$$
$$= \{B \in \mathcal{O}(X); \mu(B) = \mu(A) = \mu(B \cap A)\}.$$

Consider

$$\Sigma(\mu) := \big\{[A]_\mu; A \in \mathcal{O}(X)\big\}.$$

Then $\Sigma(\mu)$ is a Boolean σ-algebra under the following operations:

$$[A]_\mu \backslash [B]_\mu = [A \backslash B]_\mu \tag{5.1}$$
$$[A]_\mu \cap [B]_\mu = [A \cap B]_\mu \tag{5.2}$$

for any $A, B \in \mathcal{O}(X)$. Define $\tilde{\mu} : \Sigma(\mu) \to \mathcal{B}(\mathcal{H})$ by

$$\tilde{\mu}([A]_\mu) = \mu(A) \quad \text{for all } A \in \mathcal{O}(X),$$

which is well defined by virtue of the very definition of $[A]_\mu$. If there is no possibility of confusion, we shall still denote $\tilde{\mu}$ by μ only.

Definition 5.1 ([12]). For $i = 1, 2$, let X_i be two measurable spaces and let $\mathcal{H}$ be a Hilbert space. Two POVMs $\mu_i : \mathcal{O}(X_i) \to \mathcal{B}(\mathcal{H})$ are called *measure isomorphic* and denoted by $\mu_1 \cong \mu_2$, if there exists a Boolean isomorphism $\Phi : \Sigma(\mu_1) \to \Sigma(\mu_2)$ i.e. Φ is bijective and both Φ and Φ^{-1} preserve the operations in (5.1) and (5.2):

$$\Phi\left([A_1]_{\mu_1} \backslash [B_1]_{\mu_1}\right) = \Phi([A_1]_{\mu_1}) \backslash \Phi([B_1]_{\mu_1}),$$
$$\Phi\left([A_1]_{\mu_1} \cap [B_1]_{\mu_1}\right) = \Phi\left([A_1]_{\mu_1}\right) \cap \Phi\left([B_1]_{\mu_1}\right) \; etc. \tag{5.3}$$

such that $\mu_1(A_1) = \mu_2\left(\Phi([A_1]_{\mu_1})\right)$ for all $A_1, B_1 \in \mathcal{O}(X_1)$.

The following theorem compares some natural properties of POVMs under measure isomorphism.

Theorem 5.2. *Let $\mu_i : \mathcal{O}(X_i) \to \mathcal{B}(\mathcal{H})$, $i = 1, 2$ be two normalized POVMs such that they are measure isomorphic. Then we have the following:*

(1) μ_1 is a spectral measure if and only if μ_2 is a spectral measure.
(2) μ_1 is atomic (non-atomic) if and only if μ_2 is atomic (non-atomic).
(3) μ_1 is C^*-extreme (extreme) in $\mathcal{P}_{\mathcal{H}}(X_1)$ if and only if μ_2 is C^*-extreme (extreme) in $\mathcal{P}_{\mathcal{H}}(X_2)$.

Proof. Let $\Phi : \Sigma(\mu_1) \to \Sigma(\mu_2)$ be a Boolean isomorphism satisfying $\mu_1(A_1) = \mu_2(\Phi([A]_{\mu_1}))$ for all $A_1 \in \mathcal{O}(X_1)$. By symmetry, it is enough to prove the statements in just one direction.

(1) This is straightforward by isomorphism. If μ_2 is a spectral measure then for any $A_1 \in \mathcal{O}(X_1)$, $\mu_2(\Phi([A_1]_{\mu_1}))$ is a projection. Since $\mu_1(A_1) = \mu_2\left(\Phi([A_1]_{\mu_1})\right)$, it follows that $\mu_1(A_1)$ is a projection and hence μ_1 is a spectral measure.

(2) Firstly we claim that if A_1 is an atom for μ_1, then A_2 is an atom for μ_2 for any $A_2 \in \Phi([A_1]_{\mu_1})$. To see this, first note that $\mu_2(A_2) = \mu_1(A_1) \neq 0$. Let $A_2' \subseteq A_2$ be a measurable subset. Then for any $A_1' \in \Phi^{-1}([A_2']_{\mu_2})$, we have

$$\Phi\left([A_1' \cap A_1]_{\mu_1}\right) = \Phi\left([A_1']_{\mu_1}\right) \cap \Phi\left([A_1]_{\mu_1}\right) = [A_2']_{\mu_2} \cap [A_2]_{\mu_2}$$
$$= [A_2' \cap A_2]_{\mu_2} = [A_2']_{\mu_2} = \Phi([A_1']_{\mu_1})$$

and hence $[A_1' \cap A_1]_{\mu_1} = [A_1']_{\mu_1}$, which in turn implies

$$\mu_1(A_1' \cap A_1) = \mu_1(A_1'). \tag{5.4}$$

But since A_1 is atomic for μ_1, we have

$$\text{either } \mu_1(A_1' \cap A_1) = 0 \;\; \text{or} \;\; \mu_1(A_1' \cap A_1) = \mu_1(A_1)$$

and therefore from (5.4),

$$\text{either } \mu_1(A_1') = 0 \;\; \text{or} \;\; \mu_1(A_1') = \mu_1(A_1).$$

Since $A_1 \in \Phi^{-1}([A_2]_{\mu_2})$ and $A_1' \in \Phi^{-1}([A_2']_{\mu_2})$, it follows that

$$\text{either } \mu_2(A_2') = 0 \;\; \text{or} \;\; \mu_2(A_2') = \mu_2(A_2).$$

This shows our claim that A_2 is an atom for μ_2.

Now assume that μ_1 is atomic. To show that μ_2 is atomic, let $A_2 \in \mathcal{O}(X_2)$ be such that $\mu_2(A_2) \neq 0$. If $A_1 \in \Phi^{-1}([A_2]_{\mu_2})$, then $\mu_1(A_1) = \mu_2(A_2) \neq 0$. Since μ_1 is

atomic, A_1 contains an atom for μ_1, say A_1'. Fix $A_2' \in \Phi([A_1']_{\mu_1})$. Then A_2' is an atom for μ_2 by the claim above. As above we show that $\mu_2(A_2' \cap A_2) = \mu_2(A_2')$, which implies that $A_2' \cap A_2$ is an atom for μ_2 contained in A_2. This proves that μ_2 is atomic. Similarly if μ_1 is non-atomic, then there is no atom for μ_1, and again it follows from the claim above that there is no atom for μ_2, which is equivalent to saying that μ_2 is non-atomic.

(3) Assume that μ_2 is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X_2)$. To show that μ_1 is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X_1)$, let $\mu_1(\cdot) = \sum_{j=1}^n T_j^* \mu_1^j(\cdot) T_j$ be a proper C^*-convex combination in $\mathcal{P}_{\mathcal{H}}(X_1)$. For each j, define $\mu_2^j : \mathcal{O}(X_2) \to \mathcal{B}(\mathcal{H})$ by

$$\mu_2^j(A_2) = \mu_1^j\left(\Phi^{-1}\left([A_2]_{\mu_2}\right)\right) \quad \text{for all } A_2 \in \mathcal{O}(X_2).$$

For μ_2^j to be well defined, we need to show that $\mu_1^j(A_1) = \mu_1^j(A_1')$ for any $A_1, A_1' \in \Phi^{-1}([A_2]_{\mu_2})$. So fix $A_1, A_1' \in \Phi^{-1}([A_2]_{\mu_2})$. Then $[A_1]_{\mu_1} = [A_1']_{\mu_1}$ and hence, we get

$$\mu_1(A_1 \backslash A_1') = 0 = \mu_1(A_1' \backslash A_1).$$

Therefore, since $T_j^* \mu_1^j(\cdot) T_j \leq \mu_1(\cdot)$, it follows that

$$T_j^* \mu_1^j(A_1 \backslash A_1') T_j = 0 = T_j^* \mu_1^j(A_1' \backslash A_1) T_j$$

which, as T_j is invertible, yields

$$\mu_1^j(A_1 \backslash A_1') = 0 = \mu_1^j(A_1' \backslash A_1).$$

This implies the requirement for well-definedness of μ_2^j. Also note that

$$\mu_1^j(A_1) = \mu_1^j\left(\Phi^{-1}(\Phi([A_1]_{\mu_1})\right) = \mu_2^j(\Phi([A_1]_{\mu_1})), \tag{5.5}$$

for all $A_1 \in \mathcal{O}(X_1)$. Further for any $A_2 \in \mathcal{O}(X_2)$, we have

$$\sum_{j=1}^n T_j^* \mu_2^j(A_2) T_j = \sum_{j=1}^n T_j^* \mu_1^j\left(\Phi^{-1}([A_2]_{\mu_2})\right) T_j = \mu_1\left(\Phi^{-1}([A_2]_{\mu_2})\right) = \mu_2(A_2).$$

Subsequently, since μ_2 is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X_2)$, there exists an unitary operator $U_j \in \mathcal{B}(\mathcal{H})$ such that $\mu_2(\cdot) = U_j^* \mu_2^j(\cdot) U_j$ for each j. It then follows for all $A_1 \in \mathcal{O}(X_1)$, that

$$\mu_1(A_1) = \mu_2(\Phi([A_1]_{\mu_1})) = U_j^* \mu_2^j(\Phi([A_1]_{\mu_1})) U_j = U_j^* \mu_1^j(A_1) U_j,$$

where the last equality is due to (5.5). This proves that μ_1 is unitarily equivalent to each μ_1^j which consequently implies that μ_1 is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X_1)$. That μ_1 is extreme if and only if μ_2 is extreme follows similarly. □

In the proof of part (2) of the theorem above, we observed the following:

Proposition 5.3. *Let $\mu_i : \mathcal{O}(X_i) \to \mathcal{B}(\mathcal{H})$, $i = 1, 2$ be two measure isomorphic POVMs with Boolean isomorphism $\Phi : \Sigma(\mu_1) \to \Sigma(\mu_2)$. Then A_1 is an atom for μ_1 if and only if any representative of $\Phi([A_1]_{\mu_1})$ is an atom for μ_2.*

Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM. We say μ is *countably generated* if there exists a countable collection of subsets $\mathcal{F} \subseteq \mathcal{O}(X)$ such that for any $A \in \mathcal{O}(X)$, there exists $B \in \sigma(\mathcal{F})$ satisfying $[A]_\mu = [B]_\mu$. Here $\sigma(\mathcal{F})$ denotes the σ-algebra generated by $\mathcal{F}$. The following result has been borrowed from [3].

Theorem 5.4 (Proposition 59, [3]). *If $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ is a countably generated POVM, then μ is measure isomorphic to a POVM $\nu : \mathcal{O}([0, 1]) \to \mathcal{B}(\mathcal{H})$.*

Recall that when X is a separable metric space, then $\mathcal{O}(X)$ is its Borel σ-algebra and in this case, any POVM on X is countably generated. What the theorem above basically says is that, to study C^*-extreme points in $\mathcal{P}_\mathcal{H}(X)$ for a separable metric space X, it is sufficient to just characterize the C^*-extreme points in $\mathcal{P}_\mathcal{H}([0, 1])$ in view of Theorem 5.2. This result will also help us find an example (see Example 7.9) of a C^*-extreme point in $\mathcal{P}_\mathcal{H}(X)$ which is not spectral, when $\mathcal{H}$ is infinite dimensional.

Now we consider measure isomorphism of POVMs induced from a bimeasurable map. Recall that for measurable spaces X_1 and X_2, a function $f : X_1 \to X_2$ is called *measurable* if $f^{-1}(A_2) \in \mathcal{O}(X_1)$ whenever $A_2 \in \mathcal{O}(X_2)$. Note that for any measurable space X and a measurable subset $Y \subseteq X$, Y itself inherits the natural measurable space structure from X with the σ algebra $\{A \cap Y; A \in \mathcal{O}(X)\}$.

Theorem 5.5. *For $i = 1, 2$, let X_i be two measurable spaces and let $Y_i \subseteq X_i$ be measurable subsets. Let $f : Y_1 \to Y_2$ be a bijective map such that both f and f^{-1} are measurable. Given a normalized POVM $\mu_1 : \mathcal{O}(X_1) \to \mathcal{B}(\mathcal{H})$ satisfying $\mu_1(A_1) = \mu_1(A_1 \cap Y_1)$ for all $A_1 \in \mathcal{O}(X_1)$, define $\mu_2 : \mathcal{O}(X_2) \to \mathcal{B}(\mathcal{H})$ by $\mu_2(A_2) = \mu_1\left(f^{-1}(A_2 \cap Y_2)\right)$ for all $A_2 \in \mathcal{O}(X_2)$. Then μ_1 and μ_2 are measure isomorphic.*

Proof. We claim that the map $\Phi : \Sigma(\mu_1) \to \Sigma(\mu_2)$ defined by

$$\Phi([A_1]_{\mu_1}) = [f(A_1 \cap Y_1)]_{\mu_2} \quad \text{for all } A_1 \in \mathcal{O}(X_1), \tag{5.6}$$

is a Boolean isomorphism. First note that

$$\mu_1(A_1) = \mu_1(A_1 \cap Y_1) = \mu_1\left(f^{-1}(f(A_1 \cap Y_1))\right) = \mu_2(f(A_1 \cap Y_1)) \tag{5.7}$$

for all $A_1 \in \mathcal{O}(X_1)$. This implies that $\mu_1(A_1) = 0$ if and only if $\mu_2(f(A_1 \cap Y_1)) = 0$ for any $A_1 \in \mathcal{O}(X_1)$. Therefore if $[A_1]_{\mu_1} = [A_1']_{\mu_1}$ for some $A_1, A_1' \in \mathcal{O}(X_1)$, then $[f(A_1 \cap Y_1)]_{\mu_2} = [f(A_1' \cap Y_1)]_{\mu_2}$. This proves the well-definedness of Φ. Similarly by symmetry, we prove that Φ is injective. That Φ is onto is straightforward by noting that

$$\Phi\left([f^{-1}(A_2 \cap Y_2)]_{\mu_1}\right) = [A_2 \cap Y_2]_{\mu_2} = [A_2]_{\mu_2}$$

for any $A_2 \in \mathcal{O}(X_2)$. This shows that Φ is a Boolean isomorphism as claimed. Further from (5.6) and (5.7), we have

$$\mu_2(\Phi([A_1]_{\mu_1})) = \mu_2(f(A_1 \cap Y_1)) = \mu_1(A_1)$$

for any $A_1 \in \mathcal{O}(X_1)$. Thus we conclude that μ_1 and μ_2 are measure isomorphic. $\quad\square$

Now we apply these results to the study of C^*-extreme POVMs. Consider the map $g : [0, 1) \to \mathbb{T}$ given by $g(t) = e^{2\pi i t}$ for $t \in [0, 1)$, where $\mathbb{T}$ is the unit circle. It is clear that g is a bijective map such that both g and g^{-1} are Borel measurable. Therefore for any Hilbert space $\mathcal{H}$, normalized POVMs $\mu \in \mathcal{P}_\mathcal{H}([0, 1])$ with $\mu(\{1\}) = 0$ are in one-to-one correspondence with $\mathcal{P}_\mathcal{H}(\mathbb{T})$ through measure isomorphism, by Theorem 5.5. In particular, since singletons under non-atomic POVMs have zero measure, it follows that non-atomic POVMs in $\mathcal{P}_\mathcal{H}([0, 1])$ are measure isomorphic to non-atomic POVMs in $\mathcal{P}_\mathcal{H}(\mathbb{T})$.

Next if X is a separable metric space, then non-atomic POVMs in $\mathcal{P}_\mathcal{H}(X)$ are measure isomorphic to non-atomic POVMs in $\mathcal{P}_\mathcal{H}([0, 1])$ from Theorem 5.4 and Theorem 5.2, which in turn are measure isomorphic to non-atomic POVMs in $\mathcal{P}_\mathcal{H}(\mathbb{T})$ as seen above. Thus we conclude in view of Theorem 5.2 that, characterizing the non-atomic C^*-extreme points in $\mathcal{P}_\mathcal{H}(X)$ is equivalent to characterizing non-atomic C^*-extreme points in $\mathcal{P}_\mathcal{H}([0, 1])$ or $\mathcal{P}_\mathcal{H}(\mathbb{T})$. Also we already know the structure of atomic C^*-extreme points. Therefore what we observed from the discussion above and Corollary 4.11 is that, to characterize C^*-extreme points of $\mathcal{P}_\mathcal{H}(X)$, it is enough to understand the behaviour of C^*-extreme points of $\mathcal{P}_\mathcal{H}([0, 1])$ or $\mathcal{P}_\mathcal{H}(\mathbb{T})$.

6. POVMs on Topological Spaces

The results presented in this article so far have been for POVMs on general measurable spaces. Our attention now shifts toward the particular case of topological spaces. For the whole section, we assume that X is a Hausdorff topological space. As mentioned earlier, in this case $\mathcal{O}(X)$ will denote the Borel σ-algebra of X.

6.1. Regular POVMs. An additional property of a POVM that can be studied when X is a topological space, is that of regularity. The assumption of regularity shall be useful once we discuss the correspondence between POVMs and completely positive maps in Sect. 7. Recall that a positive measure λ is regular if it is inner regular (or tight) with respect to compact subsets and outer regular with respect to open subsets:

$$\lambda(A) = \sup\{\lambda(E) : E \text{ compact with } E \subseteq A\}$$
$$= \inf\{\lambda(G) : G \text{ open with } A \subseteq G\},$$

for every $A \in \mathcal{O}(X)$.

Definition 6.1. A POVM $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ on a topological space X is said to be *regular* if $\mu_{h,h}$ as defined in equation (2.1), is a regular positive measure for each $h \in \mathcal{H}$.

The issue of regularity does not arise for complete separable metric spaces (Theorem 3.2, [29]), as all Borel measures are automatically regular.

The following lemma says that regularity is preserved under the minimal Naimark dilation.

Lemma 6.2. *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM with the minimal Naimark dilation $(\pi, V, \mathcal{H}_\pi)$. Then μ is regular if and only if π is regular.*

Proof. If π is regular then, since $\mu_{h,h} = \pi_{Vh,Vh}$ for each $h \in \mathcal{H}$, it is clear that μ is regular. For the converse, assume that μ is regular. First note that, if $k = \pi(B)Vh$ for some $B \in \mathcal{O}(X)$, $h \in \mathcal{H}$, then for any $A \in \mathcal{O}(X)$,

$$\pi_{k,k}(A) = \langle \pi(B)Vh, \pi(A)\pi(B)Vh \rangle = \langle h, V^*\pi(A)\pi(B)Vh \rangle = \mu_{h,h}(A \cap B).$$

Since $A \mapsto \mu_{h,h}(A \cap B)$ is regular, it follows that $\pi_{\pi(B)Vh,\pi(B)Vh}$ is regular. Consequently, $\pi_{k,k}$ is regular for all $k \in \mathrm{span}\{\pi(A)Vh : A \in \mathcal{O}(X), h \in \mathcal{H}\}$. Now fix $\epsilon > 0$ and $B \in \mathcal{O}(X)$. Then for general $k \in \mathcal{H}_\pi$, let $\{k_0\}$ be in $\mathrm{span}\{\pi(A)Vh : A \in \mathcal{O}(X), h \in \mathcal{H}\}$ such that

$$\|k - k_0\| < \sqrt{\epsilon}/2.$$

Since π_{k_0,k_0} is regular as shown above, there is a compact subset C and an open subset O with $C \subseteq B \subseteq O$ such that

$$\langle k_0, \pi(O\backslash C)k_0 \rangle < \epsilon/4.$$

Thus

$$\langle k, \pi(O\backslash C)k \rangle = \|\pi(O\backslash C)^{1/2}k\|^2 \leq 2\|\pi(O\backslash C)^{1/2}k_0\|^2 + 2\|\pi(O\backslash C)^{1/2}(k_0 - k)\|^2$$
$$\leq 2\langle k_0, \pi(O\backslash C)k_0 \rangle + 2\|k_0 - k\|^2 < 2\left(\epsilon/4 + \epsilon/4\right) = \epsilon.$$

Since ϵ and B are arbitrary, we conclude that $\pi_{k,k}$ is regular. $\qquad\square$

Remark 6.3. If μ is a regular POVM, then it is easy to check that $T^*\mu(\cdot)T$ is also regular for any $T \in \mathcal{B}(\mathcal{H})$. Moreover, if ν is a POVM such that $\nu \leq \mu$, then ν is also regular.

6.2. Regular atomic and non-atomic POVMs. We now discuss the structure of atomic and non-atomic regular POVMs. Just like that in classical theory, we show that every atom for a regular POVM is concentrated on a singleton up to a set of measure 0 and that every atomic regular POVM is concentrated on a countable subset. First step in that direction is the following lemma.

Lemma 6.4. *Let $\pi : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_\pi)$ be a regular spectral measure satisfying $\pi(A) = I_{\mathcal{H}_\pi}$ or 0 for each $A \in \mathcal{O}(X)$ (here $\mathcal{H}_\pi$ could be non-separable). Then there exists a unique $x \in X$ such that $\pi = \delta_x(\cdot)I_{\mathcal{H}_\pi}$, where δ_x denotes the Dirac measure concentrated at x.*

Proof. For each $A \in \mathcal{O}(X)$, let $\lambda(A) = 0$ or 1 accordingly so that $\pi(A) = \lambda(A)I_{\mathcal{H}_\pi}$. Clearly λ is a regular probability measure, as π is regular (e.g. $\lambda = \pi_{h,h}$ for any unit vector $h \in \mathcal{H}_\pi$).

Whence by inner regularity, there is a compact subset $C \subseteq X$ such that $\lambda(C) > 0$ and thus, $\lambda(C) = 1$. We claim to find an element $x \in C$ such that $\lambda = \delta_x$. Suppose this is not the case, then $\lambda(\{x\}) = 0$ for each $x \in C$ (otherwise, $\lambda(\{x\}) = 1 = \lambda(C)$ for some x). Therefore it follows from outer regularity of λ, that there is an open subset E_x containing x such that $\lambda(E_x) < 1/2$ and thus, $\lambda(E_x) = 0$. Since $\{E_x\}_{x \in C}$ is an open cover for the compact subset C, there exist finitely many points $x_1, \ldots, x_n \in C$ such that $C \subseteq \cup_{i=1}^n E_{x_i}$. But then we have

$$\lambda(C) \leq \sum_{i=1}^n \lambda(E_{x_i}) = 0,$$

leading us to a contradiction. Thus $\lambda = \delta_x$ for some $x \in X$ and hence $\pi = \delta_x(\cdot)I_{\mathcal{H}_\pi}$. The uniqueness is obvious as $\lambda(X) = \lambda(\{x\}) = 1$. $\qquad\square$

It is well-known that the lemma above fails to be true (even on compact Hausdorff spaces) for finite positive measures, if we drop the regularity assumption (see Example 7.1.3, [5]).

The following theorem and the subsequent corollary give characterization of all atomic and non-atomic regular POVMs.

Theorem 6.5. *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be an atomic regular POVM. Then there exists a countable subset $\{x_n\}$ of X and positive operators $\{T_n\}$ in $\mathcal{B}(\mathcal{H})$ such that*

$$\mu(A) = \sum_n \delta_{x_n}(A) T_n \tag{6.1}$$

for each $A \in \mathcal{O}(X)$.

Proof. Let $(\pi, V, \mathcal{H}_\pi)$ be the minimal Naimark dilation for μ. Since μ is atomic, π is also atomic by Proposition 2.14. Also π is regular by Lemma 6.2. We claim that there exists a countable subset $\{x_n\}$ of X and orthogonal projections $\{P_n\}$ on $\mathcal{H}_\pi$ such that $\pi(A) = \sum_n \delta_{x_n}(A) P_n$ for all $A \in \mathcal{O}(X)$. Then the required assertion will follow by taking $T_n = V^* P_n V \in \mathcal{B}(\mathcal{H})$.

Let $\{B_i\}_{i \in I}$ be a maximal collection of mutually disjoint atoms for π, whose existence is ensured by Zorn's lemma. Note that, since $\pi(B_i) \neq 0$, we have $\mu(B_i) \neq 0$ for all $i \in I$ by Proposition 2.6. Hence it follows from Remark 2.3 that I must be countable. Furthermore for each $A \in \mathcal{O}(X)$, we have

$$\pi(A) = \sum_{i \in I} \pi(A \cap B_i), \tag{6.2}$$

otherwise there would exist an atom for π, say $A_1 \subseteq A \backslash (\cup_i (A \cap B_i))$ which is disjoint to each B_i, violating the maximality of the collection $\{B_i\}_{i \in I}$.

Now for each $i \in I$, we set $P_i = \pi(B_i)$ and $\mathcal{H}_i = \operatorname{ran}(P_i)$. Note that each $\mathcal{H}_i$ is a reducing subspace for $\pi(A)$ for all $A \in \mathcal{O}(X)$ and therefore, the map $\pi_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_i)$ given by

$$\pi_i(A) = \pi(A \cap B_i)_{|\mathcal{H}_i} = \pi(A)_{|\mathcal{H}_i} \quad \text{for all } A \in \mathcal{O}(X),$$

is a well defined regular spectral measure. Also for each $A \in \mathcal{O}(X)$, as B_i is an atom for π, we have

$$\text{either } \pi(A \cap B_i) = 0 \text{ or } \pi(A \cap B_i) = \pi(B_i)$$

that is

$$\text{either } \pi_i(A) = 0 \text{ or } \pi_i(A) = I_{\mathcal{H}_i}.$$

Therefore by Lemma 6.4 there is an element $x_i \in B_i$ such that $\pi_i = \delta_{x_i}(\cdot) I_{\mathcal{H}_i}$. Equivalently for each $A \in \mathcal{O}(X)$, we have

$$\pi(A) P_i = \delta_{x_i}(A) P_i$$

and hence (6.2) yields

$$\pi(A) = \sum_{i \in I} \pi(A \cap B_i) = \sum_{i \in I} \pi(A) \pi(B_i) = \sum_{i \in I} \delta_{x_i}(A) P_i.$$

This shows our claim, completing the proof.   □

Corollary 6.6. *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a regular POVM. Then*

(1) for any atom B for μ, there exists a (unique) $x \in B$ such that $\mu(B) = \mu(\{x\})$.
(2) μ is atomic if and only if there exists a countable subset $Y \subseteq X$ such that $\mu(Y) = \mu(X)$.
(3) μ is non-atomic if and only if $\mu(\{x\}) = 0$ for all $x \in X$.

Proof. The proof of (1) is actually ingrained in the proof of Theorem 6.5; if B is an atomic subset for μ, then it is atomic for π and then we actually showed above that $\pi(B) = \pi(\{x\})$ for some $x \in B$. To prove (2), first note that any POVM concentrated on a countable subset is atomic and hence the 'if' part follows. The converse follows from Theorem 6.5, by taking $Y = \{x_n\}$. The 'only if' of Part (3) is trivial. To prove the 'if' part of (3), since every atom is concentrated on a singleton by Part (1), the hypothesis implies that μ has no atom, which is equivalent to saying that μ is non-atomic. $\quad\square$

Corollary 6.7. *Let $\{\mu_n\}$ be a countable collection of regular POVMs and let $\mu = \oplus_n \mu_n$. Then μ is atomic (non-atomic) if and only if each μ_n is atomic (non-atomic).*

Proof. We use Corollary 6.6 to prove the assertions. If μ is atomic, then there is a countable subset Y such that $\mu(Y) = \mu(X)$. In particular $\mu_n(Y) = \mu_n(X)$ for each n, which implies that μ_n is atomic. Conversely if each μ_n is atomic, then $\mu_n(Y_n) = \mu_n(X)$ for some countable subset Y_n. If $Y = \cup_n Y_n$, then Y is countable and $\mu(Y) = \mu(X)$, concluding that μ is atomic. The equivalence of non-atomicity follows similarly. $\quad\square$

6.3. Regular C^*-extreme POVMs. For any topological space X and a Hilbert space $\mathcal{H}$, denote the collection of all regular normalized POVMs from $\mathcal{O}(X)$ to $\mathcal{B}(\mathcal{H})$ by $\mathcal{RP}_{\mathcal{H}}(X)$. Note that $\mathcal{RP}_{\mathcal{H}}(X) \subseteq \mathcal{P}_{\mathcal{H}}(X)$ and $\mathcal{RP}_{\mathcal{H}}(X)$ is itself a C^*-convex set in the sense that

$$\sum_{i=1}^{n} T_i^* \mu_i(\cdot) T_i \in \mathcal{RP}_{\mathcal{H}}(X),$$

whenever $\mu_i \in \mathcal{RP}_{\mathcal{H}}(X)$ and T_i's are C^*-coefficients for $1 \leq i \leq n$. In a fashion similar to Definition 3.2, we can define C^*-extreme points of $\mathcal{RP}_{\mathcal{H}}(X)$. The following proposition says that, for a regular normalized POVM μ, it does not matter whether we are considering C^*-extremity of μ in $\mathcal{RP}_{\mathcal{H}}(X)$ or in $\mathcal{P}_{\mathcal{H}}(X)$.

Proposition 6.8. *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a normalized regular POVM. Then μ is C^*-extreme (extreme) in $\mathcal{P}_{\mathcal{H}}(X)$ if and only if μ is C^*-extreme (extreme) in $\mathcal{RP}_{\mathcal{H}}(X)$.*

Proof. If we show that every proper C^*-convex combination for μ in $\mathcal{P}_{\mathcal{H}}(X)$ is also a proper C^*-convex combination in $\mathcal{RP}_{\mathcal{H}}(X)$ and vice versa, then we are done. So let $\mu(\cdot) = \sum_{i=1}^{n} T_i^* \mu_i(\cdot) T_i$ be a proper C^*-convex combination in $\mathcal{P}_{\mathcal{H}}(X)$ for $\mu_i \in \mathcal{P}_{\mathcal{H}}(X)$. Note that, since $T_i^* \mu_i(\cdot) T_i \leq \mu(\cdot)$ for each i, it follows from Remark 6.3 that $T_i^* \mu_i(\cdot) T_i$ is regular. Again by the same remark, since

$$\mu_i(\cdot) = T^{*^{-1}} \left(T^* \mu_i(\cdot) T_i \right) T_i^{-1},$$

it follows that μ_i is regular. Thus $\mu_i \in \mathcal{RP}_{\mathcal{H}}(X)$, which shows that $\sum_{i=1}^{n} T_i^* \mu_i(\cdot) T_i$ is also a proper C^*-convex combination for μ in $\mathcal{RP}_{\mathcal{H}}(X)$. Since $\mathcal{RP}_{\mathcal{H}}(X) \subseteq \mathcal{P}_{\mathcal{H}}(X)$, the converse of the claim is immediate. The assertions about extreme points follow similarly. $\quad\square$

We have already seen the following result for countable measurable spaces in Theorem 3.11 without the assumption of regularity. The extension to uncountable discrete spaces requires regularity in a crucial way.

Proposition 6.9. *Let X be a discrete (possibly uncountable) space. Then every regular POVM on X is atomic. Moreover, a normalized POVM in $\mathcal{RP}_{\mathcal{H}}(X)$ is C^*-extreme if and only if it is spectral.*

Proof. Firstly let λ be a regular Borel positive measure on X. By regularity of λ, for each $n \in \mathbb{N}$ there is a compact subset C_n such that $\lambda(X \backslash C_n) < 1/n$. Set $C = \cup_n C_n$. Since X is discrete, each of C_n is a finite subset and hence C is countable. Note that

$$\lambda(X \backslash C) \leq \lambda(X \backslash C_n) \leq 1/n,$$

for each n and hence, $\lambda(X \backslash C) = 0$. This says that every regular Borel positive measure on X is concentrated on a countable subset and so it is atomic.

Now let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a regular POVM. Then as observed above, $\mu_{h,h}$ is concentrated on a countable subset for each $h \in \mathcal{H}$. Let $\{h_n\}$ be an orthonormal basis for $\mathcal{H}$. Then for each $n \in \mathbb{N}$, there are countable subsets, say B_n such that

$$\mu_{h_n,h_n}(X \backslash B_n) = 0.$$

Set $B = \cup_n B_n$, then for each $n \in \mathbb{N}$, we have

$$\mu_{h_n,h_n}(X \backslash B) \leq \mu_{h_n,h_n}(X \backslash B_n) = 0.$$

Consequently for all $h \in \mathcal{H}$, we have

$$\mu_{h,h}(X \backslash B) = \langle h, \mu(X \backslash B)h \rangle = \sum_n |\langle h_n, h \rangle|^2 \langle h_n, \mu(X \backslash B)h_n \rangle = 0$$

and hence $\mu(X \backslash B) = 0$. Since B is countable, we conclude that μ is concentrated on a countable subset and so μ is atomic. Thus if μ is a C^*-extreme point in $\mathcal{RP}_{\mathcal{H}}(X)$, then it is spectral by Theorem 3.11. $\qquad\square$

Remark 6.10. Let $\tilde{X} = X \cup \{\infty\}$ be the one-point compactification of a discrete space X and let $\mu : \mathcal{O}(\tilde{X}) \to \mathcal{B}(\mathcal{H})$ be a normalized regular POVM. Then the restriction $\mu_{|\mathcal{O}(X)}$ of μ to $\mathcal{O}(X)$ is also regular and hence concentrated on a countable subset, as seen in Proposition 6.9. In particular, μ itself is concentrated on a countable subset and hence is atomic. Therefore, we conclude from Theorem 3.11 that any regular normalized POVM on $\tilde{X}$ is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(\tilde{X})$ if and only if it is spectral.

6.4. Topology on $\mathcal{P}_{\mathcal{H}}(X)$. As earlier X is a topological space and $POVM_{\mathcal{H}}(X)$ denotes the collection of all POVMs from $\mathcal{O}(X)$ to $\mathcal{B}(\mathcal{H})$. Now we define a topology on this set. We shall call this topology as 'bounded-weak' inspired from a topology defined on the collection of all completely positive maps on a C^*-algebra with the same name. The reason for this nomenclature will be apparent in the next section. We have observed that the set $\mathcal{P}_{\mathcal{H}}(X)$ of normalized POVMs is a C^*-convex set. Our aim now is to show a Krein–Milman type theorem for C^*-convexity in this topology on $\mathcal{P}_{\mathcal{H}}(X)$.

Let $C_b(X)$ denote the space of all bounded continuous functions on X. Recall that $\mu_{h,k}$ is the complex measure as in (2.1) for any POVM μ. We define the topology by defining convergence of nets.

Definition 6.11. Given a net μ^i and μ in $POVM_{\mathcal{H}}(X)$, we say $\mu^i \to \mu$ in $POVM_{\mathcal{H}}(X)$ in *bounded weak topology* if

$$\int_X f d\mu^i_{h,k} \to \int_X f d\mu_{h,k}$$

for all $f \in C_b(X)$ and $h, k \in \mathcal{H}$.

Notice that the topology on $POVM_{\mathcal{H}}(X)$ is the smallest topology which makes the maps: $\mu \mapsto \int_X f d\mu_{h,k}$ from $POVM_{\mathcal{H}}(X)$ to $\mathbb{C}$, continuous for all $f \in C_b(X)$ and $h, k \in \mathcal{H}$. It is then immediate to verify that, for a given $\mu \in POVM_{\mathcal{H}}(X)$, sets of the form

$$O = \left\{ \nu \in POVM_{\mathcal{H}}(X); \left| \int_X f_i d\nu_{h_i,k_i} - \int_X f_i d\mu_{h_i,k_i} \right| < \epsilon, 1 \le i \le n \right\}, \quad (6.3)$$

where $f_i \in C_b(X)$, $h_i, k_i \in \mathcal{H}$ for $1 \le i \le n$, $\epsilon > 0$, form a basis around μ in $POVM_{\mathcal{H}}(X)$. The definition here reminds us the weak topology considered in classical probability theory. Moreover, we shall see in Sect. 7 that this definition is directly connected to the bounded weak topology on the collection of completely positive maps on a commutative C^*-algebra.

It should be added here that one can define a topology on $POVM_{\mathcal{H}}(X)$ in several ways. For example, for a net μ_i of POVMs and a POVM μ, we could define the convergence $\mu_i \to \mu$ by saying that $\mu_i(A) \to \mu(A)$ in WOT (or σ-weak topology) for all $A \in \mathcal{O}(X)$. This topology is certainly stronger than the bounded weak topology defined above. This topology has been considered in [23]. We could have also defined a topology just by considering $C_c(X)$, the space of all compactly supported continuous functions, instead of $C_b(X)$ in the definition. In this case, we would get a weaker topology than we originally defined. Nevertheless in this case, one can show along the lines of classical probability theory that this topology agrees with bounded weak topology on $\mathcal{P}_{\mathcal{H}}(X)$ whenever X is a locally compact Hausdorff space.

Our main focus for this topology is the set of normalized POVMs. In general, the set $\mathcal{P}_{\mathcal{H}}(X)$ is not Hausdorff; for an example, one can consider the classically famous Dieudonné measure λ (which is not regular) on the compact Hausdorff space $X = [0, \omega_1]$ equipped with order topology, where ω_1 is the first uncountable ordinal (see Example 7.1.3, [5]). One can show that $\int_X f d\lambda = f(\omega_1) = \int_X f d\delta_{\omega_1}$ for all $f \in C_b(X)$ and hence the distinct elements $\lambda(\cdot)I_{\mathcal{H}}$ and $\delta_{\omega_1}(\cdot)I_{\mathcal{H}}$ in $\mathcal{P}_{\mathcal{H}}(X)$ are not separated by open subsets. However the topology restricted to $\mathcal{RP}_{\mathcal{H}}(X)$ is Hausdorff whenever X is a locally compact Hausdorff space, which is a consequence of uniqueness of regular Borel measures in Riesz-Markov theorem.

Remark 6.12. As in classical probability theory, for a locally compact Hausdorff space (more generally for completely regular space, see Lemma 8.9.2, [5]), the set $\{\delta_x(\cdot)I_{\mathcal{H}}; x \in X\}$ is closed in $\mathcal{RP}_{\mathcal{H}}(X)$ and it is homeomorphic to X. Using this or otherwise, one can show that $\mathcal{RP}_{\mathcal{H}}(X)$ is compact if and only if X is compact.

6.5. A Krein–Milman type theorem. Now we move on to prove the main result of this section. It is well known that, in a locally convex topological vector space, a convex compact set is the closure of convex hull of its extreme points. This is known as Krein–Milman theorem. We here establish a similar kind of result for C^*-convexity in the sense that $\mathcal{P}_{\mathcal{H}}(X)$ is the closure of C^*-convex hull of its C^*-extreme points. A Krein–Milman

type theorem was proved in [16] when X is a compact Hausdorff space and $\mathcal{H}$ is a finite dimensional Hilbert space. We generalize it to arbitrary topological spaces and arbitrary Hilbert spaces. Moreover, in our case the compactness of $\mathcal{P}_{\mathcal{H}}(X)$ is not required. We first consider the following proposition, whose proof follows the same argument as normally used in classical measure theory. We provide the proof for the sake of completeness.

Proposition 6.13. *Let X be a topological space and $\mathcal{H}$ a Hilbert space. Then the collection of all normalized POVMs concentrated on finite subsets is dense in $\mathcal{P}_{\mathcal{H}}(X)$.*

Proof. Let $\mu \in \mathcal{P}_{\mathcal{H}}(X)$, and E be a typical open set in $\mathcal{P}_{\mathcal{H}}(X)$ containing μ of the form

$$E = \left\{ \nu \in \mathcal{P}_{\mathcal{H}}(X); \left| \int_X f_i \, d\nu_{h_i,k_i} - \int_X f_i \, d\mu_{h_i,k_i} \right| < \epsilon, 1 \leq i \leq n \right\},$$

for some fixed $f_i \in C_b(X)$, $h_i, k_i \in \mathcal{H}$, $i = 1, \ldots, n$ and $\epsilon > 0$. We shall obtain an element in E concentrated on a finite subset, which will imply the required result. Now for each $i \in \{1, \ldots, n\}$, get simple functions g_i on X satisfying

$$\sup_{x \in X} |f_i(x) - g_i(x)| < \epsilon/2M,$$

where M is a positive constant with $M > \sup_i \|h_i\| \|k_i\|$. Since g_i's are simple functions, there is a finite partition $\{A_{ij}\}$ of X and scalars $\{c_{ij}\} \subseteq \mathbb{C}$ (where j varies over some finite indexing set, say Λ_i for each $1 \leq i \leq n$) such that

$$g_i = \sum_{j \in \Lambda_i} c_{ij} \chi_{A_{ij}}$$

for each i. Pick $x_{ij} \in A_{ij}$ and set

$$\nu = \sum_{i=1}^{n} \sum_{j \in \Lambda_i} \delta_{x_{ij}}(\cdot) \mu(A_{ij}).$$

It is clear that ν is a POVM concentrated on a finite subset. Also we have

$$\nu(X) = \sum_{i=1}^{n} \sum_{j \in \Lambda_i} \mu(A_{ij}) = \mu(X) = I_{\mathcal{H}},$$

and hence ν is normalized. We claim that $\nu \in E$. Firstly note that

$$\int_X f \, d\nu = \sum_{i=1}^{n} \sum_{j \in \Lambda_i} f(x_{ij}) \mu(A_{ij})$$

for any bounded Borel measurable function f on X (here $\int_X f \, d\nu \in \mathcal{B}(\mathcal{H})$ is the operator satisfying $\langle h, (\int_X f \, d\nu) k \rangle = \int_X f \nu_{h,k}$ for all $h, k \in \mathcal{H}$). Therefore for each $m \in \{1, \ldots, n\}$, we have

$$\int_X g_m \, d\nu = \sum_{i=1}^{n} \sum_{j \in \Lambda_i} g_m(x_{ij}) \mu(A_{ij}) = \sum_{j \in \Lambda_m} c_{mj} \mu(A_{mj}) = \int_X g_m \, d\mu.$$

Thus we get the following:

$$\left| \int_X f_i \, dv_{h_i,k_i} - \int_X f_i \, d\mu_{h_i,k_i} \right| \leq \left| \int_X f_i \, dv_{h_i,k_i} - \int_X g_i \, dv_{h_i,k_i} \right| + \left| \int_X g_i \, dv_{h_i,k_i} - \int_X g_i \, d\mu_{h_i,k_i} \right|$$

$$+ \left| \int_X g_i \, d\mu_{h_i,k_i} - \int_X f_i \, d\mu_{h_i,k_i} \right|$$

$$\leq \int_X |f_i - g_i| \, d|v_{h_i,k_i}| + \int_X |g_i - f_i| \, d|\mu_{h_i,k_i}|$$

$$\leq \left(\sup_{x \in X} |f_i(x) - g_i(x)| \right) \left(|v_{h_i,k_i}|(X) + |\mu_{h_i,k_i}|(X) \right)$$

$$\leq (\epsilon/2M) \left(2\|h_i\| \|k_i\| \right)$$

$$< \epsilon$$

for $i = 1, \ldots, n$, where $|\mu_{h_i,k_i}|$ and $|v_{h_i,k_i}|$ denote the total variation of the complex measures μ_{h_i,k_i} and v_{h_i,k_i} respectively and we have used the fact that $|\mu_{h_i,k_i}|(X) \leq \|h_i\| \|k_i\|$, which is straightforward to verify. It then follows that $v \in E$, completing the proof. $\qquad \square$

Definition 6.14. For a given subset $\mathcal{M}$ of $\mathcal{P}_{\mathcal{H}}(X)$, the *C*-convex hull* of $\mathcal{M}$ is the set defined by

$$\left\{ \sum_{i=1}^n T_i^* \mu_i(\cdot) T_i \, : \, \mu_i \in \mathcal{M}, \, T_i \in \mathcal{B}(\mathcal{H}) \text{ for } 1 \leq i \leq n \text{ such that } \sum_{i=1}^n T_i^* T_i = I_{\mathcal{H}} \right\}. \tag{6.4}$$

Theorem 6.15 (Krein–Milman type theorem). *Let X be a topological space and $\mathcal{H}$ a Hilbert space. Then the C*-convex hull of Dirac spectral measures (i.e. $\delta_x(\cdot)I_{\mathcal{H}}$ for $x \in X$) is dense in $\mathcal{P}_{\mathcal{H}}(X)$. In particular, the C*-convex hull of all C*-extreme points is dense in $\mathcal{P}_{\mathcal{H}}(X)$.*

Proof. Fix $\mu \in \mathcal{P}_{\mathcal{H}}(X)$. By Proposition 6.13, there is a net $\mu_i \in \mathcal{P}_{\mathcal{H}}(X)$ such that $\mu_i \to \mu$ in $\mathcal{P}_{\mathcal{H}}(X)$ and each μ_i is concentrated on a finite subset. Therefore if we show that each μ_i is in the C*-convex hull of Dirac spectral measures, then we are done. So assume without loss of generality, that $\mu \in \mathcal{P}_{\mathcal{H}}(X)$ is concentrated on a finite subset, say $\{x_1, \ldots, x_n\}$. If $T_i = \mu(\{x_i\})$, then it is immediate that

$$\mu = \sum_{i=1}^n \delta_{x_i}(\cdot) T_i.$$

Set $S_i = T_i^{1/2} \in \mathcal{B}(\mathcal{H})$ for each i. Then

$$\sum_{i=1}^n S_i^* S_i = \sum_{i=1}^n T_i = \mu(X) = I_{\mathcal{H}}$$

and

$$\mu(\cdot) = \sum_{i=1}^n S_i^* \delta_{x_i}(\cdot) S_i,$$

which confirms that μ is a C*-convex combination of Dirac spectral measures. $\qquad \square$

It is obvious that Dirac spectral measures are regular. Therefore, Theorem 6.15 along with Proposition 6.8 give us the following version of Krein–Milman theorem for regular POVMs. Its usefulness shall be apparent when we discuss unital completely positive maps in the next section.

Corollary 6.16. *Let X be a topological space and $\mathcal{H}$ a Hilbert space. Then the C^*-convex hull of all regular spectral measures (in particular, regular C^*-extreme points) is dense in $\mathcal{RP}_{\mathcal{H}}(X)$.*

7. Applications to Completely Positive Maps

We now apply the results we have obtained in previous sections for POVMs, to the theory of completely positive maps on unital commutative C^*-algebras. That there is a strong relationship between these two topics is folklore.

If $\mathcal{A}$ is a commutative unital C^*-algebra then by Gelfand–Naimark theorem, there is a compact Hausdorff space X (called *spectrum* of $\mathcal{A}$) such that $\mathcal{A} = C(X)$, the space of all continuous functions on X. Therefore for the rest of this section, we assume that X is a compact Hausdorff space.

7.1. Completely positive maps. Like before let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators on a Hilbert space $\mathcal{H}$. For any C^*-algebra $\mathcal{A}$, a linear map $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is called *positive* if $\phi(a) \geq 0$ in $\mathcal{B}(\mathcal{H})$ whenever $a \geq 0$ in $\mathcal{A}$. The map ϕ is called a *completely positive (CP) map* if $\phi \otimes id_n : \mathcal{A} \otimes M_n \to \mathcal{B}(\mathcal{H}) \otimes M_n$ is a positive map for every $n \in \mathbb{N}$ (here, id_n stands for the identity map on $n \times n$ matrix algebra M_n). The well-known Stinespring's theorem (Theorem 4.1, [33]) ensures that, if $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a completely positive map, then there exists a triple $(\psi, V, \mathcal{K})$ where $\mathcal{K}$ is a Hilbert space, $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is a unital $*$-homomorphism and $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that

$$\phi(a) = V^* \psi(a) V \quad \text{for all } a \in \mathcal{A}, \tag{7.1}$$

and satisfies the minimality condition: $\mathcal{K} = [\psi(\mathcal{A})V\mathcal{H}]$. Moreover any such triple is unique up to unitary equivalence. In our case, the algebra $C(X)$ being commutative, complete positivity of linear maps on $C(X)$ is same as positivity (Theorem 3.11, [33]).

7.2. Correspondence between POVMs and CP maps. Let X be a compact Hausdorff space and $\mathcal{H}$ a Hilbert space. We now review the correspondence between regular POVMs on X and completely positive maps on $C(X)$ (see Chapter 4, [33]). Because most of the subsequent results hinge upon this correspondence, we give a detailed description.

Given a regular POVM $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$, consider for any $f \in C(X)$, the map $B_f : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ defined by

$$B_f(h, k) = \int_X f \, d\mu_{h,k} \quad \text{for all } h, k \in \mathcal{H}$$

where $\mu_{h,k}$ denotes the complex measure, as in (2.1). It is straightforward to check that B_f is a sesquilinear form satisfying $\|B_f\| \leq \|f\| \|\mu(X)\|$ and therefore, by Riesz Theorem (*Theorem II.2.2,* [8]) we obtain a unique bounded operator, call it $\phi_\mu(f) \in \mathcal{B}(\mathcal{H})$, satisfying $B_f(h, k) = \langle h, \phi_\mu(f)k \rangle$. Further it is immediate that $\phi_\mu(f) \geq 0$ in

$\mathcal{B}(\mathcal{H})$, whenever $f \geq 0$ in $C(X)$. Hence, the induced map $\phi_\mu : C(X) \to \mathcal{B}(\mathcal{H})$ defines a completely positive map via the assignment

$$\langle h, \phi_\mu(f)k \rangle = \int_X f d\mu_{h,k} \quad \text{for all } f \in C(X) \text{ and } h, k \in \mathcal{H}. \tag{7.2}$$

On the other hand, given a completely positive map $\phi : C(X) \to \mathcal{B}(\mathcal{H})$, consider for each $h, k \in \mathcal{H}$, the bounded linear functional on $C(X) := f \mapsto \langle h, \phi(f)k \rangle$. Then by the application of Riesz-Markov representation theorem, we obtain a unique regular Borel measure $\nu_{h,k}$ satisfying

$$\langle h, \phi(f)k \rangle = \int_X f d\nu_{h,k} \quad \text{for all } f \in C(X)$$

and $\|\nu_{h,k}\| \leq \|\phi\| \|h\| \|k\|$. Now for each bounded Borel measurable function g, consider the map: $(h, k) \mapsto \int_X g d\nu_{h,k}$ from $\mathcal{H} \times \mathcal{H}$ to $\mathbb{C}$, which is sesquilinear as above and bounded by $\|\phi\| \|g\|$. Hence again by Riesz Theorem, we obtain a unique bounded operator $\tilde{\phi}(g) \in \mathcal{B}(\mathcal{H})$ satisfying

$$\langle h, \tilde{\phi}(g)k \rangle = \int_X g d\nu_{h,k} \quad \text{for all } h, k \in \mathcal{H}. \tag{7.3}$$

Note that $\tilde{\phi}(g) \geq 0$ in $\mathcal{B}(\mathcal{H})$ whenever $g \geq 0$ in $B(X)$, the collection of all bounded Borel measurable functions on X. In particular for $A \in \mathcal{O}(X)$, if we set

$$\mu_\phi(A) = \tilde{\phi}(\chi_A), \tag{7.4}$$

where $\chi_A \in B(X)$ is the characteristic function of the subset A, then $\mu_\phi(A)$ is a positive operator in $\mathcal{B}(\mathcal{H})$ and satisfies

$$\nu_{h,k}(A) = \langle h, \mu_\phi(A)k \rangle \quad \text{for all } h, k \in \mathcal{H}.$$

Because $\nu_{h,h}$ is a regular Borel positive measure for each $h \in \mathcal{H}$, it is immediate that μ_ϕ defines a regular POVM which satisfies the equality $\mu_\phi(X) = \phi(1)$, where 1 denotes the constant function 1 on X.

Remark 7.1. For any POVM (not necessarily regular) μ, one can define a completely positive map ϕ satisfying (7.2) in a similar way. However, the regular measure μ_ϕ corresponding to ϕ that we got above, could significantly be different than the original μ. More precisely, there may exist more than one Borel POVM on a compact Hausdorff space X (certainly, non-metrizable), say μ_1 and μ_2, such that $\int_X f d\mu_1 = \int_X f d\mu_2$ for all $f \in C(X)$ (see the discussion just before Remark 6.12). Therefore to maintain uniqueness, we shall always assume the POVM to be regular whenever we talk about the correspondence between a POVM and a completely positive map.

The following theorem summarises some basic properties of this correspondence. See (Proposition 4.5, [33]), [20, 21] for some discussions on this.

Theorem 7.2. *Let X be a compact Hausdorff space and let $\mathcal{H}$ be a Hilbert space. Then the correspondence described above between $\mathcal{B}(\mathcal{H})$ valued regular POVMs on X and completely positive maps on $C(X)$, satisfies the following:*

(1) $\phi_{\mu_\phi} = \phi$ and $\mu_{\phi_\mu} = \mu$.
(2) $\phi(1) = \mu_\phi(X)$.

(3) μ is a projection valued measure if and only if ϕ_μ is a $*$-homomorphism.
(4) $\phi_{\mu_1+\mu_2} = \phi_{\mu_1} + \phi_{\mu_2}$ and $\mu_{\phi_1+\phi_2} = \mu_{\phi_1} + \mu_{\phi_2}$.
(5) $\phi_{T^*\mu(\cdot)T} = T^*\phi_\mu(\cdot)T$ and $\mu_{T^*\phi(\cdot)T} = T^*\mu_\phi(\cdot)T$ for any $T \in \mathcal{B}(\mathcal{H})$.

Proof. Part (1) is just uniqueness of the correspondence and part (2) follows from the discussion above. To show (3), first assume that ϕ_μ is a $*$-homomorphism. Then for all $f, g \in C(X)$ and $h, k \in \mathcal{H}$, we have

$$\int_X fg\, d\mu_{h,k} = \langle h, \phi_\mu(fg)k\rangle = \langle h, \phi_\mu(f)\phi_\mu(g)k\rangle = \int_X f\, d\mu_{h,\phi_\mu(g)k}.$$

Since $f \in C(X)$ is arbitrary, it follows from uniqueness of regular Borel measures in Riesz-Markov theorem that $g\, d\mu_{h,k} = d\mu_{h,\phi_\mu(g)k}$, as complex measures. Equivalently for any $A \in \mathcal{O}(X)$, we have

$$\int_A g\, d\mu_{h,k} = \mu_{h,\phi_\mu(g)k}(A),$$

that is

$$\int_X g\chi_A\, d\mu_{h,k} = \langle h, \mu(A)\phi_\mu(g)k\rangle = \langle \mu(A)h, \phi_\mu(g)k\rangle = \int_X g\, d\mu_{\mu(A)h,k}.$$

Again, since $g \in C(X)$ is arbitrary, we conclude that $\chi_A\, d\mu_{h,k} = d\mu_{\mu(A)h,k}$, as complex measures. Equivalently for any $B \in \mathcal{O}(X)$, we get

$$\int_X \chi_{A\cap B}\, d\mu_{h,k} = \int_X \chi_A\chi_B\, d\mu_{h,k} = \mu_{\mu(A)h,k}(B) = \langle \mu(A)h, \mu(B)k\rangle,$$

which further implies

$$\langle h, \mu(A \cap B)k\rangle = \langle h, \mu(A)\mu(B)k\rangle.$$

Since $h, k \in \mathcal{H}$ are arbitrary, we conclude that

$$\mu(A \cap B) = \mu(A)\mu(B) \quad \text{for all } A, B \in \mathcal{O}(X),$$

which shows that μ is a projection valued measure. The converse of the statement follows just by reversing of the argument above.

Part (4) directly follows from the assignment in (7.2). To show part (5): let $T \in \mathcal{B}(\mathcal{H})$ and set $\nu(\cdot) = T^*\mu(\cdot)T$. For any $h, k \in \mathcal{H}$ and $B \in \mathcal{O}(X)$, then

$$\langle h, \nu(B)k\rangle = \langle h, T^*\mu(B)Tk\rangle = \langle Th, \mu(B)Tk\rangle$$

which equivalently says $\nu_{h,k} = \mu_{Th,Tk}$, as complex measures. Therefore for any $f \in C(X)$, we have

$$\langle h, \phi_\nu(f)k\rangle = \int_X f\, d\nu_{h,k} = \int_X f\, d\mu_{Th,Tk} = \langle Th, \phi_\mu(f)Tk\rangle = \langle h, T^*\phi_\mu(f)Tk\rangle$$

which proves that $\phi_\nu = T^*\phi_\mu(\cdot)T$. The other equality follows similarly. $\qquad\square$

It is crucial that for a compact Hausdorff space X, if μ is a regular POVM with a Naimark dilation $(\pi, V, \mathcal{H}_\pi)$ then $(\phi_\pi, V, \mathcal{H}_\pi)$ is a Stinespring dilation for the corresponding CP map ϕ_μ (follows directly from part (5) of Theorem 7.2). Further, minimality conditions match:

$$[\pi(\mathcal{O}(X))V\mathcal{H}] = [\phi_\pi(C(X))V\mathcal{H}] \tag{7.5}$$

and therefore, the Stinespring dilation $\phi_\mu = V^*\phi_\pi(\cdot)V$ is minimal if and only if the Naimark dilation $\mu = V^*\pi(\cdot)V$ is minimal. Here we have some additional technical properties of this correspondence which are quite useful for us.

Proposition 7.3. *Let X be a compact Hausdorff space and $\mu : \mathcal{O}(X) \rightarrow \mathcal{B}(\mathcal{H})$ a regular POVM. Then $\mu(\mathcal{O}(X))' = \phi_\mu(C(X))'$. Moreover, $\mu(A) \in WOT\text{-}\phi_\mu(C(X))$ and $\phi_\mu(f) \in WOT\text{-}\overline{span}\,\mu(\mathcal{O}(X))$ for all $A \in \mathcal{O}(X)$ and $f \in C(X)$ and in particular, $WOT\text{-}\overline{\phi_\mu(C(X))} = WOT\text{-}\overline{span}\,\mu(\mathcal{O}(X))$.*

Proof. First assume $T \in \mu(\mathcal{O}(X))'$. Then $\mu(A)T = T\mu(A)$ for all $A \in \mathcal{O}(X)$ and hence

$$\langle T^*h, \mu(A)k\rangle = \langle h, T\mu(A)k\rangle = \langle h, \mu(A)Tk\rangle,$$

for all $h, k \in \mathcal{H}$, which is equivalent to $\mu_{T^*h,k} = \mu_{h,Tk}$, as complex measures. Therefore for all $f \in C(X)$, it follows that

$$\langle T^*h, \phi_\mu(f)k\rangle = \int_X f\,d\mu_{T^*h,k} = \int_X f\,d\mu_{h,Tk} = \langle h, \phi_\mu(f)Tk\rangle.$$

Since $h, k \in \mathcal{H}$ are arbitrary, we conclude that

$$T\phi_\mu(f) = \phi_\mu(f)T \quad \text{for all } f \in C(X),$$

which implies $T \in \phi_\mu(C(X))'$. Thus we have proved the inclusion $\mu(\mathcal{O}(X))' \subseteq \phi_\mu(C(X))'$. The other way of the inclusion is similarly proved just by reversing the implications above.

Now let $(\pi, V, \mathcal{H}_\pi)$ be the minimal Naimark dilation for μ. To show that $\mu(A) \in WOT\text{-}\overline{\phi_\mu(C(X))}$ for $A \in \mathcal{O}(X)$, firstly note that

$$\pi(\mathcal{O}(X))'' = \phi_\pi(C(X))'',$$

the double commutant of the respective sets in $\mathcal{B}(\mathcal{H}_\pi)$, which follows from first part of the proof. Therefore, since $\pi(A) \in \pi(\mathcal{O}(X))$ and $\pi(\mathcal{O}(X)) \subseteq \pi(\mathcal{O}(X))'' = \phi_\pi(C(X))''$, it follows from double commutant theorem (Theorem IX.6.4, [8]) for the $*$-algebra $\phi_\pi(C(X))$, that there is a net $\{f_i\}$ in $C(X)$ such that

$$\phi_\pi(f_i) \rightarrow \pi(A) \quad \text{in WOT.}$$

This implies

$$\phi_\mu(f_i) = V^*\phi_\pi(f_i)V \rightarrow V^*\pi(A)V = \mu(A) \quad \text{in WOT}$$

and so we conclude that $\mu(A) \in WOT\text{-}\overline{\phi_\mu(C(X))}$. Other assertions follow similarly. $\square$

7.3. C^*-extreme points of UCP maps on commutative C^*-algebras. For a unital C^*-algebra $\mathcal{A}$ and a Hilbert space $\mathcal{H}$, let $UCP_{\mathcal{H}}(\mathcal{A})$ denote the collection of all unital completely positive maps from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$. It is clear that $UCP_{\mathcal{H}}(\mathcal{A})$ is a convex set. The seminal paper by Arveson [1] studies the extreme points of $UCP_{\mathcal{H}}(\mathcal{A})$ and provides an abstract characterization. Several authors have looked into the classical convexity ([7,31,32,37] and [4]) of $UCP_{\mathcal{H}}(\mathcal{A})$ and its subclasses. Many others have considered different versions of convexity on $UCP_{\mathcal{H}}(\mathcal{A})$, e.g. [9,14,15,17,18,25,26,38] and [19].

The main focus of this paper has been on the notion of C^*-convexity. Farenick and Morenz [15] first studied the C^*-convexity structure of $UCP_{\mathcal{H}}(\mathcal{A})$. They gave a complete characterization of all C^*-extreme points of $UCP_{\mathcal{H}}(\mathcal{A})$, whenever $\mathcal{H}$ is finite dimensional. In [17], an abstract characterization of all C^*-extreme points were given, which we have presented in Theorem 3.3 in the language of POVMs. In [19], Gregg obtained a necessary criterion for C^*-extreme points in $UCP_{\mathcal{H}}(\mathcal{A})$, when $\mathcal{A}$ is a commutative unital C^*-algebra. We carry forward this investigation of C^*-convexity structure of $UCP_{\mathcal{H}}(C(X))$ by using the tools that we have developed for POVMs and its correspondence with completely positive maps.

More formally, $UCP_{\mathcal{H}}(C(X))$ is a C^*-convex set in the sense that

$$\sum_{i=1}^{n} T_i^* \phi_i(\cdot) T_i \in UCP_{\mathcal{H}}(C(X))$$

whenever $\phi_i \in UCP_{\mathcal{H}}(C(X))$ and $T_i \in \mathcal{B}(\mathcal{H})$ with $\sum_{i=1}^{n} T_i^* T_i = I_{\mathcal{H}}$. In a way similar to C^*-extreme points for POVMs in Definition 3.2, we define C^*-extreme points of $UCP_{\mathcal{H}}(C(X))$ (see [15]) as follows:

Definition 7.4. A map $\phi \in UCP_{\mathcal{H}}(C(X))$ is *C^*-extreme* if, whenever $\phi = \sum_{i=1}^{n} T_i^* \phi_i(\cdot) T_i$ for $\phi_i \in UCP_{\mathcal{H}}(C(X))$ with invertible operators $T_i \in \mathcal{B}(\mathcal{H})$ satisfying $\sum_{i=1}^{n} T_i^* T_i = I_{\mathcal{H}}$, then ϕ_i is unitarily equivalent to ϕ i.e. $\phi = U_i^* \phi_i(\cdot) U_i$ for some unitary operator $U_i \in \mathcal{B}(\mathcal{H})$ for every i.

The correspondence of regular POVMs and completely positive maps described above clearly preserves classical as well as C^*-convexity structures. Recall that $\mathcal{RP}_{\mathcal{H}}(X)$ denotes the collection of all regular Borel normalized POVMs on X.

Theorem 7.5. *A normalized regular POVM μ is C^*-extreme (extreme) in $\mathcal{RP}_{\mathcal{H}}(X)$ (or in $\mathcal{P}_{\mathcal{H}}(X)$) if and only if ϕ_μ is C^*-extreme (extreme) in $UCP_{\mathcal{H}}(C(X))$.*

Proof. The proof follows from Theorem 7.2, because classical, C^*-convex combinations and unitary equivalences are preserved under the correspondence. □

Following the discussions above, we are now ready to deduce some results for $UCP_{\mathcal{H}}(C(X))$. As noticed in Proposition 6.8, a regular normalized POVM μ is a C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ if and only if μ is a C^*-extreme point in $\mathcal{RP}_{\mathcal{H}}(X)$. Therefore, it follows from Theorem 7.5 that μ is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X)$ if and only if ϕ_μ is C^*-extreme in $UCP_{\mathcal{H}}(C(X))$. Thus, whenever X is a compact Hausdorff space, we have got freedom to bring back all the results on C^*-extreme points in $\mathcal{P}_{\mathcal{H}}(X)$ into the language of $UCP_{\mathcal{H}}(C(X))$. We frequently make use of Theorem 7.2 and Theorem 7.5. Before going forward, we recall the following known fact.

Theorem 7.6. (Proposition 1.2, [15]) *Every unital $*$-homomorphism is a C^*-extreme point in $UCP_{\mathcal{H}}(C(X))$.*

Now let X be a countable compact Hausdorff space. Then we saw in Theorem 3.11 that every C^*-extreme point in $\mathcal{P}_\mathcal{H}(X)$ is spectral. Since spectral measures correspond to unital $*$-homomorphisms, here is the corresponding result.

Theorem 7.7. *Let $\mathcal{A}$ be a commutative unital C^*-algebra with countable spectrum and let ϕ be a map in $UCP_\mathcal{H}(\mathcal{A})$. Then ϕ is C^*-extreme if and only if ϕ is a $*$-homomorphism.*

We apply this result to the C^*-algebra generated by a single normal operator to have the following.

Example 7.8. Let $N \in \mathcal{B}(\mathcal{K})$ be a normal operator on a Hilbert space $\mathcal{K}$ with countable spectrum $\sigma(N)$ (in particular, when N is compact). It is known that for such a normal operator, a subspace $\mathcal{H} \subseteq \mathcal{K}$ is invariant for N if and only if it is reducing for N(Theorem 1.23, [34]). Consider the unital completely positive map $\phi_N : C^*(N) \to \mathcal{B}(\mathcal{H})$ defined by $\phi_N(T) = P_\mathcal{H} T_{|\mathcal{H}}$ for all $T \in C^*(N)$, where $C^*(N)$ is the unital C^*-algebra generated by N. It is easy to verify that ϕ_N is a $*$-homomorphism if and only if $\mathcal{H}$ is a reducing subspace for N. Thus since $C^*(N)$ is isomorphic to $C(\sigma(N))$ as C^*-algebra and $\sigma(N)$ is countable, the argument above along with Theorem 7.7 show that the following conditions are equivalent:

(1) ϕ_N is a C^*-extreme point in $UCP_\mathcal{H}(C^*(N))$.
(2) ϕ_N is a $*$-homomorphism.
(3) $\mathcal{H}$ is an invariant subspace of N.
(4) $\mathcal{H}$ is a co-invariant subspace of N.
(5) $\mathcal{H}$ is a reducing subspace of N.

Next using the results in Sect. 5, we provide here an example of a C^*-extreme point in $\mathcal{P}_\mathcal{H}(X)$ which is not spectral, whenever X is an uncountable compact metric space and $\mathcal{H}$ an infinite dimensional Hilbert space.

Example 7.9. Consider the normalized POVM $\nu : \mathcal{O}(\mathbb{T}) \to \mathcal{B}(\mathcal{H}^2)$ defined by

$$\nu(A) = P_{\mathcal{H}^2} M_{\chi_A|_{\mathcal{H}^2}} \quad \text{for all } A \in \mathcal{O}(\mathbb{T}),$$

where $\mathcal{H}^2$ denotes the Hardy space on the unit circle $\mathbb{T}$. Here M_f denotes the multiplication operator on $L^2(\mathbb{T})$ for any $f \in L^\infty(\mathbb{T})$. Then the corresponding unital completely positive map $\phi_\nu : C(\mathbb{T}) \to \mathcal{B}(\mathcal{H}^2)$ is given by

$$\phi_\nu(f) = P_{\mathcal{H}^2} M_{f|_{\mathcal{H}^2}} \quad \text{for all } f \in C(\mathbb{T}).$$

It is known (Example 2, [15]) that ϕ_ν is a C^*-extreme point in $UCP_\mathcal{H}(C(\mathbb{T}))$ and therefore, ν is C^*-extreme in $\mathcal{P}_{\mathcal{H}^2}(\mathbb{T})$ by Theorem 7.5. Also note that ν is not spectral, since ϕ_ν is not a $*$-homomorphism. Now let X be an uncountable compact metric space. Then by well-known theorems of Borel isomorphism (Theorem 2.12, [29]), there exists a Borel isomorphism $f : \mathbb{T} \to X$. Define the normalized POVM $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}^2)$ by

$$\mu(A) = \nu(f^{-1}(A)) \quad \text{for all } A \in \mathcal{O}(X). \tag{7.6}$$

Then Theorem 5.5 along with Theorem 5.2 imply that μ is a C^*-extreme point in $\mathcal{P}_{\mathcal{H}^2}(X)$ and is not spectral. Thus, since any infinite dimensional separable Hilbert space is isomorphic to $\mathcal{H}^2$, what we have shown is that whenever X is an uncountable compact metric space and $\mathcal{H}$ an infinite dimensional Hilbert space, then $\mathcal{P}_\mathcal{H}(X)$ contains a C^*-extreme point which is not spectral. The assertion above can be applied to Polish spaces as well.

Let E be an uncountable compact subset of $\mathbb{C}$. Then E is a compact metric space. We consider the normalized POVM $\mu : \mathcal{O}(E) \to \mathcal{B}(\mathcal{H}^2)$ constructed in Example 7.9, which is already in the minimal Naimark dilation form $\mu(\cdot) = V^* \pi(\cdot) V$. If $N = \int_E z \, d\pi \in \mathcal{B}(\mathcal{H}_\pi)$, then N is a normal operator with spectrum E. Also the corresponding completely positive map $\phi_\mu : C^*(N) \to \mathcal{B}(\mathcal{H}^2)$ is of the form $\phi_\mu(T) = P_{\mathcal{H}^2} T_{|_{\mathcal{H}^2}}$ for $T \in C^*(N)$. Thus we have got an example of a completely positive map of the form ϕ_N as discussed in Example 7.8, which is C^*-extreme but not a $*$-homomorphism.

Now let $\mathcal{A}$ be a separable commutative unital C^*-algebra. Then its spectrum is a separable compact Hausdorff space (Theorem V.6.6, [8]) and hence metrizable, which is to say $\mathcal{A} = C(X)$ for a compact metric space X. Therefore, Example 7.9 and Theorem 7.5 give us the following result for a separable commutative unital C^*-algebra with uncountable spectrum.

Theorem 7.10. *Let $\mathcal{A}$ be a separable commutative unital C^*-algebra with uncountable spectrum and let $\mathcal{H}$ be an infinite dimensional separable Hilbert space. Then $UCP_\mathcal{H}(\mathcal{A})$ contains a C^*-extreme point which is not a $*$-homomorphism.*

The theorem above fails to be true if the separability assumption is removed, as we see below. If X is a discrete space and $\tilde{X}$ denotes its one-point compactification, then we saw in Remark 6.10 that every regular POVM in $\mathcal{P}_\mathcal{H}(\tilde{X})$ is atomic, and hence every C^*-extreme point in $\mathcal{RP}_\mathcal{H}(\tilde{X})$ is spectral. Equivalently, every C^*-extreme point in $UCP_\mathcal{H}(C(\tilde{X}))$ is a $*$-homomorphism by Theorem 7.5. Note that, whenever X is an uncountable discrete space, then $\tilde{X}$ is a non-separable compact Hausdorff space and in particular, $C(\tilde{X})$ is a non separable C^*-algebra (Theorem V.6.6, [8]). Thus the assumption of separability of the C^*-algebra $\mathcal{A}$ in Theorem 7.10 is crucial. We have obtained the following:

Theorem 7.11. *Let $\mathcal{A}$ be a commutative unital C^*-algebra whose spectrum is a one-point compactification of a discrete space. Then every C^*-extreme point in $UCP_\mathcal{H}(\mathcal{A})$ is a $*$-homomorphism.*

Next let $\phi : C(X) \to \mathcal{B}(\mathcal{H})$ be a unital completely positive map such that $\phi(C(X))$ is commutative. Then WOT-$\overline{\phi(C(X))}$ is commutative. Since WOT-$\overline{\phi(C(X))}$ = WOT-$\overline{\text{span}} \mu_\phi(\mathcal{O}(X))$ by Proposition 7.3, it follows that WOT-$\overline{\text{span}} \mu_\phi(\mathcal{O}(X))$ is commutative. In particular, $\mu_\phi(\mathcal{O}(X))$ is commutative. Therefore if ϕ is a C^*-extreme point in $UCP_\mathcal{H}(C(X))$ with commutative range, then μ_ϕ is a C^*-extreme point in $\mathcal{P}_\mathcal{H}(X)$ with commutative range. Then it follows from Theorem 3.9 that μ_ϕ is spectral and hence, ϕ is a $*$-homomorphism. Thus we have got the following result. A similar result for extreme points with commutative range in $UCP_\mathcal{H}(C(X))$ holds true (see Corollary 3.6, [37]).

Theorem 7.12. *Let $\mathcal{A}$ be a commutative unital C^*-algebra and $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ a unital completely positive map with commutative range. Then ϕ is C^*-extreme in $UCP_\mathcal{H}(\mathcal{A})$ if and only if ϕ is a $*$-homomorphism.*

We now discuss the bounded-weak topology on $UCP_\mathcal{H}(C(X))$ and how it is connected to the topology on $\mathcal{P}_\mathcal{H}(X)$ defined earlier (which we called bounded weak topology as well). The bounded-weak topology (see [1,33]) on $UCP_\mathcal{H}(C(X))$ is given by the convergence: for a net $\{\phi_i\}$ and ϕ in $UCP_\mathcal{H}(C(X))$,

$$\phi_i \to \phi \text{ if and only if } \phi_i(f) \to \phi(f) \text{ in WOT}$$

for all $f \in C(X)$.

For a net μ^i and $\mu \in \mathcal{RP}_{\mathcal{H}}(X)$, since $\phi_\mu(f) = \int_X f \, d\mu$ for all $f \in C(X)$, it follows that $\mu^i \to \mu$ in $\mathcal{RP}_{\mathcal{H}}(X)$ if and only if $\phi_{\mu^i}(f) \to \phi_\mu(f)$ in WOT for all $f \in C(X)$. The following proposition is just a rephrasing of the definition of the topology on regular POVMs, which effectively says that $\mathcal{RP}_{\mathcal{H}}(X)$ and $UCP_{\mathcal{H}}(C(X))$ are topologically homeomorphic. Recall that by Riesz–Markov representation theorem, the space of all regular Borel complex measures $M(X)$ on X is Banach space dual of $C(X)$.

Proposition 7.13. *Let μ^i be a net in $\mathcal{RP}_{\mathcal{H}}(X)$ and $\mu \in \mathcal{RP}_{\mathcal{H}}(X)$. Then the following are equivalent:*

(1) $\mu^i \to \mu$ in $\mathcal{RP}_{\mathcal{H}}(X)$ (and, in $\mathcal{P}_{\mathcal{H}}(X)$).
(2) $\phi_{\mu^i} \to \phi_\mu$ in bounded-weak topology in $UCP_{\mathcal{H}}(C(X))$.
(3) $\mu^i_{h,k} \to \mu_{h,k}$ in weak*-topology on $M(X)$ for all $h, k \in \mathcal{H}$.

In [15], a Krein–Milman type theorem was proved for $UCP_{\mathcal{H}}(\mathcal{A})$ with respect to bounded-weak topology, for arbitrary unital C^*-algebra $\mathcal{A}$ but finite-dimensional Hilbert space $\mathcal{H}$. Here we consider commutative unital C^*-algebras and arbitrary Hilbert spaces and give a similar kind of result for $UCP_{\mathcal{H}}(C(X))$. As in Definition 6.14, we define the C^*-convex hull of a subset $\mathcal{N} \subseteq UCP_{\mathcal{H}}(C(X))$ by

$$\left\{ \sum_{i=1}^n T_i^* \phi_i(\cdot) T_i; \ \phi_i \in \mathcal{N}, T_i \in \mathcal{B}(\mathcal{H}) \text{ such that } \sum_{i=1}^n T_i^* T_i = I_{\mathcal{H}} \right\}.$$

The following version of Krein–Milman type theorem for commutative unital C^*-algebras follows from Corollary 6.16, Theorem 7.5 and Theorem 7.2.

Theorem 7.14. *Let $\mathcal{A}$ be a commutative unital C^*-algebra and $\mathcal{H}$ a Hilbert space. Then the C^*-convex hull of the collection of all unital $*$-homomorphisms (in particular, C^*-extreme points) is dense in $UCP_{\mathcal{H}}(\mathcal{A})$ with respect to bounded-weak topology.*

8. Conclusion

Our original interest was to study C^*-convexity and C^*-extreme points in the setting of unital completely positive maps on unital commutative C^*-algebras. For this purpose, we have taken recourse in the well-known correspondence between such maps and POVMs on compact spaces. While doing so, we thought it could be of independent interest to study C^*-convexity in the setting of POVMs. So we analyze POVMs on general measurable spaces and get several interesting and basic results. Naimark's dilation theorem plays a crucial role in our investigation of POVMs, just as Stinespring's dilation theorem does for completely positive maps. Below we highlight some of our main results.

The abstract characterizations of C^*-extreme POVMs in Theorem 3.3 and Corollary 3.6 are the building blocks for all the forthcoming results. Our first major result is Theorem 3.8, which says that for a C^*-extreme POVM $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ and $E \in \mathcal{O}(X)$, if $\mu(E)$ commutes with $\mu(A)$ for all $A \subseteq E$, then $\mu(E)$ is a projection. The significance of this theorem should be clear from the following consequences:

- All C^*-extreme POVMs with commutative ranges are spectral (Theorem 3.9).
- All atomic C^*-extreme POVMs are spectral, and hence all C^*-extreme POVMs on countable spaces are spectral (Theorem 3.11).
- If $\dim \mathcal{H} < \infty$, then all C^*-extreme POVMs are spectral (Theorem 3.13).

We next study mutually disjoint POVMs and behaviour of C^*-convexity under their direct sums. Here we show the following:

- Any C^*-extreme POVM decomposes uniquely into a direct sum of an atomic C^*-extreme POVM and a non-atomic C^*-extreme POVM such that they are mutually disjoint (Theorem 4.9).

In essence, this implies that in order to get complete picture of C^*-extreme POVMs, it suffices to understand non-atomic C^*-extreme POVMs, given the fact that we have already characterized atomic C^*-extreme POVMs.

Our next main result is a version of Krein–Milman theorem for the C^*-convexity of POVMs on topological spaces. We define an appropriate topology on the C^*-convex space $\mathcal{P}_{\mathcal{H}}(X)$ of normalized POVMs, and prove in Theorem 6.15 that

- $\mathcal{P}_{\mathcal{H}}(X)$ is closure of C^*-convex hull of the set of its C^*-extreme points.

Finally, we apply our observations about POVMs on compact Hausdorff spaces X to the study of C^*-convexity of the space $UCP_{\mathcal{H}}(C(X))$ of unital completely positive maps on the commutative C^*-algebra $C(X)$. In particular, we have the following:

- If X is countable (in particular, when $C(X) = \mathbb{C}^n$), then every C^*-extreme points of $UCP_{\mathcal{H}}(C(X))$ is a $*$-homomorphism (Theorem 7.7).
- If X is uncountable, then $UCP_{\mathcal{H}}(C(X))$ contains a C^*-extreme point which is not a $*$-homomorphism (Theorem 7.10).
- All C^*-extreme points in $UCP_{\mathcal{H}}(C(X))$ with commutative ranges are $*$-homomorphisms (Theorem 7.12).
- (A Krein–Milman type theorem) The space $UCP_{\mathcal{H}}(C(X))$ is closure in bounded-weak topology of C^*-convex hull of its C^*-extreme points (Theorem 7.14).

We mention here in the passing that the study of POVMs on compact Hausdorff spaces as done in Sect. 7 extends easily to POVMs on locally compact Hausdorff spaces. Indeed if X is a locally compact non-compact Hausdorff space, then the set of contractive POVMs:

$$\mathcal{CP}_{\mathcal{H}}(X) = \{\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}); \mu \text{ is a POVM and } \mu(X) \leq I_{\mathcal{H}}\}$$

forms a C^*-convex set. Any μ here extends to a normalized POVM $\tilde{\mu}$ on the Borel σ-algebra of the one point compactification $\tilde{X} = X \bigcup \{\infty\}$, by taking $\tilde{\mu}(\infty) = 1 - \mu(X)$. This correspondence between contractive POVMs on X and normalized POVMs on $\tilde{X}$ is bijective and preserves basic properties such as C^*-convexity, regularity, atomicity etc. Hence results can be easily translated back from the compact case.

We conclude with a question. Our hope is that getting an answer to this question may shed more light on the structure of C^*-extreme points of UCP maps on commutative C^*-algebras with non-metrizable spectrum. We have shown that any C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(\mathbb{N})$ is spectral, where $\mathbb{N}$ is the set of natural numbers. It is also known that any unital completely positive map on $l^\infty (= l^\infty(\mathbb{N}))$ corresponds to finitely additive positive operator valued measure on $\mathbb{N}$, whereas (countably additive) POVMs correspond to the normal CP maps on l^∞ and hence all normal C^*-extreme points are $*$-homomorphic. It is not clear as of now how C^*-extreme points in the collection of all finitely additive POVMs behave. Approaching another way, the spectrum of l^∞ is of course the Stone-Čech compactification of $\mathbb{N}$. Unfortunately this space is not metrizable and our result on existence of a non-homomorphic C^*-extreme point (Theorem 7.10) is not applicable and so we are left with the following question:

Question 8.1. *Are C^*-extreme unital completely positive maps on the C^*-algebra l^∞ always $*$-homomorphisms?*

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