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Essays on Random Social Choice Theory

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Contents

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Listing of figures

To baba and maa.

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1

Introduction

This thesis comprises of six chapters related to random social choice theory. We provide a brief introduction of the chapters below.

1.1 An Extreme Point Characterization of Strategy-proof and Unanimous Probabilistic Rules over Binary Restricted Domains

In this chapter, we show that every strategy-proof and unanimous probabilistic rule on a binary restricted domain has binary support, and is a probabilistic mixture of strategy-proof and unanimous deterministic rules. Examples of binary restricted domains are single-dipped domains, which are of interest when considering the location of public bads. We also provide an extension to infinitely many alternatives.

1.2 A Characterization of Random Min-max Domains and Its Applications

In this chapter, we show that a random rule on a top-connected single-peaked domain is unanimous and strategy-proof if and only if it is a random min-max rule. As a by-product of this result, it follows that a top-connected single-peaked domain is tops-only for random rules. We further provide a characterization of the random min-max domains.

1.3 Formation of Committees through Random Voting Rules

In this chapter, we consider the problem of choosing a committee from a set of finite candidates based on the preferences of the agents in a society. The preference of an agent over a candidate is binary in the sense that either she wants the candidate to be included in a(ny) committee or she does not - she is never indifferent. A collection of preferences of an agent, one for each candidate, is extended to a preference over all subsets of candidates (i.e., potential committees) in a separable manner. Separability means if an agents wants a particular candidate to be in some committee, then she wants her to be in *every* committee.

1.4 A unified characterization of the randomized strategy-proof rules

In this chapter, we show that a large class of restricted domains such as single-peaked, single-crossing, single-dipped, tree-single-peaked with top-set along a path, Euclidean, multi-peaked, intermediate ([58]), etc., can be characterized by using betweenness property, and we present a unified characterization of unanimous and strategy-proof random rules on these domains. We do separate analysis for both the cases where the number of alternatives is finite or infinite. As corollaries of our result, we show that the domains we consider in this paper satisfy tops-onlyness and deterministic extreme point property.

1.5 Restricted Probabilistic Fixed Ballot Rules and Hybrid Domains

In this chapter, we study Random Social Choice Functions (or RSCFs) in a standard ordinal mechanism design model. We introduce a new preference domain called a hybrid domain which includes as special cases as the complete domain and the single-peaked domain. We characterize the class of unanimous and strategy-proof RSCFs on these domains and refer to them as Restricted Probabilistic Fixed Ballot Rules (or RPFBRs). These RSCFs are not necessarily decomposable, i.e., cannot be written as a convex combination of their deterministic counterparts. We identify a necessary and sufficient condition under which decomposability holds for anonymous RPFBRs. Finally, we provide an axiomatic justification of hybrid domains and show that every connected domain satisfying some mild conditions is a hybrid domain where the RPFBR characterization still prevails.

1.6 Unanimous and strategy-proof probabilistic rules for single-peaked preference profiles on graphs

In this chapter, we consider the problem where finitely many agents have preferences on a finite set of alternatives, single-peaked with respect to a connected graph with these alternatives as vertices. A

probabilistic rule assigns to each preference profile a probability distribution over the alternatives. First, all unanimous and strategy-proof probabilistic rules are characterized when the graph is a tree. These rules are uniquely determined by their outcomes at those preference profiles where all peaks are on leafs of the tree, and thus extend the known case of a line graph. Second, it is shown that every unanimous and strategy-proof probabilistic rule is random dictatorial if and only if the graph has no leafs. Finally, the two results are combined to obtain a general characterization for every connected graph by using its block tree representation.

2

An Extreme Point Characterization of Strategy-proof and Unanimous Probabilistic Rules over Binary Restricted Domains

2.1 Introduction

Suppose that in choosing between red and white wine, half of the dinner party is in favor of red wine while the other half prefers white wine. In this situation a deterministic (social choice) rule has to choose one of the two alternatives, where a fifty-fifty lottery seems to be more fair. In general, for every preference profile a probabilistic rule selects a lottery over the set of alternatives. [57] provides a characterization of all strategy-proof probabilistic rules over the complete domain of preferences (see also [98]). In particular, if in addition a rule is unanimous, then it is a probabilistic mixture of deterministic rules. This result implies that in order to analyze probabilistic rules it is sufficient to study deterministic rules only.

In [81] it is shown that if preferences are single-peaked over a finite set of alternatives then every strategy-proof and unanimous probabilistic rule is a mixture of strategy-proof and unanimous deterministic rules.[1] The same is true for the multi-dimensional domain with lexicographic preferences ([33]). But it is not necessarily true for all dictatorial domains ([35]), in particular, there are domains where all strategy-proof and unanimous deterministic rules are dictatorial but not all strategy-proof and unanimous probabilistic rules are random dictatorships.

A binary restricted domain over two alternatives x and y is a domain of preferences where the top alternative(s) of each preference belong(s) to the set $\{x, y\}$ (we allow for indifferences); and moreover, for every preference with top x there is a preference with top y such that the only alternatives weakly preferred to y in the former and x in the latter preference, are x and y.

Outstanding examples of binary restricted domains are domains of single-dipped preferences with respect to a given ordering of the alternatives. Single-dipped preferences are of central interest in situations where the location of an obnoxious facility (public bad) has to be determined by voting: think of deciding on the location of a garbage dump along a road, such that every inhabitant has a single dip (his house, or the school of his children, etc.) and prefers a location for the garbage dump as far away as possible from this dip. [79] have shown the equivalence between individual and group strategy-proofness in subdomains of single-dipped preferences. They characterize a general class of strategy-proof deterministic rules. In [68] the problem of locating a single public bad along a line segment when agents' preferences are single-dipped, is studied. In particular, all strategy-proof and unanimous deterministic rules are characterized. In [15] it is shown that, when all single-dipped preferences are admissible, the range of a strategy-proof and unanimous deterministic rule contains at most two alternatives. The paper also provides examples of sub-domains admitting strategy-proof rules with larger ranges. [7] consider group strategy-proofness under single-dipped preferences when agents become satiated: above a certain distance from their dips they become indifferent, and thus they go beyond the binary restricted domain. Further works on strategy-proofness under single-dipped preferences include [77], [78] [65], and [28]. For strong Nash implementation under single-dipped preferences see [105]. There is also a literature on this topic when side payments are allowed, e.g., [67] or [92].

In the present paper we show that every strategy-proof and unanimous probabilistic rule over a binary restricted domain with top alternatives x and y has binary support, i.e., for every preference profile probability 1 is assigned to $\{x, y\}$. We also show that if a strategy-proof and unanimous probabilistic rule has binary support then it can be written as a convex combination of deterministic rules. Moreover, we present a complete characterization of such rules, by using so-called admissible collections of committees.

Closely related papers are [66] and [84]. [66] include a characterization of all strategy-proof surjective deterministic rules for the case of two alternatives with indifferences allowed. Their Theorem 3 is close to our Theorem 2.3.5 – our theorem is slightly more general since we allow for more than two alternatives.

[1] [46] characterize such probabilistic rules for single-peaked preferences where the set of alternatives is the real line.

[84] show that every probabilistic rule is a convex combination of deterministic rules if there are only two alternatives and no indifferences are allowed.

The paper is organized as follows. The next section introduces the model and definitions. Section 2.3 contains the main results, Section 2.4 contains an application to single-dipped preference domains, and Section 2.5 presents an extension to the case where the number of alternatives may be infinite.

2.2 Preliminaries

Let A be a finite set of at least two alternatives and let $N = \{1, \ldots, n\}$ be a finite set of at least two agents. Subsets of N are called *coalitions*. Let $\mathbb{W}(A)$ be the set of (*weak*) *preferences* over A.[2] By P and I we denote the asymmetric and symmetric parts of $R \in \mathbb{W}(A)$. For $R \in \mathbb{W}(A)$ by $\tau(R)$ we denote set of the first ranked alternative(s) in R, i.e., $\tau(R) = \{x \in A : xRy \text{ for all } y \in A\}$. In general, the notation $\mathcal{D}$ will be used for a set of admissible preferences for an agent $i \in N$. As is clear from the notation, we assume the same set of admissible preferences for every agent. A *preference profile*, denoted by $R_N = (R_1, \ldots, R_n)$, is an element of $\mathcal{D}^n$, the Cartesian product of n copies of $\mathcal{D}$. For a coalition S, R_S denotes the restriction of R_N to S. For notational convenience we often denote a singleton set $\{z\}$ by z.

Definition 2.2.1 *A deterministic rule (DR) is a function* $f : \mathcal{D}^n \to A$.

Definition 2.2.2 *A DR f is unanimous if $f(R_N) \in \cap_{i=1}^n \tau(R_i)$ for all $R_N \in \mathcal{D}^n$ such that $\cap_{i=1}^n \tau(R_i) \neq \emptyset$.*

Agent $i \in N$ *manipulates* DR f *at* $R_N \in \mathcal{D}^n$ *via* R_i' if $f(R_i', R_{N\setminus i}) P_i f(R_N)$.

Definition 2.2.3 *A DR f is strategy-proof if for all $i \in N$, $R_N \in \mathcal{D}^n$, and $R_i' \in \mathcal{D}$, i does not manipulate f at R_N via R_i'.*

Definition 2.2.4 *A probabilistic rule (PR) is a function $\Phi : \mathcal{D}^n \to \triangle A$ where $\triangle A$ is the set of probability distributions over A. A* strict *PR is a PR that is not a DR.*

Observe that a deterministic rule can be identified with a probabilistic rule by assigning probability 1 to the chosen alternative.

For $a \in A$ and $R_N \in \mathcal{D}^n$, $\Phi_a(R_N)$ denotes the probability assigned to a by $\Phi(R_N)$. For $B \subseteq A$, we denote $\Phi_B(R_N) = \sum_{a \in B} \Phi_a(R_N)$.

Definition 2.2.5 *A PR Φ is unanimous if $\Phi_{\cap_{i=1}^n \tau(R_i)}(R_N) = 1$ for all $R_N \in \mathcal{D}^n$ such that $\cap_{i=1}^n \tau(R_i) \neq \emptyset$.*

[2] I.e., for all $R \in \mathbb{W}(A)$ and $x, y, z \in A$, we have xRy or yRx (completeness), and xRy and yRz imply xRz (transitivity). Note that reflexivity (xRx for all $x \in A$) is implied.

Definition 2.2.6 *For $R \in \mathcal{D}$ and $x \in A$, the upper contour set of x at R is the set $U(x, R) = \{y \in X : yRx\}$. In particular, $x \in U(x, R)$.*

Agent $i \in N$ *manipulates* PR Φ *at* $R_N \in \mathcal{D}^n$ *via* R_i' if $\Phi_{U(x,R_i)}(R_i', R_{N\setminus i}) > \Phi_{U(x,R_i)}(R_i, R_{N\setminus i})$ for some $x \in A$.

Definition 2.2.7 *A PR Φ is strategy-proof if for all $i \in N$, $R_N \in \mathcal{D}^n$, and $R_i' \in \mathcal{D}$, i does not manipulate Φ at R_N via R_i'.*

In other words, strategy-proofness of a PR means that a deviation results in a (first order) stochastically dominated lottery for the deviating agent.

For PRs Φ^j, $j = 1, \ldots, k$ and nonnegative numbers λ^j, $j = 1, \ldots, k$, summing to 1, we define the PR $\Phi = \sum_{j=1}^{k} \Phi^j$ by $\Phi_x(R_N) = \sum_{j=1}^{k} \lambda^j \Phi_x^j(R_N)$ for all $R_N \in \mathcal{D}^n$ and $x \in A$. We call Φ a *convex combination* of the PRs Φ^j.

Definition 2.2.8 *A domain $\mathcal{D}$ is said to be a deterministic extreme point domain if every strategy-proof and unanimous PR on $\mathcal{D}^n$ can be written as a convex combination of strategy-proof and unanimous DRs on $\mathcal{D}^n$.*

For $a \in A$, let $\mathcal{D}^a = \{R \in \mathcal{D} : \tau(R) = a\}$.

Definition 2.2.9 *Let $x, y \in A$, $x \neq y$. A domain $\mathcal{D}$ is a binary restricted domain over $\{x, y\}$ if*

(i) for all $R \in \mathcal{D}$, $\tau(R) \in \{\{x\}, \{y\}, \{x, y\}\}$,

(ii) for all $a, b \in \{x, y\}$ with $a \neq b$, and for each $R \in \mathcal{D}^a$, there exists $R' \in \mathcal{D}^b$ such that $U(b, R) \cap U(a, R') = \{a, b\}$.

Condition (ii) in the definition of a binary restricted domain is used in the proof of Proposition 2.3.1 below. There, we also provide an example (see Remark 2.3.4) to show that this condition cannot be dispensed with.

We conclude this section with the following definition.

Definition 2.2.10 *Let $x, y \in A$, $x \neq y$. A domain $\mathcal{D}$ is a* binary support domain *over $\{x, y\}$ if $\Phi_{\{x,y\}}(R_N) = 1$ for every $R_N \in \mathcal{D}^n$ and every strategy-proof and unanimous PR Φ on $\mathcal{D}^n$.*[3]

[3]Note that this domain is identified with the type of strategy-proof and unanimous PRs that it admits.

2.3 Results

In this section we present the main results of this paper. First we show that every binary support domain is a deterministic extreme point domain (Corollary 2.3.1). Next we show that every binary restricted domain is a binary support domain (Theorem 2.3.3). Consequently, every binary restricted domain is a deterministic extreme point domain (Corollary 2.3.2). Next, we characterize the set of all strategy-proof and unanimous probabilistic rules on such binary restricted domains.

2.3.1 Binary support domains are deterministic extreme point domains

First we establish a necessary and sufficient condition for a domain to be a deterministic extreme point domain.

Theorem 2.3.1 *A domain $\mathcal{D}$ is a deterministic extreme point domain if and only if every strategy-proof and unanimous strict PR on $\mathcal{D}^n$ is a convex combination of two other distinct strategy-proof and unanimous PRs.*

Proof:

First, let $\mathcal{D}$ be an arbitrary domain. Observe that every PR Φ can be identified with a vector in $\mathbb{R}^{pm}$, where p is the number of different preference profiles, i.e., the number of elements of $\mathcal{D}^n$, and m is the number of elements of A. Compactness and convexity of a set of PRs are equivalent to convexity and compactness of the associated subset of $\mathbb{R}^{pm}$.

We show that the set of all strategy-proof and unanimous probabilistic rules $\mathcal{S}$ over $\mathcal{D}^n$ is compact and convex.

For convexity, let $\Phi', \Phi'' \in \mathcal{S}$ and $0 \leq \alpha \leq 1$, and let the PR Φ be defined by $\Phi(R_N) = \alpha\Phi'(R_N) + (1-\alpha)\Phi''(R_N)$ for all $R_N \in \mathcal{D}^n$. Clearly, Φ is unanimous. For strategy-proofness, let $i \in N$, $R_N \in \mathcal{D}^n$ and $R'_i \in \mathcal{D}$. Then, for all $b \in A$, by strategy-proofness of Φ' and Φ'' we have $\Phi'_{U(b,R_i)}(R'_i, R_{N\setminus i}) \leq \Phi_{U(b,R_i)}(R_N)$ and $\Phi''_{U(b,R_i)}(R'_i, R_{N\setminus i}) \leq \Phi''_{U(b,R_i)}(R_N)$, so that

$$\alpha\Phi'_{U(b,R_i)}(R'_i, R_{N\setminus i}) + (1-\alpha)\Phi''_{U(b,R_i)}(R'_i, R_{N\setminus i}) \leq \alpha\Phi'_{U(b,R_i)}(R_N) + (1-\alpha)\Phi''_{U(b,R_i)}(R_N),$$

hence $\Phi_{U(b,R_i)}(R'_i, R_{N\setminus i}) \leq \Phi_{U(b,R_i)}(R_N)$. Thus, Φ is strategy-proof, and $\mathcal{S}$ is convex.

For closedness, consider a sequence Φ^k, $k \in \mathbb{N}$, in $\mathcal{S}$ such that $\lim_{k\to\infty} \Phi^k = \Phi$, i.e., for all $x \in A$ and $R_N \in \mathcal{D}^n$, $\lim_{k\to\infty} \Phi^k_x(R_N) = \Phi_x(R_N)$. It is easy to see that Φ is unanimous. Suppose that Φ were not strategy-proof. Then there exist $i \in N$, $R_N \in \mathcal{D}^n$ and $R'_i \in \mathcal{D}$ such that for some $b \in A$, $\Phi_{U(b,R_i)}(R'_i, R_{N\setminus i}) > \Phi_{U(b,R_i)}(R_N)$. This means there exists $k \in \mathbb{N}$ such that $\Phi^k_{U(b,R_i)}(R'_i, R_{N\setminus i}) > \Phi^k_{U(b,R_i)}(R_N)$. This contradicts strategy-proofness of Φ^k. So, $\mathcal{S}$ is closed. Clearly, $\mathcal{S}$ is bounded. Thus, it is compact.

Since $\mathcal{S}$ is compact and convex, by the Theorem of Krein-Milman (e.g., [90]) it is the convex hull of its (non-empty set of) extreme points. Now, for the if-part of the theorem, for a domain $\mathcal{D}$ satisfying the premise, no strict PR is an extreme point. Thus, $\mathcal{D}$ is a deterministic extreme point domain. In fact, it is also easy to see that every strategy-proof and unanimous deterministic rule is an extreme point of $\mathcal{S}$.

For the only-if part, let $\mathcal{D}$ be a deterministic extreme point domain and let Φ be a strategy-proof and unanimous strict PR on $\mathcal{D}^n$. Then there are $\lambda^1, \ldots, \lambda^k$, $k \geq 2$, with $\lambda^i > 0$ for all $i = 1, \ldots, k$ and $\sum_{i=1}^k \lambda^i = 1$, and strategy-proof and unanimous DRs $f^1, \ldots, f^k$ on $\mathcal{D}^n$ with $f^i \neq f^j$ for $i \neq j$, such that $\Phi = \sum_{i=1}^k \lambda^i f^i$. We define $\Phi' = \sum_{i=2}^k \frac{\lambda^i}{1-\lambda^1} f^i$. Then $\Phi = (1-\lambda^1)\Phi' + \lambda^1 f^1$, and Φ' and f^1 are distinct strategy-proof and unanimous PRs different from Φ. ■

In the following theorem we show that if a strategy-proof and unanimous strict PR has binary support, then it can be written as a convex combination of two other strategy-proof and unanimous PRs.

Theorem 2.3.2 *Let $\Phi : \mathcal{D}^n \to \Delta(A)$ be a strategy-proof and unanimous strict PR and let $x, y \in A$ such that $\Phi_{\{x,y\}}(R_N) = 1$ for all $R_N \in \mathcal{D}^n$. Then there exist strategy-proof and unanimous PRs Φ', Φ'' with $\Phi' \neq \Phi''$ such that $\Phi(R_N) = \frac{1}{2}\Phi'(R_N) + \frac{1}{2}\Phi''(R_N)$ for all $R_N \in \mathcal{D}^n$.*

Proof: Note that $\Phi_{\{x,y\}}(R_N) = 1$ for all $R_N \in \mathcal{D}^n$ implies that $\Phi(R_N)$ is completely determined by $\Phi_x(R_N)$ for all $R_N \in \mathcal{D}^n$. Since Φ is a strict PR, there exists $R'_N \in \mathcal{D}^n$ such that $\Phi_x(R'_N) = p \in (0,1)$. Let $C = \{R_N \in \mathcal{D}^n : \Phi_x(R_N) \neq p\}$. Since C is finite set, there is an $\varepsilon \in (0,p)$ such that for all $R_N \in C$, $\Phi_x(R_N) \notin [p-\varepsilon, p+\varepsilon]$. We define Φ' and Φ'' with support $\{x,y\}$ by

$$\Phi'_x(R_N) = \begin{cases} \Phi_x(R_N) \text{ if } R_N \in C \\ \Phi_x(R_N) + \varepsilon \text{ otherwise} \end{cases} \quad \text{and } \Phi''_x(R_N) = \begin{cases} \Phi_x(R_N) \text{ if } R_N \in C \\ \Phi_x(R_N) - \varepsilon \text{ otherwise.} \end{cases}$$

Clearly, $\Phi' \neq \Phi''$ and $\Phi(R_N) = \frac{1}{2}\Phi'(R_N) + \frac{1}{2}\Phi''(R_N)$ for all $R_N \in \mathcal{D}^n$. Unanimity of Φ' and Φ'' follows from unanimity of Φ. We show that Φ' and Φ'' are strategy-proof. We consider only Φ', the proof for Φ'' is analogous. Let $i \in N$, $R_N \in \mathcal{D}^n$ and $Q_i \in \mathcal{D}$. Write $Q_N = (Q_i, R_{N\setminus i})$. We consider the following cases.
Case 1 $R_N, Q_N \notin C$. Then $\Phi'_x(R_N) = p + \varepsilon = \Phi'_x(Q_N)$. So i does not manipulate Φ' at R_N via Q_i.
Case 2 $R_N, Q_N \in C$. Then $\Phi'_x(R_N) = \Phi_x(R_N)$ and $\Phi'_x(Q_N) = \Phi_x(Q_N)$. Since i does not manipulate Φ at R_N via Q_i, this implies that i does not manipulate Φ' at R_N via Q_i.
Case 3 $R_N \notin C$, $Q_N \in C$. Then $\Phi'_x(R_N) = \Phi_x(R_N) + \varepsilon$ and $\Phi'_x(Q_N) = \Phi_x(Q_N) \notin [\Phi_x(R_N) - \varepsilon, \Phi_x(R_N) + \varepsilon]$. If xP_iy (where P_i is the asymmetric part of R_i), then by strategy-proofness of Φ, $\Phi'_x(Q_N) = \Phi_x(Q_N) \leq \Phi_x(R_N) = \Phi'_x(R_N) - \varepsilon < \Phi'_x(R_N)$, so that i does not manipulate Φ' at R_N via Q_i. If yP_ix, then by strategy-proofness of Φ, $\Phi'_x(Q_N) = \Phi_x(Q_N) \geq \Phi_x(R_N) + \varepsilon = \Phi'_x(R_N)$, so that i does not manipulate Φ' at R_N via Q_i.

Case 4 $R_N \in C$, $Q_N \notin C$. If xP_iy then by strategy-proofness of Φ and the choice of ε,
$\Phi'_x(Q_N) = \Phi_x(Q_N) + \varepsilon \leq (\Phi_x(R_N) - \varepsilon) + \varepsilon = \Phi_x(R_N) = \Phi'_x(R_N)$, so that i does not manipulate Φ' at R_N via Q_i. If yP_ix, then by strategy-proofness of Φ,
$\Phi'_y(Q_N) = \Phi_y(Q_N) - \varepsilon \leq \Phi_y(R_N) - \varepsilon = \Phi'_y(R_N) - \varepsilon < \Phi'_y(R_N)$, so that i does not manipulate Φ' at R_N via Q_i. ■

Theorems 2.3.2 and 2.3.1 imply the following result.

Corollary 2.3.1 *Every binary support domain is a deterministic extreme point domain.*

2.3.2 Binary restricted domains are binary support domains

The main result of this subsection is the following theorem.

Theorem 2.3.3 *Every binary restricted domain is a binary support domain.*

We first prove the result for two agents and then use induction to prove it for an arbitrary number of agents.

Proposition 2.3.1 *Let $\mathcal{D}$ be a binary restricted domain over $\{x, y\}$, and let $\Phi : \mathcal{D}^2 \to \triangle A$ be a strategy-proof and unanimous PR. Then $\Phi_{\{x,y\}}(R_N) = 1$ for all $R_N \in \mathcal{D}^2$.*

Proof: By unanimity of Φ it is sufficient to consider the case where $R_N = (R_1, R_2)$ with $R_1 \in \mathcal{D}^x$ and $R_2 \in \mathcal{D}^y$.

First assume that $U(y, R_1) \cap U(x, R_2) = \{x, y\}$. Suppose that $\Phi_B(R_N) > 0$ for $B = A \setminus U(y, R_1)$. Then agent 1 manipulates at R_N via some $R'_1 \in \mathcal{D}^y$, since by unanimity $\Phi_y(R'_1, R_2) = 1$ and y is strictly preferred to (every element of) $A \setminus U(y, R_1)$ at the preference R_1 of agent 1. Hence, we must have $\Phi_B(R_N) = 0$ for $B = A \setminus U(y, R_1)$. Similarly one shows that $\Phi_{B'}(R_N) = 0$ for $B' = A \setminus U(x, R_2)$. Since $U(y, R_1) \cap U(x, R_2) = \{x, y\}$, we have $\Phi_{\{x,y\}}(R_N) = 1$.

Next, suppose that $U(y, R_1) \cap U(x, R_2) \neq \{x, y\}$. This, by the definition of a binary restricted domain, means that there exist $R'_1 \in \mathcal{D}^x$ and $R'_2 \in \mathcal{D}^y$ such that $U(y, R_1) \cap U(x, R'_2) = \{x, y\}$ and $U(y, R'_1) \cap U(x, R_2) = \{x, y\}$. By the first part of the proof we have $\Phi_{\{x,y\}}(R_1, R'_2) = 1$ and $\Phi_{\{x,y\}}(R'_1, R_2) = 1$. Let $\Phi_x(R_1, R'_2) = \varepsilon$ and $\Phi_x(R'_1, R_2) = \varepsilon'$. Since $R_1, R'_1 \in \mathcal{D}^x$ and $R_2, R'_2 \in \mathcal{D}^y$, strategy-proofness implies $\Phi_x(R'_1, R'_2) = \Phi_x(R_1, R'_2) = \varepsilon$ and $\Phi_y(R'_1, R'_2) = \Phi_y(R'_1, R_2) = 1 - \varepsilon'$. This means $\Phi_{\{x,y\}}(R'_1, R'_2) = \varepsilon + 1 - \varepsilon'$, which implies $\varepsilon \leq \varepsilon'$. By a similar argument it follows that $\varepsilon' \leq \varepsilon$. Hence, $\varepsilon = \varepsilon'$. Finally, again since $R_1, R'_1 \in \mathcal{D}^x$ and $R_2, R'_2 \in \mathcal{D}^y$, we have by strategy-proofness that $\Phi_x(R_1, R_2) = \Phi_x(R'_1, R_2) = \varepsilon$ and $\Phi_y(R_1, R_2) = \Phi_y(R_1, R'_2) = 1 - \varepsilon$, and hence $\Phi_{\{x,y\}}(R_1, R_2) = 1$, completing the proof. ■

Remark 2.3.4 *Condition (ii) in Definition 2.2.9 of a binary restricted domain cannot be omitted. Let* $A = \{x, y, z\}$, $N = \{1, 2\}$, *and let* $\mathcal{D} = \{R, R'\} \subseteq \mathbb{W}(A)$ *with* $xPzPy$ *and* $yP'zP'x$ *(*P *and* P' *are the asymmetric parts of* R *and* R'*, respectively). Hence,* $\mathcal{D}$ *is not a binary restricted domain over* $\{x, y\}$*, since (ii) in Definition 2.2.9 is not fulfilled. Let* $(\alpha, \beta, \gamma) \in \Delta(A)$ *be the lottery with probabilities on x, y, and z, respectively. Define the PR* Φ *by:* $\Phi(R_N) = (1, 0, 0)$ *if* $R_N = (R, R)$, $\Phi(R_N) = (0, 1, 0)$ *if* $R_N = (R', R')$*, and* $\Phi(R_N) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ *in the two other cases. Then clearly* Φ *is unanimous and strategy-proof. Hence,* $\mathcal{D}$ *is not a binary support domain.*

The following proposition treats the case with more than two agents.

Proposition 2.3.2 *Let* $n \geq 3$*, let* $\mathcal{D}$ *be binary restricted domain over* $\{x, y\}$*, and let* $\Phi : \mathcal{D}^n \to \triangle A$ *be a strategy-proof and unanimous PR. Then* $\Phi_{\{x,y\}}(R_N) = 1$ *for all* $R_N \in \mathcal{D}^n$*.*

Proof: As before, $N = \{1, \ldots, n\}$ is the set of agents. We prove the result by induction. Assume that the proposition holds for all sets with $k < n$ agents.

Let $N^* = \{1, 3, \ldots, n\}$ and define the PR $g : \mathcal{D}^{n-1} \to \triangle A$ for the set of agents N^* as follows: For all $R_{N^*} = (R_1, R_3, \ldots, R_n) \in \mathcal{D}^{n-1}$,

$$g(R_1, R_3, \ldots, R_n) = \Phi(R_1, R_1, R_3, \ldots, R_n).$$

Claim 1 $g_{\{x,y\}}(R_{N^*}) = 1$ for all $R_{N^*} \in \mathcal{D}^{n-1}$.

To prove this claim, first observe that g inherits unanimity from Φ. We show that g also inherits strategy-proofness. It is easy to see that agents other than 1 do not manipulate g since Φ is strategy-proof. Let $(R_1, R_3, \ldots, R_n) \in \mathcal{D}^{n-1}$ and $Q_1 \in \mathcal{D}$. For all $b \in A$, we have

$$\begin{aligned} g_{U(b,R_1)}(R_1, R_3, \ldots, R_n) &= \Phi_{U(b,R_1)}(R_1, R_1, R_3, \ldots, R_n) \\ &\geq \Phi_{U(b,R_1)}(Q_1, R_1, R_3, \ldots, R_n) \\ &\geq \Phi_{U(b,R_1)}(Q_1, Q_1, R_3, \ldots, R_n) \\ &= g_{U(b,R_1)}(Q_1, R_3, \ldots, R_n), \end{aligned}$$

where the inequalities follow from strategy-proofness of Φ. The proof of Claim 1 is now complete by the induction hypothesis.[4]

Thus, by Claim 1, we have $\Phi_{\{x,y\}}(R_N) = 1$ for all $R_N \in \mathcal{D}^n$ with $R_1 = R_2$. Our next claim shows that the same holds if $\tau(R_1) = \tau(R_2)$.

[4]We have included the proof of Claim 1 for completeness. It can also be found in [98].

Claim 2 Let R_N be a preference profile such that $\tau(R_1) = \tau(R_2)$. Then $\Phi_{\{x,y\}}(R_N) = 1$.

To prove this claim, first suppose that $\tau(R_1) = \tau(R_2) = \{x, y\}$. Then, if $\Phi_{\{x,y\}}(R_N) < 1$, player 1 manipulates at R_N via R_2 since by Claim 1, $\Phi_{\{x,y\}}(R_2, R_2, R_{N\setminus\{1,2\}}) = 1$. Now consider the case $\tau(R_1) = \tau(R_2) \in \{x, y\}$, say $\tau(R_1) = \tau(R_2) = x$. By Claim 1 we have $\Phi_{\{x,y\}}(R_1, R_1, R_{N\setminus\{1,2\}}) = \Phi_{\{x,y\}}(R_2, R_2, R_{N\setminus\{1,2\}}) = 1$. Moreover, since $\tau(R_1) = \tau(R_2) = x$ we have by strategy-proofness $\Phi_x(R_1, R_1, R_{N\setminus\{1,2\}}) = \Phi_x(R_1, R_2, R_{N\setminus\{1,2\}}) = \Phi_x(R_2, R_2, R_{N\setminus\{1,2\}}) = \varepsilon$ (say).

Since $\mathcal{D}$ is a binary restricted domain, if $\tau(R_i) \neq y$ for all $i \in N \setminus \{1, 2\}$, then by unanimity $\Phi_{\{x,y\}}(R_N) = \Phi_x(R_N) = 1$, and we are done. Now suppose there is $i \in N \setminus \{1, 2\}$ such that $\tau(R_i) = y$. Let $R \in \mathcal{D}$ be such that $\tau(R) = y$ and $U(x, R) \cap U(y, R_1) = \{x, y\}$. Such an R exists since $\mathcal{D}$ is a binary restricted domain. Consider the preference profile $\bar{R}_{N\setminus\{1,2\}}$ of the agents in $N \setminus \{1, 2\}$ defined as follows: for all $i \in N \setminus \{1, 2\}$

$$\bar{R}_i = \begin{cases} R \text{ if } \tau(R_i) = y \\ R_i \text{ otherwise.} \end{cases}$$

By Claim 1, $\Phi_{\{x,y\}}(R_1, R_1, \bar{R}_{N\setminus\{1,2\}}) = \Phi_{\{x,y\}}(R_2, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1$. Since $\tau(R_1) = \tau(R_2) = x$, we have by strategy-proofness $\Phi_x(R_1, R_1, \bar{R}_{N\setminus\{1,2\}}) = \Phi_x(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = \Phi_x(R_2, R_2, \bar{R}_{N\setminus\{1,2\}})$. We show $\Phi_x(R_1, R_1, \bar{R}_{N\setminus\{1,2\}}) = \varepsilon$. First we claim that $\Phi_y(R_1, R_1, R_{N\setminus\{1,2\}}) = \Phi_y(R_1, R_1, \bar{R}_{N\setminus\{1,2\}})$. To see this, consider a player $i \in N \setminus \{1, 2\}$ such that $R_i \neq \bar{R}_i$. Then $\tau(R_i) = \tau(\bar{R}_i) = y$, hence by strategy-proofness we have $\Phi_y(R_1, R_1, R_i, R_{N\setminus\{1,2,i\}}) = \Phi_y(R_1, R_1, \bar{R}_i, R_{N\setminus\{1,2,i\}})$. By repeating this argument, $\Phi_y(R_1, R_1, R_{N\setminus\{1,2\}}) = \Phi_y(R_1, R_1, \bar{R}_{N\setminus\{1,2\}})$. Hence, since $\Phi_{\{x,y\}}(R_1, R_1, \bar{R}_{N\setminus\{1,2\}}) = 1$, we obtain $\Phi_x(R_1, R_1, \bar{R}_{N\setminus\{1,2\}}) = \varepsilon$.

Using similar logic it follows that $\Phi_y(R_1, R_2, R_{N\setminus\{1,2\}}) = \Phi_y(R_1, R_2, \bar{R}_{N\setminus\{1,2\}})$. We complete the proof by showing $\Phi_y(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1 - \varepsilon$. For this, since $\Phi_x(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = \varepsilon$, it suffices to show that $\Phi_{\{x,y\}}(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1$. Suppose that $\Phi_B(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) > 0$ for $B = A \setminus U(y, R_1)$. Then agent 1 manipulates at $(R_1, R_2, \bar{R}_{N\setminus\{1,2\}})$ via R_2 since $\Phi_{\{x,y\}}(R_2, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1$. Thus, $\Phi_{U(y,R_1)}(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1$. Next we show that $\Phi_{U(x,R)}(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1$. If not, consider $i \in N \setminus \{1, 2\}$ such that $\bar{R}_i = R$. Let R'_i be such that $\tau(R'_i) = x$. Then by strategy-proofness $\Phi_{U(x,R)}(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) \geq \Phi_{U(x,R)}(R_1, R_2, R'_i, \bar{R}_{N\setminus\{1,2,i\}})$. By sequentially changing the preferences of the players in $N \setminus \{1, 2\}$ with y at the top in this manner we construct a preference profile $\hat{R}$ such that $\tau(\hat{R}_i) = x$ for all $i \in N$ and $\Phi_{U(x,R)}(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) \geq \Phi_{U(x,R)}(\hat{R}) = 1$. Hence $\Phi_{U(x,R)}(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1$.

Since $\Phi_{U(y,R_1)}(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1$, $\Phi_{U(x,R)}(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1$, and $U(y, R_1) \cap U(x, R) = \{x, y\}$, we have $\Phi_{\{x,y\}}(R_1, R_2, R_{N\setminus\{1,2\}}) = 1$. This completes the proof of Claim 2.

We can now complete the proof of the proposition. Let $R_N \in \mathcal{D}^n$ be an arbitrary preference profile. We show that $\Phi_{\{x,y\}}(R_N) = 1$. In view of Claim 2, we may assume $\tau(R_1) \neq \tau(R_2)$. Note that if $\tau(R_i) = \{x, y\}$

for some $i \in \{1,2\}$ and $\Phi_{A\setminus\{x,y\}}(R_N) > 0$, then agent i manipulates at R_N via R_j, where $j \in \{1,2\}, j \neq i$, since by Claim 1 we have $\Phi_{\{x,y\}}(R_j, R_j, R_{N\setminus\{1,2\}}) = 1$. So we may assume without loss of generality that $\tau(R_1) = x$ and $\tau(R_2) = y$.

Suppose $U(y,R_1) \cap U(x,R_2) = \{x,y\}$. If $\Phi_{A\setminus U(x,R_2)}(R_N) > 0$, then agent 2 manipulates at R_N via R_1 since, by Claim 1, $\Phi_{\{x,y\}}(R_1, R_1, R_{N\setminus\{1,2\}}) = 1$. Thus, $\Phi_{U(x,R_2)}(R_N) = 1$, and similarly one proves $\Phi_{U(y,R_1)}(R_N) = 1$. Together with $U(y,R_1) \cap U(x,R_2) = \{x,y\}$, this implies $\Phi_{\{x,y\}}(R_N) = 1$.

Finally, suppose $U(y,R_1) \cap U(x,R_2) \neq \{x,y\}$. Since $\mathcal{D}$ is a binary restricted domain there exist $R_1' \in \mathcal{D}^x$ and $R_2' \in \mathcal{D}^y$ such that $U(y,R_1) \cap U(x,R_2') = \{x,y\}$ and $U(y,R_1') \cap U(x,R_2) = \{x,y\}$. Since $\tau(R_1) = \tau(R_1') = x$ and $\tau(R_2) = \tau(R_2') = y$, by strategy-proofness we have $\Phi_x(R_1, R_2, R_{N\setminus\{1,2\}}) = \Phi_x(R_1', R_2, R_{N\setminus\{1,2\}})$ and $\Phi_y(R_1, R_2, R_{N\setminus\{1,2\}}) = \Phi_y(R_1, R_2', R_{N\setminus\{1,2\}})$. By a similar argument as in the last paragraph of proof of Proposition 2.3.1 we have $\Phi_x(R_1, R_2', R_{N\setminus\{1,2\}}) = \Phi_x(R_1', R_2, R_{N\setminus\{1,2\}})$. Hence, $\Phi_{\{x,y\}}(R_1, R_2, R_{N\setminus\{1,2\}}) = \Phi_{\{x,y\}}(R_1, R_2', R_{N\setminus\{1,2\}})$. However, $\Phi_{\{x,y\}}(R_1, R_2', R_{N\setminus\{1,2\}}) = 1$ since $U(y,R_1) \cap U(x,R_2') = \{x,y\}$, which completes the proof of the proposition. ■

Theorem 2.3.3 now follows from Propositions 2.3.1 and 2.3.2. Moreover, we have the following consequence of Theorem 2.3.3 and Corollary 2.3.1.

Corollary 2.3.2 *Every binary restricted domain is a deterministic extreme point domain.*

2.3.3 Characterization of strategy-proof and unanimous rules

In this subsection we give a characterization of all strategy-proof and unanimous PRs on a binary restricted domain. In view of Corollary 2.3.2, it will be sufficient to give a characterization of strategy-proof and unanimous DRs on a binary restricted domain.

Throughout this subsection let $\mathcal{D}$ be a binary restricted domain over $\{x,y\}$. For $R_N \in \mathcal{D}^n$, by $N^x(R_N)$ we denote the set of agents $i \in N$ such that $\tau(R_i) = x$; by $N^{xy}(R_N)$ the set of agents $i \in N$ such that $\tau(R_i) = \{x,y\}$; and we define

$$\mathcal{I}(R_N) = \{Q_N \in \mathcal{D}^n : N^{xy}(Q_N) = N^{xy}(R_N) \text{ and } R_i = Q_i \text{ for every } i \in N^{xy}(R_N)\}.$$

Thus, $\mathcal{I}(R_N)$ is the (equivalence) class of all preference profiles that share with R_N the set of agents who are indifferent between x and y and have the same preference as in R_N.

For $R_N \in \mathcal{D}^N$ a *committee* $\mathcal{W}(R_N)$ is a set of subsets of N such that:

(1) If $N^{xy}(R_N) = N$ then $\mathcal{W}(R_N) = \emptyset$ or $\mathcal{W}(R_N) = \{\emptyset\}$.

(2) If $N^{xy}(R_N) \neq N$ then $\mathcal{W}(R_N) \subseteq 2^{N\setminus N^{xy}(R_N)}$ satisfies

(i) $\emptyset \notin \mathcal{W}(R_N)$ and $N \setminus N^{xy}(R_N) \in \mathcal{W}(R_N)$,

(ii) for all $S, T \subseteq N \setminus N^{xy}(R_N)$, if $S \subseteq T$ and $S \in \mathcal{W}(R_N)$, then $T \in \mathcal{W}(R_N)$.

In case (2) in the above definition, a committee is a simple game, elements of $\mathcal{W}(R_N)$ are called *winning coalitions*, and other subsets of $N \setminus N^{x,y}(R_N)$ are called *losing coalitions*.

A collection of committees $\mathcal{W} = \{\mathcal{W}(R_N) : R_N \in \mathcal{D}^n\}$ is an *admissible collection of committees* (ACC) if the following three conditions hold:

a) For all $R_N, Q_N \in \mathcal{D}^n$, if $Q_N \in \mathcal{I}(R_N)$ then $\mathcal{W}(Q_N) = \mathcal{W}(R_N)$.

b) For all $R_N \in \mathcal{D}^n$, $i \in N \setminus N^{xy}(R_N)$, $R'_i \in \mathcal{D}$ such that $\tau(R'_i) = \{x, y\}$, and $C \in \mathcal{W}(R_N)$, if $i \notin C$, then $C \in \mathcal{W}(R_{N \setminus i}, R'_i)$.

c) For all $R_N \in \mathcal{D}^n$, $i \in N \setminus N^{xy}(R_N)$, $R'_i \in \mathcal{D}$ such that $\tau(R'_i) = \{x, y\}$, and $C \notin \mathcal{W}(R_N)$, if $i \in C$, then $C \setminus \{i\} \notin \mathcal{W}(R_{N \setminus i}, R'_i)$.

Thus, a collection of committees is admissible if a) each committee depends only on the set of indifferent agents and their preferences; b) if a coalition is winning and an agent not belonging to it becomes indifferent, then the coalition stays winning; and c) if a coalition is losing and an agent belonging to it becomes indifferent, then the coalition without that agent stays losing. Observe that a), b), and c) are trivially fulfilled if $N^{xy}(R_N) = N$, i.e., if all agents are indifferent. In particular, in that case $\mathcal{I}(R_N) = \{R_N\}$.

With an ACC $\mathcal{W}$ we associate a DR $f_{\mathcal{W}}$ as follows: for every $R_N \in \mathcal{D}^n$,

$$f_{\mathcal{W}}(R_N) = \begin{cases} x & \text{if } N^x(R_N) \in \mathcal{W}(R_N) \\ y & \text{if } N^x(R_N) \notin \mathcal{W}(R_N). \end{cases}$$

We now show that every strategy-proof and unanimous DR is of the form $f_{\mathcal{W}}$. We just outline the proof since it is rather standard, and, moreover, the theorem is almost equivalent to Theorem 3 in [66]. A (nonessential) difference is that the last mentioned result is formulated for the case where $A = \{x, y\}$, so that all preference profiles with the same indifferent agents are equivalent, making our condition a) on an ACC redundant.

Theorem 2.3.5 *Let $\mathcal{D}$ be a binary restricted domain. A DR f on $\mathcal{D}^n$ is strategy-proof and unanimous if and only if there is an ACC $\mathcal{W}$ such that $f = f_{\mathcal{W}}$.*

Proof: For the only-if part, let f be a strategy-proof and unanimous DR. For each $R_N \in \mathcal{D}^n$ we define the set $\mathcal{W}_f(R_N)$ of coalitions as follows. If $N^{xy}(R_N) = N$ then $\mathcal{W}_f(R_N) = \{\emptyset\}$ if $f(R_N) = x$ and $\mathcal{W}_f(R_N) = \emptyset$ otherwise. If $N^{xy}(R_N) \neq N$ then for every $C \subseteq N \setminus N^{xy}(R_N)$, $C \in \mathcal{W}_f(R_N)$ if and only if there is a

$Q_N \in \mathcal{I}(R_N)$ such that $f(Q_N) = x$ and $C = N^x(Q_N)$. Then $\mathcal{W}_f(R_N)$ is a committee for each $R_N \in \mathcal{D}^N$ by unanimity and strategy-proofness of f. Also, the collection $\mathcal{W}_f = \{\mathcal{W}_f(R_N) : R_N \in \mathcal{D}^n\}$ is an ACC: a) follows directly by definition of the committees $\mathcal{W}_f(R_N)$; and b) and c) follow from unanimity and strategy-proofness of f. Finally, it is straightforward to check that $f = f_{\mathcal{W}_f}$.

For the if-part, let $\mathcal{W}$ be an ACC. Then it is easy to check that $f = f_{\mathcal{W}}$ is strategy-proof and unanimous.
■

By Corollary 2.3.2 and Theorem 2.3.5 we obtain the following result.

Corollary 2.3.3 *Let $\mathcal{D}$ be a binary restricted domain. A PR f on $\mathcal{D}^n$ is strategy-proof and unanimous if and only if it is a convex combination of DRs of the form $f = f_{\mathcal{W}}$ for ACCs $\mathcal{W}$.*

Remark 2.3.6 *The set of winning coalitions $\mathcal{W}(R_N)$ may indeed depend on the preference profile of the indifferent agents, i.e., the agents in $\mathcal{I}(R_N)$. Here is an example. Let $N = \{1, 2, 3\}$, $A = \{x, y, v, w\}$ and define: $\mathcal{W}(R_N) = \{\{1, 3\}, N\}$ if $N^{xy}(R_N) = \emptyset$; $\mathcal{W}(R_N) = \{\{1, 3\}\}$ if $N^{xy}(R_N) = \{2\}$; $\mathcal{W}(R_N) = \{\{2, 3\}\}$ if $N^{xy}(R_N) = \{1\}$; $\mathcal{W}(R_N) = \{\{1, 2\}\}$ if $N^{xy}(R_N) = \{3\}$ and vR_3w; $\mathcal{W}(R_N) = \{\{1\}, \{1, 2\}\}$ if $N^{xy}(R_N) = \{3\}$ and wP_3v; and $\mathcal{W}(R_N) = \{\emptyset\}$ if $N^{xy}(R_N) = N$. Then it is straightforward to verify that $f_{\mathcal{W}}$ is strategy-proof and unanimous.*

2.4 Application to single-dipped preferences

In this section we apply our results to single-dipped domains and characterize all strategy-proof and unanimous PRs on such a domain.

Definition 2.4.1 *A preference of agent $i \in N$, $R_i \in \mathbb{W}(A)$, is* single-dipped *on A relative to a linear ordering $\succ$ of the set of alternatives if*

(i) R_i has a unique minimal element $d(R_i)$, the dip *of R_i and*

(ii) for all $y, z \in A$, $[d(R_i) \succeq y \succ z$ or $z \succ y \succeq d(R_i)] \Rightarrow zP_iy$.

Let $\mathcal{D}_\succ$ denote the set of all single-dipped preferences relative to the ordering $\succ$, and let $\mathcal{R}_\succ \subseteq \mathcal{D}_\succ$. Clearly $\mathcal{D}_\succ$ is a binary restricted domain. Moreover, $\mathcal{R}_\succ$ is a binary restricted domain if it satisfies condition (ii) in Definition 2.2.9, the definition of a binary restricted domain. Hence, by Corollary 2.3.2 and Theorem 2.3.5 we obtain the following result.

Corollary 2.4.1 *Let $\succ$ be a linear ordering over A and let $\mathcal{R}_\succ \subseteq \mathcal{D}_\succ$ satisfy (ii) in Definition 2.2.9. Then a PR on $\mathcal{R}^n_\succ$ is strategy-proof and unanimous if and only if it it is a convex combination of DRs on $\mathcal{R}^n_\succ$ of the form $f = f_{\mathcal{W}}$ for ACCs $\mathcal{W}$.*

Consider a single-dipped domain where the alternatives are assumed to be equidistant from each other and preference is consistent with the distance from the dip. More precisely, when the distance of an alternative from the dip of an agent is higher than that of another alternative, the agent prefers the former alternative to the latter. Call such a domain a 'distance single-dipped domain'. If ties between equidistant alternatives are broken in both ways, then such a domain is again a binary restricted domain, and Corollary 2.4.1 applies. However, if ties are broken in favor of the left side (or of the right side) only, then the domain is no longer a binary restricted domain. Indeed, in Example 2.4.2 we show that there exists a strategy-proof and unanimous PR that does not have binary support.

Example 2.4.2 *Consider the distance single-dipped domain presented in the table below. There are two agents and four alternatives: think of the alternatives as located on a line in the ordering* $x_1 < x_2 < x_3 < x_4$ *with equal distances. Ties are always broken in favor of the left alternative. It is not hard to verify that the PR given in the table (probabilities in the order* x_1, x_2, x_3, x_4*, and* $0 < \beta < \alpha < 1$*,* $0 < \gamma < \varepsilon < 1$ *arbitrary) is strategy-proof and unanimous, but does not have binary support.*

1/2	$x_1x_2x_3x_4$	$x_4x_3x_2x_1$	$x_4x_1x_3x_2$	$x_1x_2x_4x_3$
$x_1x_2x_3x_4$	$(1,0,0,0)$	$(\alpha-\beta,\beta,0,1-\alpha)$	$(\alpha,0,0,1-\alpha)$	$(1,0,0,0)$
$x_4x_3x_2x_1$	$(\varepsilon-\gamma,\gamma,0,1-\varepsilon)$	$(0,0,0,1)$	$(0,0,0,1)$	$(\varepsilon-\gamma,\gamma,0,1-\varepsilon)$
$x_4x_1x_3x_2$	$(\varepsilon,0,0,1-\varepsilon)$	$(0,0,0,1)$	$(0,0,0,1)$	$(\varepsilon,0,0,1-\varepsilon)$
$x_1x_2x_4x_3$	$(1,0,0,0)$	$(\alpha-\beta,\beta,0,1-\alpha)$	$(\alpha,0,0,1-\alpha)$	$(1,0,0,0)$

Remark 2.4.3 Other examples of binary restricted domains are single-peaked domains where each peak can only be one of two fixed adjacent alternatives, or certain single-crossing domains with only two alternatives that can serve as top alternative. These domains, however, are of limited interest within the single-peaked and single-crossing domains, respectively.

Of course, there are binary restricted domains which are much larger than and considerably different from single-dipped domains – an obvious example is the domain of all preferences with x or y or both on top, or any subdomain including a preference with x on top and y second and a preference with y on top and x second.

2.5 Infinitely many alternatives

In this section we assume that the set of alternatives A may be an infinite set, for instance a closed interval in $\mathbb{R}$. We assume A to be endowed with a σ-algebra of measurable sets; only preferences in $\mathbb{W}(A)$ for which the upper contour sets $U(x,R)$, $x \in A$, are measurable, are considered. A PR Φ assigns to an admissible preference profile a probability distribution over the measurable space A, hence a probability to every measurable set. The set of all such probability distributions will still be denoted as $\Delta(A)$. For a measurable set $B \subseteq A$, $\Phi_B(R_N)$ denotes the probability assigned to B if the preference profile is R_N. All the introduced concepts and definitions extend in a straightforward manner to this setting. In particular, Definitions 2.2.1–2.2.7, 2.2.9, and 2.2.10 are literally the same. Also Propositions 2.3.1 and 2.3.2 are still valid, and therefore Theorem 2.3.3 still holds: a binary restricted domain over $\{x, y\}$ $(x, y \in A)$ is a binary support domain. The purpose of this section is to provide a characterization of all strategy-proof and unanimous PRs on a binary restricted domain.

Let $\mathcal{D}$ be a binary restricted domain over $\{x, y\}$ for some $x, y \in A$. We use some of the notations introduced in Section 2.3.3. For $R_N \in \mathcal{D}^n$ with $N^{xy}(R_N) = N$ we let $h(R_N) = h(R_N)(\emptyset) \in [0, 1]$ and for $R_N \in \mathcal{D}^n$ with $N^{xy}(R_N) \neq N$ we let $h(R_N) : 2^{N \setminus N^{xy}(R_N)} \to [0, 1]$ satisfy $h(R_N)(\emptyset) = 0$, $h(R_N)(N \setminus N^{xy}(R_N)) = 1$, and $h(R_N)(C) \leq h(R_N)(C')$ for all $C, C' \subseteq N \setminus N^{xy}(R_N)$ with $C \subseteq C'$; we assume, moreover, that $h(Q_N) = h(R_N)$ whenever $Q_N \in \mathcal{I}(R_N)$ and that

$$h(R_N)(C \setminus i) \leq h(R'_N)(C \setminus i) \leq h(R_N)(C)$$

whenever $i \in N \setminus N^{xy}(R_N)$, $R'_N = (R_{N\setminus i}, R'_i)$ for some R'_i with $\tau(R'_i) = \{x, y\}$, and $C \subseteq N \setminus N^{xy}(R_N)$ with $i \in C$. Observe that such an h generalizes the concept of an admissible collection of committees: we call h a *probabilistic admissible collection of committees* (PACC). For $R_N \in \mathcal{D}^n$ with $N^{xy}(R_N) \neq N$, the number $h(R_N)(C)$ can be interpreted as the probability that a coalition C is winning given a profile with $N^{xy}(R_N)$ as the set of agents who are indifferent between x and y and having $R_{N^{xy}(R_N)}$ as preference profile; specifically, if C is the set of agents with x on top, then this probability will be assigned to x. If $N^{xy}(R_N) = N$, then $h(R_N) = h(R_N)(\emptyset)$ is the probability assigned to x.

We say that a PR Φ on $\mathcal{D}^n$ *is associated with a* PACC h if (i) $\Phi_{\{x,y\}}(R_N) = 1$ for all $R_N \in \mathcal{D}^n$; (ii) $\Phi_x(R_N) = h(R_N)(N^x(R_N))$ for all $R_N \in \mathcal{D}^n$.

We have the following result.

Theorem 2.5.1 *Let $\mathcal{D}$ be a binary restricted domain over $\{x, y\}$. A PR Φ on $\mathcal{D}^n$ is strategy-proof and unanimous if and only if it is associated with a PACC.*

Proof: For the if-part, let PR Φ be a associated with a PACC h. We show that Φ is unanimous and strategy-proof.

We first show that Φ is unanimous. Consider a profile $R_N \in \mathcal{D}^n$ such that $\cap_{i \in N} \tau(R_i) \neq \emptyset$. If $\tau(R_i) = \{x, y\}$ for all $i \in N$ then unanimity holds by definition. Suppose $\cap_{i \in N} \tau(R_i) = x$. Then $N^x(R_N) = N \setminus N^{xy}(R_N)$. Since $h(R_N)(N \setminus N^{xy}(R_N)) = 1$, we have $\Phi_x(R_N) = 1$. If $\cap_{i \in N} \tau(R_i) = y$ then $N^x(R_N) = \emptyset$ which implies $\Phi_x(R_N) = h(R_N)(\emptyset) = 0$. So, $\Phi_y(R_N) = 1$.

Next we show that Φ is strategy-proof. Consider a profile $R_N \in \mathcal{D}^n$. We only need to consider $i \in N \setminus N^{xy}(R_N)$. Let $R'_i \in \mathcal{D}$ and write $R'_N = (R_{N\setminus i}, R'_i)$. We distinguish four cases and each time show that i cannot improve by R'_i. (i) If $\tau(R_i) = x$ and $\tau(R'_i) = y$ then $\Phi_x(R_N) = h(R_N)(N^x(R_N)) \geq h(R'_N)(N^x(R_N) \setminus i) = h(R'_N)(N^x(R'_N)) = \Phi_x(R'_N)$ by definition of h. (ii) If $\tau(R_i) = y$ and $\tau(R'_i) = x$ then $\Phi_x(R_N) = h(R_N)(N^x(R_N)) \leq h(R'_N)(N^x(R_N)) = h(R'_N)(N^x(R'_N)) = \Phi_x(R'_N)$. This implies $\Phi_y(R_N) \geq \Phi_y(R'_N)$. (iii) If $\tau(R_i) = x$ and $\tau(R'_i) = \{x, y\}$, then, since $N^x(R_N) \setminus i = N^x(R'_i, R_{N\setminus i})$, we have $\Phi_x(R_N) = h(R_N)(N^x(R_N)) \geq h(R'_N)(N^x(R'_N)) = \Phi_x(R'_N)$. (iv) Finally, if $\tau(R_i) = y$ and $\tau(R'_i) = \{x, y\}$, then $\Phi_x(R_N) = h(R_N)(N^x(R_N)) \leq h(R'_N)(N^x(R'_N)) = \Phi_x(R'_N)$, which implies $\Phi_y(R'_N) \leq \Phi_y(R_N)$. This completes the proof that Φ is strategy-proof.

For the only-if part, consider a unanimous and strategy-proof PR Φ on $\mathcal{D}^n$. Then $\Phi_{\{x,y\}}(R_N) = 1$ for all $R_N \in \mathcal{D}^n$ by (the modified version of) Theorem 2.3.3. We show that Φ is associated with a PACC h. If $R_N \in \mathcal{D}^n$ with $N^{xy}(R_N) = N$, then we define $h(R_N) = h(R_N)(\emptyset) = \Phi_x(R_N)$. Now let $R_N \in \mathcal{D}^n$ with $N^{xy}(R_N) \neq N$. By strategy-proofness, $\Phi(Q_N) = \Phi(R_N)$ for all $Q_N \in \mathcal{D}^n$ with $Q_N \in \mathcal{I}(R_N)$ and $N^x(Q_N) = N^x(R_N)$. Therefore, we can define $h(R_N)(C) = \Phi_x(Q_N)$ for any $Q_N \in \mathcal{I}(R_N)$ such that $C = N^x(Q_N)$. By unanimity of Φ, $h(R_N)(\emptyset) = 0$ and $h(R_N)(N \setminus N^{xy}(R_N)) = 1$. By strategy-proofness, $h(R_N)(C) \leq h(R_N)(C')$ for all $C, C' \subseteq N \setminus N^{xy}(R_N)$ with $C \subseteq C'$.

Clearly, $h(Q_N) = h(R_N)$ whenever $R_N \in \mathcal{D}^n$ and $Q_N \in \mathcal{I}(R_N)$.

Let $R_N \in \mathcal{D}^n$, $i \in N \setminus N^{xy}(R_N)$, $R'_N = (R_{N\setminus i}, R'_i)$ for some R'_i with $\tau(R'_i) = \{x, y\}$, and $C \subseteq N \setminus N^{xy}(R_N)$ with $i \in C$. Consider $Q_N \in \mathcal{I}(R_N)$ with $N^x(Q_N) = C$. Then by strategy-proofness we have $h(R_N)(C) = \Phi_x(Q_N) \geq \Phi_x(Q_{N\setminus i}, R'_i) = h(R_{N\setminus i}, R'_i)(C \setminus i) = h(R'_N)(C \setminus i)$. Finally, consider $V_N \in \mathcal{I}(R_N)$ with $N^x(V_N) = C \setminus i$. Again by strategy-proofness we obtain $h(R_N)(C \setminus i) = \Phi_x(V_N) \leq \Phi_x(V_{N\setminus i}, R'_i) = h(R_{N\setminus i}, R'_i)(C \setminus i) = h(R'_N)(C \setminus i)$. ■

We conclude the paper with some thoughts about extending Theorem 2.3.1 and Corollary 2.3.1 to the case of infinitely many alternatives. As to extending Theorem 2.3.1, which states that a domain is a deterministic extreme point domain if and only if each strategy-proof and unanimous strict probabilistic rule can be written as a convex combination of two other strategy-proof and unanimous probabilistic rules, for the infinite case one may try and find a suitable topology on the set of all such rules so that it becomes a convex and compact subset of a topological vector space. Then, one could apply a topological version of the Krein-Milman Theorem (e.g., Theorem III.4.1 in [16]) and conclude that each strategy-proof and unanimous probabilistic rule is in the closure of the convex hull of the strategy-proof and unanimous deterministic rules. This, however, does not seem a straightforward exercise, and also does not deliver the exact analogue of Theorem 2.3.1. Next, Corollary 2.3.1 states that for the case of finitely many alternatives every binary support domain is a deterministic extreme point domain. This is a direct consequence of

Theorem 2.3.1 and Theorem 2.3.2, where the latter theorem states that every strategy-proof and unanimous strict probabilistic rule assigning positive probability to only two alternatives x and y, can be written as a convex combination of two other such rules. Again, extending this theorem to the case of infinitely many alternatives does not seem to be a sinecure: the proof for the finite case heavily uses the fact that if a probability $p \in (0, 1)$ is assigned to x at some preference profile, then we can find an interval around p such that at each other profile either probability p is assigned to x or some probability outside this interval. A proof along this line seems to break down if there are infinitely many alternatives.

3

A Characterization of Random Min-max Domains and Its Applications

3.1 Introduction

3.1.1 Background of the problem

We analyze the classical social choice problem of choosing an alternative from a set of feasible alternatives based on the preferences of individuals in a society. Such a procedure is known as a *deterministic social choice function* (DSCF). Arrow, Gibbard, and Satterthwaite have identified some desirable properties of such a DSCF such as *unanimity* and *strategy-proofness*. A DSCF is strategy-proof if a strategic individual cannot change its outcome in her favor by misreporting her preferences, and it is unanimous if, whenever all the individuals have the same most preferred alternative, that alternative is chosen. The classic [56]-[96] impossibility theorem states that if there are at least three alternatives and the preferences of the individuals are *unrestricted*, then the only DSCFs that are unanimous and strategy-proof are *dictatorial*. A DSCF is called dictatorial if there exists an individual, called the *dictator*, whose most preferred alternative is always chosen by the DSCF.

Although unanimity and strategy-proofness are desirable properties of a DSCF, the assumption of an unrestricted domain made in the Gibbard-Satterthwaite Theorem is quite strong. Not only do their exist many political and economic scenarios where preferences of individuals satisfy natural restrictions such as *single-peakedness*, but also the conclusion of the Gibbard-Satterthwaite Theorem does not apply to such restricted domains. Consequently, domain restrictions turn out to be an obvious and useful way of evading the dictatorship result in social choice theory.

The single-peaked property is commonly used in a public good location problem. Such a domain restriction occurs in an environment where strictly quasi-concave utility functions are maximized over a linear budget set. The study of single-peaked domains can be traced back to [20] where he shows that a *Condorcet winner* exists on such domains. Later, [72] and [103] show that a DSCF on a single-peaked domain is unanimous and strategy-proof if and only if it is a *min-max* rule. In a recent paper, [2] characterize all domains on which a DSCF is unanimous and strategy-proof if and only if it is a min-max rule. They call such domains min-max domains.

The horizon of social choice theory have been expanded by the concept of *random social choice functions* (RSCF). An RSCF assigns a probability distribution over the alternatives at every preference profile. Thus, RSCFs are generalization of DSCFs. The importance of RSCFs over DSCFs has been well-established in the literature (see, for example, [46], [81]).

The study of RSCFs dates back to [57] where he shows that an RSCF on the unrestricted domain is unanimous and strategy-proof if and only if it is a *random dictatorial* rule. A random dictatorial rule is a convex combination of dictatorial rules. [46] characterize the unanimous and strategy-proof random rules on maximal single-peaked domains, and [81] show that such a rule is a convex combination of min-max rules. [87] establish a similar result by using the theory of totally unimodular matrices from combinatorial integer programming.

3.1.2 Our motivation

Our motivations behind this work are as follows:

- As we have discussed earlier, single-peaked domains are very useful in modeling preferences in many practical situations. However, to the best of our knowledge, there is no characterization available in the literature of the unanimous and strategy-proof RSCFs on single-peaked domains other than the maximal single-peaked domain and minimally rich single-peaked domains. The maximal single-peaked domain requires that every single-peaked preference is present in the domain. On the other hand, minimally rich single-peaked domains require presence of 'extreme' single-peaked preferences such as the ones in which all the alternatives on the left (right) side of the top-ranked alternative are preferred to all those on the right (left) side of the same. Both these domains are quite demanding for practical purposes. This motivates us to investigate the structure of the unanimous and strategy-proof RSCFs on other single-peaked domains.

- Min-max rules are quite simple to understand, intuitively appealing, and easy to work with. They also have desirable properties like tops-onliness and anonymity (a class of min-max rules called median rules). This motivates us to find all domains on which a rule (RSCF or DSCF) is unanimous and strategy-proof if and only if it is a min-max rule.

- A domain satisfies the *deterministic extreme point* (DEP) property if every unanimous and strategy-proof RSCF on it can be written as a convex combination of the unanimous and strategy-proof DSCFs on that domain. Such a property of a domain is very useful in finding socially optimal strategy-proof RSCFs.[1] This is because, if a domain satisfies the DEP property, then the maximum expected social welfare will always be achieved by some strategy-proof DSCF. This reduces the problem of finding socially optimal strategy-proof RSCFs to that of finding socially optimal strategy-proof DSCFs. [55] characterize socially optimal strategy-proof DSCFs on regular single-crossing domains. It is worth noting that a regular single-crossing domain is single-peaked. Therefore, if such single-peaked domains satisfy the DEP property, then the same rules as found in [55] will continue to be optimal amongst the strategy-proof RSCFs. This motivates us to characterize all single-peaked domains that satisfy the DEP property.

3.1.3 Our contribution

We provide a characterization of the unanimous and strategy-proof RSCFs on top-connected single-peaked domains. For such domains, there is a prior ordering over the alternatives. The top-set of a domain consists of those alternatives that appear as a top-ranked alternative in some preference in the domain. Two alternatives a_r and a_s are called consecutive in the top-set of a domain if both of them belong to the top-set and no alternative in-between (with respect to the given prior order) them belongs to the same set. A domain is called top-connected if, for every two alternatives a_r and a_{r+s} that are consecutive in the top-set, there are two preferences P and P' such that the alternatives $a_r, a_{r+1}, \ldots, a_{r+s}$ appear successively from the top in P, and the alternatives $a_{r+s}, a_{r+s-1}, \ldots, a_r$ appear successively from the top in P'. For example, if the set of alternatives is $\{a_1, \ldots, a_{10}\}$ and the top-set of a domain is $\{a_3, a_5, a_8, a_9\}$, then, for instance, alternatives a_5 and a_8 are consecutive in the top-set of that domain. Top-connectedness for such a domain requires the presence of preferences such as $P = a_5a_6a_7a_8\ldots$ and $P' = a_8a_7a_6a_5\ldots$, where by $abc\ldots$ we denote a preference in which a is ranked first, b is ranked second, c

[1] An RSCF is socially optimal if it maximizes the sum of the expected utilities (ordinal or cardinal, depending on the model) of the individuals with respect to some prior distribution over the preferences of the individuals of the society.

is ranked third, and the other alternatives are arbitrarily ranked in the remaining positions. Note that if the top-set of a domain consists of all alternatives (such a domain is called regular in the literature), then top-connectedness requires that for every two alternatives of the form a_r and a_{r+1}, there are two preferences P and P' such that a_r is ranked first and a_{r+1} is ranked second in P, and a_{r+1} is ranked first and a_r is ranked second in P'. Clearly, top-connectedness is a mild condition for a single-peaked domain. For instance, single-peaked domains that arise from situations where alternatives are equidistant from each other and preferences are based on Euclidean distances are top-connected. Thus, our result applies to a large class of single-peaked domains of practical importance. It is worth noting that [2] provide the deterministic analogue of our results.

Owing to the importance of the min-max rules and the DEP property, we characterize all random min-max domains. An RSCF is called random min-max if it can be written as a convex combination of the min-max rules, and a domain is called random min-max if an RSCF on it is unanimous and strategy-proof if and only if it is a random min-max rule. Thus, our result shows that a large class of domains of practical importance satisfies the DEP property.

As a by-product of our result, it follows that every top-connected single-peaked domain is tops-only for random rules. [30] provide a sufficient condition for a domain to be tops-only for DSCFs, and later [31] provide the same for RSCFs. However, top-connected single-peaked domains do not satisfy any of these conditions.

As applications of our result, we obtain a characterization of the unanimous and strategy-proof RSCFs on minimally rich single-peaked domains, regular single-crossing domains, and Euclidean domains. Minimally rich single-peaked domains are introduced in [81]. Such domains arise in the problem of locating a public good where agents are 'single-minded' in the sense that either they prefer the left direction or the right direction. Thus, for such a domain, either all the alternatives on the left side of the peak are preferred to those on the right side or vice versa. Single-crossing domains are well known for their frequent applications in models of income taxation and redistribution ([89], [69]), local public goods and stratification ([102], [48], [51]), and coalition formation ([41], [64]).[2] [94] provide a characterization of the unanimous and strategy-proof deterministic rules on such domains. Here, we provide the same for random rules under regularity. Euclidean domains arise in public good location problems where agents derive their preferences based on the Euclidean distances of the alternatives from their own location (which is the peak of the preference). The practical importance of such domains is well-established in the literature. [78] consider the problem of locating a public bad over two-dimensional Euclidean space and show that under some mild condition, every unanimous and

[2] Moreover, models that study the selection of policies in the market for higher education ([52]) and the choice of constitutional and voting rules ([9]) also use single-crossing domains. [93] has a detailed exposition on various applications, interpretations, and scopes of single-crossing domains.

strategy-proof SCF on such domains is dictatorial.

3.1.4 Organization of the paper

The paper is organized as follows. In Section 3.2, we introduce the basic model. Section 3.3 provides a characterization of the unanimous and strategy-proof random rules on top-connected single-peaked domains and Section 3.4 provides a characterization of the random min-max domains. We provide some applications of our results in Section 3.5. Finally, Section 3.6 concludes the paper.

3.2 Preliminaries

Let $N = \{1, \ldots, n\}$ be a finite set of agents. Except where otherwise mentioned, $n \geq 2$. Let $A = \{a_1, \ldots, a_m\}$ be a finite set of alternatives with a prior ordering $\prec$ given by $a_1 \prec \cdots \prec a_m$. Whenever we write minimum or maximum of a subset of A, we mean it w.r.t. the ordering $\prec$ over A. By $a \preceq b$, we mean $a = b$ or $a \prec b$. For $a, b \in A$, we define $[a, b] = \{c \mid \text{ either } a \preceq c \preceq b \text{ or } b \preceq c \preceq a\}$. By (a, b), we define $[a, b] \setminus \{a, b\}$. For notational convenience, whenever it is clear from the context, we do not use braces for singleton sets, i.e., we denote the set $\{i\}$ by i.

3.2.1 Domain of preferences and their properties

A complete, antisymmetric, and transitive binary relation over A (also called a linear order) is called a preference. We denote by $\mathbb{L}(A)$ the set of all preferences over A. For $P \in \mathbb{L}(A)$ and $a, b \in A$, aPb is interpreted as "a is strictly preferred to b according to P". For $P \in \mathbb{L}(A)$, by $P(k)$ we mean the k-th ranked alternative in P, i.e., $P(k) = a$ if and only if $|\{b \in A \mid bPa\}| = k - 1$. For $P \in \mathbb{L}(A)$ and $a \in A$, the *upper contour set* of a at P, denoted by $U(a, P)$, is defined as the set of alternatives that are as good as a in P, i.e., $U(a, P) = \{b \in A \mid bPa\} \cup a$. We denote by $\mathcal{D} \subseteq \mathbb{L}(A)$ a set of admissible preferences. For $a \in A$, let $\mathcal{D}^a = \{P \in \mathcal{D} \mid P(1) = a\}$. The top-set of a domain $\mathcal{D}$ is defined as $\tau(\mathcal{D}) = \cup_{P \in \mathcal{D}} P(1)$. A domain $\mathcal{D}$ is called *regular* if $\tau(\mathcal{D}) = A$.

Definition 3.2.1 *A preference P is called single-peaked if for all $a, b \in A$, $[P(1) \preceq a \prec b$ or $b \prec a \preceq P(1)]$ implies aPb. A domain is called single-peaked if each preference in the domain is single-peaked, and is called maximal single-peaked if it contains all single-peaked preferences.*

A preference profile, denoted by $P_N = (P_1, \ldots, P_n)$, is an element of $\mathcal{D}^n = \mathcal{D} \times \cdots \times \mathcal{D}$.

3.2.2 Social choice functions and their properties

A *Random Social Choice Function* (RSCF) is a function $\phi : \mathcal{D}^n \to \Delta A$, where ΔA denotes the set of probability distributions on A.

For $B \subseteq A$ and $P_N \in \mathcal{D}^n$, we define $\phi_B(P_N) = \sum_{a \in B} \phi_a(P_N)$, where $\phi_a(P_N)$ is the probability of a at $\phi(P_N)$.

For later reference, we include the following observation.

Remark 3.2.2 *For all $L, L' \in \Delta A$ and all $P \in \mathbb{L}(A)$, if $L_{U(x,P)} \geq L'_{U(x,P)}$ and $L'_{U(x,P)} \geq L_{U(x,P)}$ for all $x \in A$, then $L = L'$.*

Definition 3.2.3 *An RSCF $\phi : \mathcal{D}^n \to \Delta A$ is called unanimous if for all $a \in A$ and all $P_N \in \mathcal{D}^n$,*

$$[P_i(1) = a \text{ for all } i \in N] \Rightarrow [\phi_a(P_N) = 1].$$

Definition 3.2.4 *An RSCF $\phi : \mathcal{D}^n \to \Delta A$ is called strategy-proof if for all $i \in N$, all $P_N \in \mathcal{D}^n$, all $P'_i \in \mathcal{D}$, and all $x \in A$,*

$$\sum_{y \in U(x,P_i)} \phi_y(P_i, P_{-i}) \geq \sum_{y \in U(x,P_i)} \phi_y(P'_i, P_{-i}).$$

Remark 3.2.5 *An RSCF is called a deterministic social choice function (DSCF) if it selects a degenerate probability distribution at every preference profile. More formally, an RSCF $\phi : \mathcal{D}^n \to \Delta A$ is called a DSCF if $\phi_a(P_N) \in \{0, 1\}$ for all $a \in A$ and all $P_N \in \mathcal{D}^n$. The concepts of unanimity and strategy-proofness for DSCFs are special cases of the corresponding definitions for RSCFs.*

Definition 3.2.6 *An RSCF $\phi : \mathcal{D}^n \to \Delta A$ is called tops-only if $\phi(P_N) = \phi(P'_N)$ for all $P_N, P'_N \in \mathcal{D}^n$ such that $P_i(1) = P'_i(1)$ for all $i \in N$.*

Next, we define the concept of uncompromisingness for an RSCF. Loosely put, it says that exaggerating behavior of an agent does not influence the outcome.

Definition 3.2.7 *An RSCF $\phi : \mathcal{D}^n \to \Delta A$ is called uncompromising if $\phi_B(P_N) = \phi_B(P'_i, P_{-i})$ for all $i \in N$, all $P_N \in \mathcal{D}^n$, all $P'_i \in \mathcal{D}$, and all $B \subseteq A$ such that $B \cap [P_i(1), P'_i(1)] = \emptyset$.*

Note that uncompromisingness implies tops-onliness. It says that if an agent moves his/her top-ranked alternative closer to or farther from an alternative x in a way so that both the initial and the final positions of his/her top-ranked alternative are different from x, then the probability assigned to x by an RSCF cannot change.

Random min-max rules

In this section, we introduce a class of random social choice functions called random min-max rules. [72] and [103] introduce the concept of min-max rules. Random min-max rules are convex combinations of these rules. Formal definitions are as follows.

Definition 3.2.8 *A DSCF f on $\mathcal{D}^n$ is called a min-max rule if for all $S \subseteq N$, there exists $\beta_S \in A$ satisfying*

$$\beta_\emptyset = a_m, \beta_N = a_1, \text{ and } \beta_T \preceq \beta_S \text{ for all } S \subseteq T$$

such that for each $P_N \in \mathcal{D}^n$

$$f(P_N) = \min_{S \subseteq N} \left[\max_{i \in S} \{P_i(1), \beta_S\} \right].$$

Note that min-max rules are tops-only by definition. In what follows, we provide an example of a min-max rule.

Example 3.2.9 *Let $A = \{a_1, \ldots, a_{10}\}$ and $N = \{1, 2, 3\}$. Consider the min-max rule, say f, with parameters as given in Table 3.2.1.*

Table 3.2.1: Parameters of the min-max rule f

β	β_1	β_2	β_3	$\beta_{\{1,2\}}$	$\beta_{\{1,3\}}$	$\beta_{\{2,3\}}$
	a_8	a_9	a_7	a_4	a_5	a_2

The outcome of the min-max rule at the profile (a_5, a_3, a_8), where a_5, a_3, and a_8 are the top ranked alternatives of agents 1, 2, and 3, respectively, is determined as follows.

$$\begin{aligned} f(P_N) &= \min_{S \subseteq \{1,2,3\}} \left[\max_{i \in S} \{P_i(1), \beta_S\} \right] \\ &= \min \left[\max\{\beta_\emptyset\}, \max\{P_1(1), \beta_1\}, \max\{P_2(1), \beta_2\}, \max\{P_3(1), \beta_3\}, \right. \\ &\qquad \max\{P_1(1), P_2(1), \beta_{\{1,2\}}\}, \max\{P_1(1), P_3(1), \beta_{\{1,3\}}\}, \max\{P_2(1), P_3(1), \beta_{\{2,3\}}\}, \\ &\qquad \left. \max\{P_1(1), P_2(1), P_3(1)\beta_{\{1,2,3\}}\} \right] \\ &= \min \left[a_{10}, a_8, a_9, a_8, a_5, a_8, a_8, a_8 \right] \\ &= a_5. \end{aligned}$$

For RSCFs $\phi^j, j = 1, \ldots, k$ and non-negative numbers $\lambda^j, j = 1, \ldots, k$, summing to 1, we define the RSCF $\phi = \sum_{j=1}^k \lambda^j \phi^j$ as $\phi_a(P_N) = \sum_{j=1}^k \lambda^j \phi^j_a(P_N)$ for all $P_N \in \mathcal{D}^n$ and all $a \in A$. We call ϕ a *convex combination* of the RSCFs ϕ^j.

Definition 3.2.10 *An RSCF $\phi : \mathcal{D}^n \to \Delta A$ is called a random min-max rule if ϕ can be written as a convex combination of some min-max rules.*

3.3 A characterization of the unanimous and strategy-proof RSCFs on top-connected single-peaked domains

Two alternatives are called consecutive in the top-set of a domain if there is no alternative from the top-set that appears in-between (with respect to the prior order $\preceq$) those two alternatives. More formally, two alternatives a_r and a_s are called consecutive in $\tau(\mathcal{D})$ if $(a_r, a_s) \cap \tau(\mathcal{D}) = \emptyset$. For a domain $\mathcal{D}$, define the top-interval $I(\mathcal{D})$ as the set of alternatives $[\min(\tau(\mathcal{D})), \max(\tau(\mathcal{D}))]$.

Definition 3.3.1 *A single-peaked domain $\mathcal{D}$ is called top-connected if for every two consecutive alternatives a_r and a_s in $\tau(\mathcal{D})$ with $\min(\tau(\mathcal{D})) \preceq a_r \prec a_s \preceq \max(\tau(\mathcal{D}))$, there exist $P \in \mathcal{D}^{a_r}$ and $P' \in \mathcal{D}^{a_s}$ such that $a_s P a_{r-1}$ if $a_{r-1} \in I(\mathcal{D})$ and $a_r P' a_{s+1}$ if $a_{s+1} \in I(\mathcal{D})$.*

Remark 3.3.2 *Note that top-connectedness does not impose any restriction (except from single-peakedness) on any preference with the top-ranked alternative as $\min(\tau(\mathcal{D}))$ or $\max(\tau(\mathcal{D}))$. To see this, take, for instance, $\min(\tau(\mathcal{D})) = a_r \prec a_s \preceq \max(\tau(\mathcal{D}))$. Definition 3.3.1 says there must exist a single-peaked preference $P \in \mathcal{D}^{a_r}$ such that $a_s P a_{r-1}$ if $a_{r-1} \in I(\mathcal{D})$. However, since $a_r = \min(\tau(\mathcal{D}))$, it must be that $a_{r-1} \notin I(\mathcal{D})$. Therefore, this condition does not apply to P. Similar logic applies to any preference with the top-ranked alternative as $\max(\tau(\mathcal{D}))$.*

For a sequence of alternatives $b_1, \ldots, b_k$, denote by $\langle b_1, \ldots, b_k \rangle \ldots$ a preference where $P(l) = b_l$ for all $l = 1, \ldots, k$. Then, the top-connectedness property of a domain $\mathcal{D}$ assures that for every two consecutive alternatives a_r and a_s in $\tau(\mathcal{D})$ with $\min(\tau(\mathcal{D})) \preceq a_r \prec a_s \preceq \max(\tau(\mathcal{D}))$, there are two single-peaked preferences P and P' such that $P = \langle a_r, a_{r+1}, \ldots, a_{s-1}, a_s \rangle \ldots$ if $a_{r-1} \in I(\mathcal{D})$ and $P' = \langle a_s, a_{s-1}, \ldots, a_{r+1}, a_r \rangle \ldots$ if $a_{s+1} \in I(\mathcal{D})$. For example, if $A = \{a_1, \ldots, a_{15}\}$ and $\tau(\mathcal{D}) = \{a_3, a_4, a_5, a_8, a_{10}\}$, then top-connectedness ensures, for instance, that preferences such as $\langle a_5, a_6, a_7, a_8 \rangle \ldots$ and $\langle a_8, a_7, a_6, a_5 \rangle \ldots$ are present in the domain. Note that as we mention in Remark 3.3.2, top-connectedness does not impose any restriction (except from single-peakedness) on the preferences with top-ranked alternatives a_3 or a_{10}. Thus, the top-connectedness property of a domain $\mathcal{D}$ guarantees that for every two consecutive alternatives a_r and a_s in $\tau(\mathcal{D})$ with $\min(\tau(\mathcal{D})) \preceq a_r \prec a_s \preceq \max(\tau(\mathcal{D}))$, there are two single-peaked preferences P and P' such that $P|_{I(\mathcal{D})} = \langle a_r, a_{r+1}, \ldots, a_{s-1}, a_s \rangle \ldots$ and $P'|_{I(\mathcal{D})} = \langle a_s, a_{s-1}, \ldots, a_{r+1}, a_r \rangle \ldots$.[3]

[3]For $P \in \mathbb{L}(A)$ and $B \subseteq A$, $P|_B \in \mathbb{L}(B)$ is defined as follows: for all $a, b \in B$, $aP|_B b$ if and only if aPb.

We provide an example of a top-connected single-peaked domain in Example 3.3.3.

Example 3.3.3 *Let $A = \{a_1, \ldots, a_{10}\}$ be the set of alternatives. Consider the top-connected single-peaked domain $\mathcal{D} = \{P_1, \ldots, P_9\}$ given in Table 3.3.1. Here, $\tau(\mathcal{D}) = \{a_3, a_4, a_7, a_9\}$.*

Table 3.3.1

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
a_3	a_3	a_4	a_4	a_4	a_7	a_7	a_9	a_9
a_4	a_2	a_3	a_5	a_5	a_6	a_8	a_{10}	a_8
a_5	a_4	a_2	a_6	a_6	a_5	a_9	a_8	a_7
a_2	a_1	a_5	a_3	a_7	a_4	a_6	a_7	a_6
a_1	a_5	a_6	a_7	a_3	a_3	a_5	a_6	a_{10}
a_6	a_6	a_1	a_8	a_2	a_2	a_4	a_5	a_5
a_7	a_7	a_7	a_9	a_8	a_8	a_3	a_4	a_4
a_8	a_8	a_8	a_{10}	a_9	a_9	a_{10}	a_3	a_3
a_9	a_9	a_9	a_2	a_1	a_1	a_2	a_2	a_2
a_{10}	a_{10}	a_{10}	a_1	a_{10}	a_{10}	a_1	a_1	a_1

It is worth noting that the number of preferences in a top-connected single-peaked domain can range from $2|\tau(\mathcal{D})| - 1$ to 2^{m-1}. Thus, the class of such domains is quite large. It should be further noted that any single-peaked domain $\mathcal{D}$ with $|\tau(\mathcal{D})| = 2$ is a top-connected single-peaked domain. This is because top-connectedness does not impose any condition on the preferences with top-ranked alternatives $\min(\tau(\mathcal{D}))$ or $\max(\tau(\mathcal{D}))$.

Our next theorem provides a characterization of the unanimous and strategy-proof RSCFs on top-connected single-peaked domains.

Theorem 3.3.4 *An RSCF on a top-connected single-peaked domain is unanimous and strategy-proof if and only if it is a random min-max rule.*

The proof of this theorem is relegated to Appendix 3.7. We provide a brief sketch of it here. First note that the if part of the theorem follows as a consequence of [72]. This is because, since every top-connected single-peaked domain is a subset of the maximal single-peaked domain, every min-max rule on such a domain is unanimous and strategy-proof. Because every random min-max rule is a convex combination of min-max rules, such rules will also be unanimous and strategy-proof on top-connected single-peaked domains.

For the only-if part of the theorem, we first prove a proposition which states that every unanimous and strategy-proof RSCF on a top-connected single-peaked domain is uncompromising. We use the method of induction on the number of agents in proving this. We start with the base case comprising of one agent. The proposition follows trivially for this case. Assuming that the proposition is true for $n-1$ agents, we proceed to prove it for n agents. First, we consider all preference profiles where two particular agents, say agents 1 and 2, have the same preferences. Since the restriction of an RSCF, say ϕ, on such profiles can be thought of an RSCF on a domain with $n-1$ agents, it follows from the induction hypothesis that the restriction of ϕ on these profiles satisfy uncompromisingness (in a suitable sense). Next, we vary the preferences of agents 1 and 2 in two steps. In the first step, we consider preferences of those agents such that they have the same top-ranked alternative and show that ϕ satisfies uncompromisingness over these profiles (in a suitable sense). In the second step, we consider arbitrary preferences of agents 1 and 2 and complete the proof of the proposition. Finally, we complete the proof of the theorem by showing that every uncompromising RSCF is a random min-max rule. In proving this, we use results from [46] and [81].

Remark 3.3.5 *It is worth noting that we do not assume tops-onlyness in addition to unanimity and strategy-proofness for the RSCFs on top-connected single-peaked domains. However, since every random min-max rule is tops-only, it follows that unanimity and strategy-proofness together guarantee tops-onlyness on such domains. [31] provide a sufficient condition for a domain to be tops-only for RSCFs.*[4] *However, top-connected single-peaked domains do not satisfy their condition.*

Remark 3.3.6 *[81] show that every unanimous and strategy-proof RSCF on a minimally rich single-peaked domain is a random min-max rule. A domain is called minimally rich if for every alternative, there are two preferences with that alternative at the top such that in one of them* all *the alternatives on the left side of the top-ranked alternative are preferred to those on the right side, and in the other one, the converse happens. To the contrary, a regular top-connected single-peaked domain requires for every alternative, two preferences with it at the top such that in one of them the alternative that is to the immediate left of the top is preferred to that on the immediate right, and in the other, the converse happens. Thus, our result improves the result in [81] in a considerable manner.*

Remark 3.3.7 *A domain $\mathcal{D}$ is said to satisfy deterministic extreme point (DEP) property if every unanimous and strategy-proof RSCF on $\mathcal{D}^n$ can be written as a convex combination of unanimous and strategy-proof DSCFs on $\mathcal{D}^n$. It follows from Theorem 3.3.4 that top-connected single-peaked domains satisfy DEP property.*

Remark 3.3.8 *[2] provide a characterization of the domains on which a DSCF is unanimous and strategy-proof if and only if it is a min-max rule. They call these domains min-max domains. It is worth mentioning that (i) they consider DSCFs, whereas we consider RSCFs, (ii) they assume the domains to be*

[4] A domain is called tops-only if every unanimous and strategy-proof RSCF on it is tops-only.

regular, whereas we allow the domains to be arbitrary, and (iii) they allow the set of admissible preferences to be different for different individuals, whereas we assume that all the individuals have the same set of admissible preferences. Thus, under the assumption that all the individuals have the same set of admissible preferences, a generalized version (to the case of non-regular domains) of the result in [2] follows as a corollary of our result.

3.4 Random min-max domains and their characterization

In this section, we introduce the concept of random min-max domains and provide a characterization of them. A domain is called random min-max if an RSCF on it is unanimous and strategy-proof if and only if it is a random min-max rule. Below, we provide a formal definition.

Definition 3.4.1 *A domain $\mathcal{D}$ is called a random min-max domain if,*

- *every random min-max rule on $\mathcal{D}^n$ is strategy-proof, and*
- *every unanimous and strategy-proof RSCF on $\mathcal{D}^n$ is a random min-max rule.*

Note that Definition 3.4.1 in particular implies that on a random min-max domain, every min-max rule is strategy-proof and every unanimous and strategy-proof DSCF is a min-max rule.

Now, we present a characterization of the random min-max domains.

Theorem 3.4.2 *A domain is a random min-max domain if and only if it is a top-connected single-peaked domain.*

It follows from Theorem 3.4.2 that if we consider a single-peaked domain that is not *top-connected, then there must be some unanimous and strategy-proof RSCF that is* not *a min-max rule, and on the other hand, if we consider a non-single-peaked domain, then some random min-max rule must be manipulable on that domain. Thus, this theorem provides the full applicability of random min-max rules as unanimous and strategy-proof random rules.*

The proof of this theorem is relegated to Appendix 5.8. We provide a sketch of it here. The if part of the theorem follows from Theorem 3.3.4. For the only-if part, we consider an arbitrary non-top-connected domain and construct a unanimous and strategy-proof RSCF (in fact, a DSCF) that is not a random min-max rule.

In the following, we provide an example to show that our assumption (which is imposed from the outset) of strict preferences is crucial for our result. In particular, we show that if a top-connected single-peaked domain allows some preferences with indifferences, then there are unanimous and strategy-proof RSCFs that are not random min-max rules.

Example 3.4.3 *Consider the following domain:*
$\mathcal{D} = \{a_1a_2a_3a_4, a_2a_1a_3a_4, a_2a_3a_1a_4, a_2a_1\overline{a_3a_4}, a_3a_2a_1a_4, a_3a_4a_2a_1, a_4a_3a_2a_1\}$. *Here, we put an overline to indicate indifferences, for instance, the preference* $a_2a_1\overline{a_3a_4}$ *implies that* a_2 *is strictly preferred to* a_1*,* a_1 *is strictly preferred to both* a_3 *and* a_4*, and* a_3 *and* a_4 *are indifferent. Note that* $\mathcal{D}$ *is a top-connected single-peaked domain with an additional preference* $a_2a_1\overline{a_3a_4}$ *(i.e.,* $\mathcal{D} \setminus \{a_2a_1\overline{a_3a_4}\}$ *is a top-connected single-peaked domain). Consider the DSCF presented in Table 3.4.1. It is left to the reader to check that it is unanimous and strategy-proof. However, since* $f(a_2a_3a_1a_4, a_4a_3a_2a_1) = a_3$ *and* $f(a_2a_1\overline{a_3a_4}, a_4a_3a_2a_1) = a_4$*,* f *is not tops-only. This, in particular, implies that* f *is not a min-max rule.*

Table 3.4.1

$1 \setminus 2$	$a_1a_2a_3a_4$	$a_2a_1a_3a_4$	$a_2a_3a_1a_4$	$a_2a_1\overline{a_3a_4}$	$a_3a_2a_1a_4$	$a_3a_4a_2a_1$	$a_4a_3a_2a_1$
$a_1a_2a_3a_4$	a_1	a_2	a_2	a_2	a_3	a_3	a_3
$a_2a_1a_3a_4$	a_2	a_2	a_2	a_2	a_3	a_3	a_3
$a_2a_3a_1a_4$	a_2	a_2	a_2	a_2	a_3	a_3	a_3
$a_2a_1\overline{a_3a_4}$	a_2	a_2	a_2	a_2	a_3	a_3	a_4
$a_3a_2a_1a_4$	a_3	a_3	a_3	a_3	a_3	a_3	a_3
$a_3a_4a_2a_1$	a_3	a_3	a_3	a_3	a_3	a_3	a_3
$a_4a_3a_2a_1$	a_3	a_3	a_3	a_4	a_3	a_3	a_4

3.5 *Applications*

As we have explained, top-connected single-peaked domains are very general in nature and many single-peaked domains of practical importance fall in this category. Here, we present a few such domains. A characterization of the unanimous and strategy-proof RSCFs on these domains follows from Theorem 3.3.4.

3.5.1 *Minimally rich single-peaked domains*

A single-peaked preference P *is called left single-peaked if* $a_j \prec P(1) \prec a_k$ *implies* a_jPa_k*. Similarly, a single-peaked preference* P *is called right single-peaked if* $a_j \prec P(1) \prec a_k$ *implies* a_kPa_j*. A domain* $\mathcal{D}$ *is minimally rich if it contains all left and right single-peaked preferences. In other words, every alternative* a_j *is the top of at least two preferences* $P, P' \in \mathcal{D}$ *where* $a_jPa_{j-1} \cdots a_1Pa_{j+1} \cdots Pa_m$ *and* $a_jP'a_{j+1} \cdots a_mP'a_{j-1} \cdots P'a_1$*. This concept was first introduced in [81].*

Lemma 3.5.1 *A minimally rich single-peaked domain is a top-connected single-peaked domain.*

The proof of this lemma is left to the reader.

3.5.2 Regular single-crossing domains

Definition 3.5.1 *A domain $\mathcal{D}$ is called a* single-crossing domain w.r.t. an ordering $<$ over $\mathcal{D}$ *if for all $a, b \in A$ and all $P, P' \in \mathcal{D}$,*

$$[a \prec b, P < P', \text{ and } bPa] \implies bP'a.$$

A domain is called single-crossing if it is single-crossing w.r.t. some ordering over the domain.

Definition 3.5.2 *A single-crossing domain $\bar{\mathcal{D}}$ is called* maximal *if there does not exist a single-crossing domain $\mathcal{D}$ such that $\bar{\mathcal{D}} \subsetneq \mathcal{D}$.*

Note that a maximal single-crossing domain with m alternatives contains $m(m-1)/2 + 1$ preferences.[5]

Lemma 3.5.2 *A regular maximal single-crossing domain is a top-connected single-peaked domain.*

The proof of this lemma is left to the reader.

3.5.3 Euclidean single-peaked domains

For ease of presentation, we assume that the set of alternatives are (finitely many) elements of the interval $[0, 1]$.[6] *Let $0 = a_1 < \cdots < a_m = 1$ be the alternatives. Assume that the individuals are located at arbitrary locations in $[0, 1]$ and derive their preferences using Euclidean distances of the alternatives from their own location. We call such preferences Euclidean. Below, we provide formal definitions of these terms.*

Definition 3.5.3 *A preference P is called Euclidean if there is $x \in [0, 1]$, called the location of P, such that for all alternatives $a, b \in A$, $|x - a| < |x - b|$ implies aPb. A domain is called Euclidean if it contains* all *Euclidean preferences.*

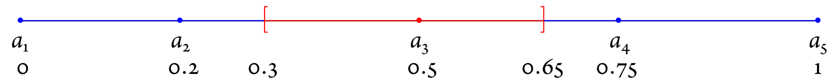
Lemma 3.5.3 *Every Euclidean domain is a top-connected single-peaked domain.*

Proof: Single-peakedness of a Euclidean domain is straight-forward. We show that such a domain is top-connected. Let $\mathcal{D}$ be a Euclidean domain. Then, it is regular by definition. Since $\mathcal{D}$ is regular, it is enough to show that for all a_r with $r \in \{1, \ldots, m-1\}$, there exist P and P′ in $\mathcal{D}$ such that $P(1) = P'(2) = a_r$ and $P(2) = P'(1) = a_{r+1}$. Consider two preferences such that both of them have locations at $\frac{a_r + a_{r+1}}{2}$. Since a_r and a_{r+1} are at equal distance from their locations, a Euclidean domain does not put any restriction on the relative preference of a_r and a_{r+1} for those preferences. So, we can have $P(1) = P'(2) = a_r$ and $P(2) = P'(1) = a_{r+1}$. This completes the proof of the lemma. ■

[5] For details see [93].

[6] With abuse of notation, we denote by $[0, 1]$ the set of real numbers in-between 0 and 1.

Figure 3.5.1: A graphic illustration of Example 3.5.4



Note that the Euclidean domains we consider are regular by definition. However, there can be Euclidean domains such that some particular alternative cannot appear as a top-ranked alternative in any preference. Such situations can occur when no individual resides in the close vicinity of that location. In the following example, we consider such a Euclidean domain and show that it admits unanimous and strategy-proof rules other than random min-max rules.

Example 3.5.4 *Suppose that the locations $a_1, \dots, a_5$ are arranged on a line as given in Figure 3.5.1. Suppose further that the individuals reside only in the region marked with blue. Note that this means the location a_3 will never be the best choice for any agent to locate a public good. The Euclidean preferences that can arise in such situation are as follows: $\{a_1a_2a_3a_4a_5, a_2a_1a_3a_4a_5, a_2a_3a_1a_4a_5, a_4a_3a_5a_2a_1, a_4a_5a_3a_2a_1, a_5a_4a_3a_2a_1\}$. In Table 3.5.1, we provide a DSCF that is unanimous and strategy-proof but not a min-max rule. To see this, assume to the contrary that it is a min-max rule. Because $f(a_5a_4a_3a_2a_1, a_1a_2a_3a_4a_5) = a_5$, it must be that $\beta_{\{2\}} = a_5$. Then, by the definition of min-max rule, it follows that $f(a_2a_1a_3a_4a_5, a_1a_2a_3a_4a_5) = a_2$, which contradicts $f(a_2a_1a_3a_4a_5, a_1a_2a_3a_4a_5) = a_1$.*

Table 3.5.1

$1\backslash 2$	$a_1a_2a_3a_4a_5$	$a_2a_1a_3a_4a_5$	$a_2a_3a_1a_4a_5$	$a_4a_3a_5a_2a_1$	$a_4a_5a_3a_2a_1$	$a_5a_4a_3a_2a_1$
$a_1a_2a_3a_4a_5$	a_1	a_1	a_1	a_1	a_1	a_1
$a_2a_1a_3a_4a_5$	a_1	a_2	a_2	a_2	a_2	a_2
$a_2a_3a_1a_4a_5$	a_1	a_2	a_2	a_2	a_2	a_2
$a_4a_3a_5a_2a_1$	a_4	a_4	a_4	a_4	a_4	a_4
$a_4a_5a_3a_2a_1$	a_4	a_4	a_4	a_4	a_4	a_4
$a_5a_4a_3a_2a_1$	a_5	a_5	a_5	a_5	a_5	a_5

3.6 Conclusion

In this paper, we have characterized the unanimous and strategy-proof random rules on a large class of single-peaked domains that we call top-connected single-peaked domains. We have shown that many single-peaked domains of practical importance fall in this class. Next, we have provided a characterization of the random min-max domains. These are the domains on which a random rule is unanimous and strategy-proof if and only if it is a random min-max rule.

An interesting problem for future work would be a characterization of unanimous and strategy-proof random rules on single-peaked domains that are not even top-connected. Tops-only property may not hold on such domains, and consequently such a characterization might turn out to be a hard problem.

Appendix

3.7 Proof of Theorem 3.3.4

Proof: (If part) Let $\mathcal{D}$ be a top-connected single-peaked domain and let ϕ be a random min-max rule. By definition ϕ is unanimous. We need to show ϕ is strategy-proof. Since $\mathcal{D}$ is a single-peaked domain, every min-max rule is strategy-proof on $\mathcal{D}^n$ ([72]). It follows by using standard arguments that every convex combination of strategy-proof deterministic rules is a strategy-proof random rule. Since ϕ is a convex combination of some min-max rules, the proof of the if part follows.

(Only-if part) Let $\mathcal{D}$ be a top-connected single-peaked domain and let $\phi : \mathcal{D}^n \to \Delta A$ be a unanimous and strategy-proof RSCF. First we prove a technical lemma which we repeatedly use in our proof.

Lemma 3.7.1 *Let $\mathcal{D}$ be a domain and let $\phi : \mathcal{D}^n \to \Delta A$ be a strategy-proof RSCF. Let $P_N \in \mathcal{D}^n$, $P'_i \in \mathcal{D}$, and $B, C \subseteq A$ be such that BP_iC, BP'_iC, and $P_i|_C = P'_i|_C$. Suppose $\phi_C(P_N) = \phi_C(P'_i, P_{-i})$ and $\phi_a(P_N) = \phi_a(P'_i, P_{-i})$ for all $a \notin B \cup C$. Then, $\phi_a(P_N) = \phi_a(P'_i, P_{-i})$ for all $a \in C$.*

Proof: First note that since $\phi_C(P_N) = \phi_C(P'_i, P_{-i})$ and $\phi_a(P_N) = \phi_a(P'_i, P_{-i})$ for all $a \notin B \cup C$, $\phi_B(P_N) = \phi_B(P'_i, P_{-i})$. Suppose $b \in C$ is such that $\phi_b(P_N) \neq \phi_b(P'_i, P_{-i})$ and $\phi_a(P_N) = \phi_a(P'_i, P_{-i})$ for all $a \in C$ with aP_ib. In other words, b is the maximal element of C according to P_i that violates the assertion of the lemma. Without loss of generality, assume that $\phi_b(P_N) < \phi_b(P'_i, P_{-i})$. However, since BP_iC, $\phi_B(P_N) = \phi_B(P'_i, P_{-i})$, and $\phi_a(P_N) = \phi_a(P'_i, P_{-i})$ for all $a \notin B$ with aP_ib, we have $\phi_{U(b,P_i)}(P_N) < \phi_{U(b,P_i)}(P'_i, P_{-i})$. This means agent i manipulates at P_N via P'_i, which is a contradiction. ■

Now we proceed to prove the only-if part. We start with a proposition.

Proposition 3.7.1 *The RSCF $\phi : \mathcal{D}^n \to \Delta A$ is uncompromising.*

Proof: Let $|N| = 1$ and let $\phi : \mathcal{D} \to \triangle A$ be a unanimous and strategy-proof RSCF. Then, unanimity implies uncompromisingness.

Assume that the theorem holds for all sets with $k < n$ agents. We prove it for n agents. Let $|N| = n$ and let $\phi : \mathcal{D}^n \to \triangle A$ be a unanimous and strategy-proof RSCF. Suppose $N^* = N \setminus \{1\}$. Define the RSCF $g : \mathcal{D}^{n-1} \to \triangle A$ for the set of voters N^* as follows: for all $P_{N^*} = (P_2, P_3, \ldots, P_n) \in \mathcal{D}^{n-1}$,

$$g(P_2, P_3, \ldots, P_n) = \phi(P_2, P_2, P_3, P_4, \ldots, P_n).$$

Evidently, g is a well-defined RSCF satisfying unanimity and strategy-proofness (See Lemma 3 in [98] for a detailed argument). Hence, by the induction hypothesis, g satisfies uncompromisingness. The proof of Proposition 3.7.1 is completed using a series of lemmas. In the next lemma, we show that ϕ is tops-only over all profiles P_N where agents 1 and 2 have the same top alternative.

Lemma 3.7.2 *Let $P_N, P'_N \in \mathcal{D}^n$ be two tops-equivalent profiles such that $P_1, P_2 \in \mathcal{D}^{a_r}$ for some $a_r \in A$. Then, $\phi(P_N) = \phi(P'_N)$.*

Proof: Note that since g is uncompromising, g satisfies tops-onlyness. Because g is tops-only and $P_1, P_2 \in \mathcal{D}^{a_r}$, we have $g(P_1, P_{-\{1,2\}}) = g(P_2, P_{-\{1,2\}})$, and hence $\phi(P_1, P_1, P_{-\{1,2\}}) = \phi(P_2, P_2, P_{-\{1,2\}})$. We show that $\phi(P_1, P_2, P_{-\{1,2\}}) = \phi(P_1, P_1, P_{-\{1,2\}})$. Using strategy-proofness of ϕ for agent 2, we have $\phi_{U(x,P_1)}(P_1, P_1, P_{-\{1,2\}}) \geq \phi_{U(x,P_1)}(P_1, P_2, P_{-\{1,2\}})$ for all $x \in A$, and using that for agent 1, we have $\phi_{U(x,P_1)}(P_1, P_2, P_{-\{1,2\}}) \geq \phi_{U(x,P_1)}(P_2, P_2, P_{-\{1,2\}})$ for all $x \in A$. Since $\phi(P_1, P_1, P_{-\{1,2\}}) = \phi(P_2, P_2, P_{-\{1,2\}})$, it follows from Remark 5.2.3 that $\phi(P_1, P_1, P_{-\{1,2\}}) = \phi(P_1, P_2, P_{-\{1,2\}})$. Using a similar logic, we have $\phi(P'_1, P'_1, P'_{-\{1,2\}}) = \phi(P'_1, P'_2, P'_{-\{1,2\}})$. Because g is tops-only and P_N, P'_N are tops-equivalent, we have $g(P_1, P_{-\{1,2\}}) = g(P'_1, P'_{-\{1,2\}})$. This implies that $\phi(P_1, P_1, P_{-\{1,2\}}) = \phi(P'_1, P'_1, P'_{-\{1,2\}})$, and hence $\phi(P_1, P_2, P_{-\{1,2\}}) = \phi(P'_1, P'_2, P'_{-\{1,2\}})$. ■

Lemma 3.7.3 *Let $1 \leq r \leq s \leq m$ and let $P_N, P'_N \in \mathcal{D}^n$ be such that $P_1, P_2 \in \mathcal{D}^{a_r}$ and $P'_1, P'_2 \in \mathcal{D}^{a_s}$, and $P_i(1) = P'_i(1)$ for all $i \neq 1, 2$. Then, $\phi_a(P_N) = \phi_a(P'_N)$ for all $a \notin [a_r, a_s]$.*

Proof: By uncompromisingness of g, we have $g_a(P_1, P_{-\{1,2\}}) = g_a(P'_1, P_{-\{1,2\}})$ for all $a \notin [a_r, a_s]$. Moreover, since g is tops-only and $P_i(1) = P'_i(1)$ for all $i \in \{3, 4, \ldots, n\}$, we have $g(P'_1, P_{-\{1,2\}}) = g(P'_1, P'_{-\{1,2\}})$. By the definition of g, $g(P_1, P_{-\{1,2\}}) = \phi(P_1, P_1, P_{-\{1,2\}})$ and $g(P'_1, P_{-\{1,2\}}) = \phi(P'_1, P'_1, P_{-\{1,2\}})$. As $P_1(1) = P_2(1)$ and $P'_1(1) = P'_2(1)$, Lemma 5.7.3 implies $\phi(P_1, P_2, P_{-\{1,2\}}) = \phi(P_1, P_1, P_{-\{1,2\}})$ and $\phi(P'_1, P'_2, P'_{-\{1,2\}}) = \phi(P'_1, P'_1, P'_{-\{1,2\}})$. Combining all these observations, we have $\phi_a(P_1, P_2, P_{-\{1,2\}}) = \phi_a(P'_1, P'_2, P'_{-\{1,2\}})$ for all $a \notin [a_r, a_s]$. ■

Lemma 3.7.4 *Let $a_r \prec a_s$ and let $P_N, P'_N \in \mathcal{D}^n$ be such that $P_1, P_2, P'_1 \in \mathcal{D}^{a_r}$ and $P'_2 \in \mathcal{D}^{a_s}$, and $P_i(1) = P'_i(1)$ for all $i \neq 1, 2$. Then, $\phi_a(P_N) = \phi_a(P'_N)$ for all $a \notin [a_r, a_s]$.*

Proof: By Lemma 5.7.3, $\phi(P_1, P_2, P_{-\{1,2\}}) = \phi(P'_1, P'_1, P'_{-\{1,2\}})$. Hence, it suffices to show that $\phi_a(P'_1, P'_1, P'_{-\{1,2\}}) = \phi_a(P'_1, P'_2, P'_{-\{1,2\}})$ for all $a \notin [a_r, a_s]$. Note that $[a_r, a_s] = U(a_s, P'_1) \cap U(a_r, P'_2)$. Therefore, we prove the above mentioned assertion for $a \notin U(a_s, P'_1)$ as the proof of the same when $a \notin U(a_r, P'_2)$ follows from symmetric arguments.

Take $a \notin U(a_s, P'_1)$. By strategy-proofness of ϕ,

$$\phi_{U(a,P'_1)}(P'_1, P'_1, P'_{-\{1,2\}}) \geq \phi_{U(a,P'_1)}(P'_1, P'_2, P'_{-\{1,2\}}) \geq \phi_{U(a,P'_1)}(P'_2, P'_2, P'_{-\{1,2\}}).$$

Moreover, by Lemma 5.7.4, $\phi_a(P'_1, P'_1, P'_{-\{1,2\}}) = \phi_a(P'_2, P'_2, P'_{-\{1,2\}})$ for all $a \notin [a_r, a_s]$, and hence $\phi_B(P'_1, P'_1, P'_{-\{1,2\}}) = \phi_B(P'_2, P'_2, P'_{-\{1,2\}})$ for all $B \subseteq A$ such that $[a_r, a_s] \subseteq B$. Since $a \notin U(a_s, P'_1)$ and $P'_1(1) = a_r$, by the definition of single-peakedness, we have $[a_r, a_s] \subseteq U(a, P'_1)$, and hence

$$\phi_{U(a,P'_1)}(P'_1, P'_1, P'_{-\{1,2\}}) = \phi_{U(a,P'_1)}(P'_1, P'_2, P'_{-\{1,2\}}). \tag{3.1}$$

Let $b \in A$ be such that bP'_1a and there is there is no $c \in A$ such that bP'_1c and cP'_1a. Then, $[a_r, a_s] \subseteq U(b, P'_1)$, and hence

$$\phi_{U(b,P'_1)}(P'_1, P'_1, P'_{-\{1,2\}}) = \phi_{U(b,P'_1)}(P'_1, P'_2, P'_{-\{1,2\}}). \tag{3.2}$$

Subtracting (5.2) from (5.1), we have $\phi_a(P'_1, P'_1, P'_{-\{1,2\}}) = \phi_a(P'_1, P'_2, P'_{-\{1,2\}})$, which completes the proof of the lemma. ■

Lemma 3.7.5 *ϕ satisfies uncompromsingness.*

Proof: First, we show ϕ satisfies uncompromising for agent $i \in \{1, 2\}$. It is sufficient to show this for agent 1. Consider P_N and P'_1 such that $P_1(1) = a_r$ and $P'_1(1) = a_s$ where $a_r \prec a_s$ and $(a_r, a_s) \cap \tau(\mathcal{D}) = \emptyset$. We show that $\phi_a(P_N) = \phi_a(P'_1, P_{-1})$ for all $a \notin [a_r, a_s]$. Consider $\bar{P}_1 \in \mathcal{D}^{a_r}$ and $\hat{P}_1 \in \mathcal{D}^{a_s}$ where $\bar{P}|_{I(\mathcal{D})} = \langle a_r, \ldots, a_s \rangle \ldots$ and $\hat{P}|_{I(\mathcal{D})} = \langle a_s, \ldots, a_r \rangle \ldots$. Without loss of generality, we assume that $P_2(1) = a_t$ where $a_s \preceq a_t$.

Claim 3.7.1 *A $\phi(P_N) = \phi(\bar{P}_1, P_{-1})$.*

Note that by Lemma 5.7.3, $\phi(P_1, P_1, P_{-\{1,2\}}) = \phi(\bar{P}_1, \bar{P}_1, P_{-\{1,2\}})$, and by Lemma 5.7.5, $\phi_a(P_1, P_1, P_{-\{1,2\}}) = \phi_a(P_1, P_2, P_{-\{1,2\}})$ for all $a \notin [a_r, a_t]$ and $\phi_a(\bar{P}_1, \bar{P}_1, P_{-\{1,2\}}) = \phi_a(\bar{P}_1, P_2, P_{-\{1,2\}})$ for all $a \notin [a_r, a_t]$. This implies $\phi_a(P_1, P_2, P_{-\{1,2\}}) = \phi_a(\bar{P}_1, P_2, P_{-\{1,2\}})$ for all $a \notin [a_r, a_t]$. By single-peakedness, we have $P_1|_{[a_r,a_t]} = P'_1|_{[a_r,a_t]}$, and therefore by applying Lemma 5.7.2 with $B = \emptyset$ and $C = [a_r, a_t]$, we have $\phi(P_1, P_2, P_{-\{1,2\}}) = \phi(\bar{P}_1, P_2, P_{-\{1,2\}})$. This completes the proof of Claim 3.7.1.

Using a similar argument as for Claim 3.7.1, we can show that $\phi(P'_1, P_{-1}) = \phi(\hat{P}_1, P_{-1})$. Thus, to show that $\phi_a(P_N) = \phi_a(P'_1, P_{-1})$ for all $a \notin [a_r, a_s]$, it is enough to show $\phi_a(\bar{P}_1, P_{-1}) = \phi_a(\hat{P}_1, P_{-1})$ for all $a \notin [a_r, a_s]$. Note that by Lemma 5.7.4, $\phi_a(\bar{P}_1, \bar{P}_1, P_{-\{1,2\}}) = \phi_a(\hat{P}_1, \hat{P}_1, P_{-\{1,2\}})$ for all $a \notin [a_r, a_s]$, and by Lemma 5.7.5, $\phi_a(\bar{P}_1, \bar{P}_1, P_{-\{1,2\}}) = \phi_a(\bar{P}_1, P_2, P_{-\{1,2\}})$ for all $a \notin [a_r, a_t]$ and

$\phi_a(\hat{P}_1, \hat{P}_1, P_{-\{1,2\}}) = \phi_a(\hat{P}_1, P_2, P_{-\{1,2\}})$ for all $a \notin [a_s, a_t]$. Combining all these observations, $\phi_a(\bar{P}_1, P_2, P_{-\{1,2\}}) = \phi_a(\hat{P}_1, P_2, P_{-\{1,2\}})$ for all $a \notin [a_r, a_t]$. Consider $b \notin [\tau(\mathcal{D})]$. Then $b \notin [a_r, a_t]$ since $[a_r, a_t] \subseteq I(\mathcal{D})$. As $\bar{P}|_{I(\mathcal{D})} = \langle a_r, \ldots, a_s \rangle \ldots$ and $\hat{P}|_{I(\mathcal{D})} = \langle a_s, \ldots, a_r \rangle \ldots$, this implies that $\phi_{[a_r,a_s]}(\bar{P}_1, P_2, P_{-\{1,2\}}) = \phi_{[a_r,a_s]}(\hat{P}_1, P_2, P_{-\{1,2\}})$. Again by single-peakedness, $\bar{P}_1|_{(a_s,a_t]} = \hat{P}_1|_{(a_s,a_t]}$. Thus, by applying Lemma 5.7.2 with $B = [a_r, a_s]$ and $C = (a_s, a_t]$, we have $\phi_a(\bar{P}_1, P_2, P_{-\{1,2\}}) = \phi_a(\hat{P}_1, P_2, P_{-\{1,2\}})$ for all $a \notin [a_r, a_s]$. This shows that ϕ is uncompromising for agents 1 and 2.

Now, we proceed to prove uncompromisingness for the other agents. It is sufficient to show this for agent 3. Consider P_N and P'_3 such that $P_3(1) = a_r$ and $P'_3(1) = a_s$, where $a_r \prec a_s$ and $(a_r, a_s) \cap \tau(\mathcal{D}) = \emptyset$. We show that $\phi_a(P_N) = \phi_a(P'_3, P_{-3})$ for all $a \notin [a_r, a_s]$. Consider $\bar{P}_3 \in \mathcal{D}^{a_r}$ and $\hat{P}_3 \in \mathcal{D}^{a_s}$, where $\bar{P}|_{I(\mathcal{D})} = \langle a_r, \ldots, a_s \rangle \ldots$ and $\hat{P}|_{I(\mathcal{D})} = \langle a_s, \ldots, a_r \rangle \ldots$. Assume $P_1(1) = a_p$ and $P_2(1) = a_q$. We distinguish two cases.

Case 1. Suppose $a_p, a_q \preceq a_r$ or $a_s \preceq a_p, a_q$.

Without loss of generality, we assume that $a_p, a_q \preceq a_r$. First we show that $\phi(P_N) = \phi(\bar{P}_3, P_{-3})$. Note that by the induction hypothesis, $\phi(P_1, P_1, P_3, P_{-\{1,2,3\}}) = \phi(P_1, P_1, \bar{P}_3, P_{-\{1,2,3\}})$. By Lemma 5.7.5, $\phi_a(P_1, P_1, P_3, P_{-\{1,2,3\}}) = \phi_a(P_1, P_2, P_3, P_{-\{1,2,3\}})$ for all $a \notin [a_p, a_q]$ and $\phi_a(P_1, P_1, \bar{P}_3, P_{-\{1,2,3\}}) = \phi_a(P_1, P_2, \bar{P}_3, P_{-\{1,2,3\}})$ for all $a \notin [a_p, a_q]$. Combining all these observations, we get $\phi_a(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \phi_a(P_1, P_2, \bar{P}_3, P_{-\{1,2,3\}})$ for all $a \notin [a_p, a_q]$. Since $a_p, a_q \preceq a_r$, by single-peakedness, $P_3|_{[a_p,a_q]} = \bar{P}_3|_{[a_p,a_q]}$. This implies that $\phi(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \phi(P_1, P_2, \bar{P}_3, P_{-\{1,2,3\}})$. Using a similar logic, we can show $\phi(P_1, P_2, P'_3, P_{-\{1,2,3\}}) = \phi(P_1, P_2, \hat{P}_3, P_{-\{1,2,3\}})$. Thus to show that $\phi_a(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \phi_a(P_1, P_2, P'_3, P_{-\{1,2,3\}})$ for all $a \notin [a_r, a_s]$, it is enough to show that $\phi_a(P_1, P_2, \bar{P}_3, P_{-\{1,2,3\}}) = \phi_a(P_1, P_2, \hat{P}_3, P_{-\{1,2,3\}})$ for all $a \notin [a_r, a_s]$. By the induction hypothesis, $\phi_a(P_1, P_1, \bar{P}_3, P_{-\{1,2,3\}}) = \phi_a(P_1, P_1, \hat{P}_3, P_{-\{1,2,3\}})$ for all $a \notin [a_r, a_s]$. Again by Lemma 5.7.5, $\phi_a(P_1, P_1, \bar{P}_3, P_{-\{1,2,3\}}) = \phi_a(P_1, P_2, \bar{P}_3, P_{-\{1,2,3\}})$ for all $a \notin [a_p, a_q]$ and $\phi_a(P_1, P_1, \hat{P}_3, P_{-\{1,2,3\}}) = \phi_a(P_1, P_2, \hat{P}_3, P_{-\{1,2,3\}})$ for all $a \notin [a_p, a_q]$. Combining all these observations, $\phi_a(P_1, P_2, \bar{P}_3, P_{-\{1,2,3\}}) = \phi_a(P_1, P_2, \hat{P}_3, P_{-\{1,2,3\}})$ for all $a \notin [a_p, a_q] \cup [a_r, a_s]$. Consider $b \notin [\tau(\mathcal{D})]$. Then $b \notin [a_p, a_q] \cup [a_r, a_s]$, and hence $\phi_b(P_1, P_2, \bar{P}_3, P_{-\{1,2,3\}}) = \phi_b(P_1, P_2, \hat{P}_3, P_{-\{1,2,3\}})$. Since $\bar{P}|_{I(\mathcal{D})} = \langle a_r, \ldots, a_s \rangle \ldots$ and $\hat{P}|_{I(\mathcal{D})} = \langle a_s, \ldots, a_r \rangle \ldots$, this implies that $\phi_{[a_r,a_s]}(P_1, P_2, \bar{P}_3, P_{-\{1,2,3\}}) = \phi_{[a_r,a_s]}(P_1, P_2, \hat{P}_3, P_{-\{1,2,3\}})$. Since $a_p, a_q \preceq a_r$, by single-peakedness $\bar{P}_3|_{[a_p,a_q]\setminus a_r} = \hat{P}_3|_{[a_p,a_q]\setminus a_r}$. Therefore, by applying Lemma 5.7.2 with $B = [a_r, a_s]$ and $C = [a_p, a_q] \setminus a_r$, we have $\phi_a(P_1, P_2, \bar{P}_3, P_{-\{1,2,3\}}) = \phi_a(P_1, P_2, \hat{P}_3, P_{-\{1,2,3\}})$ for all $a \notin [a_r, a_s]$. This completes the proof for Case 1.

Case 2. Suppose $a_p \preceq a_r \prec a_s \preceq a_q$ or $a_q \preceq a_r \prec a_s \preceq a_r$. Without loss of generality, we assume that $a_p \preceq a_r \prec a_s \preceq a_q$. First we show $\phi(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \phi(P_1, P_2, \bar{P}_3, P_{-\{1,2,3\}})$. By using uncompromisingness for agent 2, we have for all $a \notin [a_r, a_q]$,

$\phi_a(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \phi_a(P_1, P_3, P_3, P_{-\{1,2,3\}})$ and $\phi_a(P_1, P_2, \bar{P}_3, P_{-\{1,2,3\}}) = \phi_a(P_1, P_3, \bar{P}_3, P_{-\{1,2,3\}})$. Since $a_p \preceq a_r$, by Case 1, $\phi(P_1, P_3, P_3, P_{-\{1,2,3\}}) = \phi_a(P_1, P_3, \bar{P}_3, P_{-\{1,2,3\}})$. Combining all these observations, $\phi_a(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \phi_a(P_1, P_2, \bar{P}_3, P_{-\{1,2,3\}})$ for all $a \notin [a_r, a_q]$. As $a_r \prec a_q$, by single-peakedness, $P_3|_{[a_r,a_q]} = \bar{P}_3|_{[a_r,a_s]}$, and hence by strategy-proofness, $\phi(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \phi(P_1, P_2, \bar{P}_3, P_{-\{1,2,3\}})$. Similarly, we can show that $\phi(P_1, P_2, P'_3, P_{-\{1,2,3\}}) = \phi(P_1, P_2, \hat{P}_3, P_{-\{1,2,3\}})$. Thus, to show $\phi_a(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \phi_a(P_1, P_2, P'_3, P_{-\{1,2,3\}})$ for all $a \notin [a_r, a_s]$, it is enough to show that $\phi_a(P_1, P_2, \bar{P}_3, P_{-\{1,2,3\}}) = \phi_a(P_1, P_2, \hat{P}_3, P_{-\{1,2,3\}})$ for all $a \notin [a_r, a_s]$. Using an argument similar to the above and the fact that $\phi_a(P_1, P_3, \bar{P}_3, P_{-\{1,2,3\}}) = \phi_a(P_1, P_3, \hat{P}_3, P_{-\{1,2,3\}})$ for all $a \notin [a_r, a_s]$, we get $\phi_a(P_1, P_2, \bar{P}_3, P_{-\{1,2,3\}}) = \phi_a(P_1, P_2, \hat{P}_3, P_{-\{1,2,3\}})$ for all $a \notin [a_r, a_q]$. Consider $b \notin [\tau(\mathcal{D})]$. Then, $b \notin [a_r, a_q]$, and hence $\phi_b(P_1, P_2, \bar{P}_3, P_{-\{1,2,3\}}) = \phi_b(P_1, P_2, \hat{P}_3, P_{-\{1,2,3\}})$. As $\bar{P}|_{I(\mathcal{D})} = \langle a_r, \ldots, a_s \rangle \ldots$ and $\hat{P}|_{I(\mathcal{D})} = \langle a_s, \ldots, a_r \rangle \ldots$, this implies that $\phi_{[a_r,a_s]}(P_1, P_2, \bar{P}_3, P_{-\{1,2,3\}}) = \phi_{[a_r,a_s]}(P_1, P_2, \hat{P}_3, P_{-\{1,2,3\}})$. Since $a_r, a_s \preceq a_q$, by single-peakedness, $\bar{P}_3|_{(a_s,a_q]} = \hat{P}_3|_{(a_s,a_q]}$. Thus by Lemma 5.7.2 with $B = [a_r, a_s]$ and $C = (a_s, a_q]$, we have $\phi_a(P_1, P_2, \bar{P}_3, P_{-\{1,2,3\}}) = \phi_a(P_1, P_2, \hat{P}_3, P_{-\{1,2,3\}})$ for all $a \notin [a_r, a_s]$. This completes the proof for Case 2.

Since Cases 1 and 2 are exhaustive, this shows uncompromisingness for agent 3, and hence completes the proof of Lemma 3.7.5. ■

The proof of Proposition 3.7.1 follows from Lemma 3.7.5. ■

Now we complete the proof the theorem. Let $a_r \prec a_s$ be such that $a_r = \min \tau(\mathcal{D})$ and $a_s = \max \tau(\mathcal{D})$. For $S \subseteq N$ define $\beta_S = \phi(P_N)$ where $P_i(1) = a_r$ if $i \in S$ and $P_i(1) = a_s$ if $i \notin S$. Note that by the uncompromisingness of ϕ, β_S is a probability distribution on A and $\beta_S(a) = 0$ for all $a \notin [a_r, a_s]$, and all $S \subseteq N$.

First, we show that $\beta_S([a_k, a_m]) \geq \beta_{S \cup T}([a_k, a_m])$ for all $S, T \subseteq N$ and all $a_k \in A$. Suppose $\beta_S([a_k, a_m]) < \beta_{S \cup T}([a_k, a_m])$ for some $S, T \subseteq N$ and some $a_k \in A$. Without loss of generality, we can assume that $T = i$ for some $i \in N$. Let $P_{-i} \in \mathcal{D}^{n-1}$ be such that $P_j(1) = a_r$ if $j \in S$ and $P_j(1) = a_s$ if $j \notin S$. Further, let P_i, P'_i be such that $P_i(1) = a_r$ and $P'_i(1) = a_s$. By uncompromisingness, $\beta_S(a) = \beta_{S \cup i}(a) = 0$ for all $a \notin [a_r, a_s]$. Therefore, $\beta_S([a_k, a_m]) < \beta_{S \cup i}([a_k, a_m])$ implies $a_r \prec a_k \prec a_s$ and

$$\beta_S([a_k, a_s]) < \beta_{S \cup i}([a_k, a_s]). \tag{3.3}$$

Since $P_i(1) = a_r$ and $P'_i(1) = a_s$, (3.3) together with the fact that $\beta_S(a) = \beta_{S \cup i}(a) = 0$ for all $a \notin [a_r, a_s]$ implies $\phi_{U(a_{k-1}, P_i)}(P_N) < \phi_{U(a_{k-1}, P_i)}(P'_i, P_{-i})$. However, then agent i manipulates at P_N via P'_i, a contradiction. This shows that $\beta_S([a_k, a_m]) \geq \beta_{S \cup T}([a_k, a_m])$ for all $S, T \subseteq N$ and all $a_k \in A$.

Define $\hat{\beta}_S \in \Delta[a_r, a_s]$ for all $S \subseteq N$ such that $\hat{\beta}_S(a) = \beta_S(a)$ for all $a \in [a_r, a_s]$. Let $\hat{\mathcal{D}}$ be the maximal single-peaked domain over the alternatives in the interval $[a_r, a_s]$. For $P_N \in \hat{\mathcal{D}}^n$ and $a_k \in [a_r, a_s]$, we define $S(a_k, P_N) = \{i \in N \mid P_i(1) \in [a_r, a_k]\}$. Consider the RSCF $\hat{\phi} : \hat{\mathcal{D}}^n \to \Delta[a_r, a_s]$ such that for all $P_N \in \hat{\mathcal{D}}^n$ and all $a_k \in [a_r, a_s]$,

$$\hat{\phi}_{[a_r,a_k]}(P_N) = \beta_{S(a_k,P_N)}([a_r, a_k]).$$

Since $\phi_a(P_N) = \hat{\phi}_a(P_N)$ for all $a \in [a_r, a_s]$ and P_N with $P_i(1) \in \{a_r, a_s\}$ for all $i \in N$, by Proposition 1 in [81], we have $\phi_a(P_N) = \hat{\phi}_a(P_N)$ for all $a \in [a_r, a_s]$ and all $P_N \in \mathcal{D}^n$. By Theorem 4.1 in [46], $\hat{\phi}$ is unanimous and strategy-proof as $\hat{\mathcal{D}}$ is a single-peaked domain. Hence, by Theorem 3(b) in [81], $\hat{\phi}$ can be written as a convex combination of unanimous and strategy-proof DSCFs $f : \mathcal{D}^n \to A$. Again by [103], every unanimous and strategy-proof DSCFs $f : \mathcal{D}^n \to A$ is a min-max rule. By definition 3.2.10, this implies that $\hat{\phi}$ is a random min-max rule, and hence ϕ is a random min-max rule.

3.8 Proof of Theorem 3.4.2

(If part) Let $\mathcal{D}$ be a top-connected single-peaked domain. By Theorem 3.3.4, an RSCF ϕ is unanimous and strategy-proof if and only if it is a random min-max rule. Therefore, $\mathcal{D}$ is a random min-max domain, which completes the proof of the if part.

(Only-if part) Let $\mathcal{D}$ be a random min-max domain. We prove that $\mathcal{D}$ is a top-connected single-peaked domain. First we show that $\mathcal{D}$ is a single-peaked domain. Assume for contradiction that there exists $Q \in \mathcal{D}$ such that Q is not single-peaked. Without loss of generality, assume that there exist a_r, a_s with $a_r \prec a_s \prec Q(1)$ such that $a_r Q a_s$. Consider the min-max rule f on $\mathcal{D}^n$ such that $\beta_S = a_r$ for all non-empty $S \subsetneq N$. Consider the profile $P_N \in \mathcal{D}^n$ such that $P_1 = Q$ and $P_i(1) = a_s$ for all $i \neq 1$. Then, by the definition of f, $f(P_N) = a_s$. Let $P'_1 \in \mathcal{D}$ be such that $P'_1(1) = a_r$. Again, by the definition of f, $f(P'_1, P_{-1}) = a_r$. Because $a_r Q a_s$, this means agent 1 manipulates at P_N via P'_1, which contradicts that $\mathcal{D}$ is a single-peaked domain.

Now, we show that for $a_r, a_s \in \tau(\mathcal{D})$ with the property that $\min(\tau(\mathcal{D})) \prec a_r \prec a_s \prec \max(\tau(\mathcal{D}))$ and $(a_r, a_s) \cap \tau(\mathcal{D}) = \emptyset$, there exist $P \in \mathcal{D}^{a_r}$ and $P' \in \mathcal{D}^{a_s}$ such that $a_s P a_{r-1}$ and $a_r P' a_{s+1}$. Suppose not and without loss of generality assume that there exist $a_r, a_s \in \tau(\mathcal{D})$ with $a_r \prec a_s \prec \max(\tau(\mathcal{D}))$ and $(a_r, a_s) \cap \tau(\mathcal{D}) = \emptyset$ such that $a_{s+1} P a_r$ for all $P \in \mathcal{D}^{a_s}$. Consider the DSCF f on $\mathcal{D}^n$ as follows:

$$f(P_N) = \begin{cases} P_1(1) \text{ if } P_1(1) \neq a_s, \\ a_s \text{ if } P_1(1) = a_s \text{ and } a_s P_2 a_{s+1}, \\ a_{s+1} \text{ otherwise.} \end{cases}$$

It can be verified that f is unanimous and strategy-proof. We show that f is not a min-max rule. In particular, we show that f is not uncompromising. This is sufficient as every min-max rule is uncompromising. Let $P_N \in \mathcal{D}^n$ be such that $P_2(1) = \max(\tau(\mathcal{D}))$. Then, by the definition of f, $f(P_N) = a_{s+1}$ when $P_1(1) = a_s$, and $f(P_1', P_{-1}) = a_r$ when $P_1'(1) = a_r$. This clearly violates uncompromisingness for agent 1. This completes the proof of the only-if part. ■

4

Formation of Committees through Random Voting Rules

4.1 Introduction

A classic paper in the theory of mechanism design is [60]. It considered an exchange economy with at least two agents and demonstrated the impossibility of constructing an allocation rule that satisfied strategy-proofness, efficiency and individual rationality. The paper inspired an enormous and rapidly expanding literature that analyzes socially desirable goals that can be achieved in the presence of private information and strategic agents, in a wide variety of models. The present paper contributes to that literature by investigating the structure of rules that permit randomization in the well-known model of committee formation.

The committee formation model is due to [11]. The problem is one of choosing a committee from a set of available candidates based on the preferences of agents who have the responsibility of selecting the committee. The preferences of each agent are assumed to be separable, i.e. if the agent "likes" a candidate, she strictly prefers a committee where this candidate is included to one where she is excluded, the status of all other candidates remaining unchanged. A committee formation rule or a social choice function is a map that associates every collection of (separable) agent preferences with a committee. Agent preferences are private information - a fact that necessitates the elicitation of these preferences via voting. A social choice function is strategy-proof if truth-telling is an optimal strategy for each agent irrespective of her beliefs about how other agents may vote. The main result of [11] is that strategy-proof social choice functions (that additionally satisfy a weak efficiency property called unanimity) must be decomposable. In other words, the decision on each candidate's inclusion must be taken independently of the decisions on others and must be based only on preferences that agents have over the candidate (called marginal preferences). The decomposability condition on social choice function rules out many plausible rules. For instance, if there are two candidates, we could start with candidate 1 and consider candidate 2 only if 1 is not selected. [26] show that the decomposability property of strategy-proof social choice functions is very general - it holds for all multi-dimensional models with separable preferences.

In our paper we consider the same model as in [11] but analyzes committee formation rules that permit randomization. A random social choice functions is a map that associates a collection of (separable) agent preferences with a probability distribution over committees. Randomization is a natural way to resolve conflicts of interest amongst agents especially in models where compensation via monetary transfers is not feasible. The analysis of randomized mechanisms in voting models was initiated in [57]. Once randomization is allowed, the evaluation of truth-telling versus misrepresentation involves the comparison of lotteries. This evaluation typically involves domain restrictions on preferences over lotteries (i.e. all preferences over lotteries are not allowed) as a result of which the class of strategy-proof social choice functions expands (see [35]). [1]

According to our characterization result, a random social choice function is strategy proof and satisfies unanimity [2] if and only if it satisfies the properties of monotonicity and marginal decomposability. Monotonicity is a familiar property in mechanism design theory. In our model, it requires the probability of the inclusion of a candidate in every possible committee to be non-decreasing as more agents approve the candidate. Furthermore, if no agent approves a candidate, the candidate is never selected; on the other hand, if all agents approve a candidate, she is always selected.

Consider an arbitrary subset of candidates and two preference profiles where all agents agree in their opinions over this subset of candidates (they may differ in their opinion of other candidates). Marginal decomposability is satisfied if the marginal probability distribution over the subset of candidates is the same in the two profiles. Suppose there are three agents and five candidates. Consider the set of the first three candidates and two preference profiles where all agents agree in their opinions over the first three

[1] There are several ways in which this can be done. Here we follow the standard stochastic dominance approach developed in [57].

[2] A random social choice function satisfies unanimity if it picks a committee that is first-ranked by all agents, with probability one.

candidates. Pick any subset of the first three candidates, say candidates one and three. If marginal decomposability is satisfies, the probability of candidates one and three being selected in the committee at the two profiles, must be the same. Note that marginal decomposability only guarantees that marginal probabilities will be uniquely determined by marginal preferences, but does not say anything about the joint probability distribution. Thus decomposability in the sense of [26] is not guaranteed. However, marginal decomposability is equivalent to decomposability when we restrict attention to deterministic social choice functions thus getting back the decomposability result of [26] in our model.

Finally we consider the special problem of forming a committee with a number of members. A random social choice function is onto if every committee of the required size s selected with probability one at some preference profile. We show that every onto and strategy-proof RSCF in this case is a random dictatorship in an appropriate sense. This result follows from an application of the applying the main result of [57].

4.2 The Model

Let $M = \{1, \ldots, m\}$ be a finite set of m components. For each component k, $A^k = \{0, 1\}$ is the set of alternatives available in component k. For any $K \subseteq M$, $A^K = \prod_{k \in K} A^k$, denotes the set of alternatives available in components in K. The set of (multi-dimensional) alternatives is given by A^M. For ease of presentation, we write A instead of A^M. Note that the number of alternatives in A is 2^m. Throughout this paper, we do not use braces for singleton sets.

In the model M denotes the set of possible candidates from which a committee has to be formed. Thus each component refers to a possible candidate for a committee, where the numbers 0 and 1 for a component refer to the social states where the corresponding member is excluded and included in the committee, respectively. Similarly, every alternative $a = (a^1, \ldots, a^m) \in A$ refers to a committee in which the member k is present if and only if $a^k = 1$.

Let $N = \{1, \ldots, n\}$ be a set of finite set of n agents. Each agent i has a strict preference ordering P_i over the elements of A. We assume that all P_i's are separable, i.e. for all $a^{-k}, b^{-k} \in A^{M-k}$ and all $x^k, y^k \in A^k$, $(x^k, a^{-k})P_i(y^k, a^{-k})$ holds if and only if $(x^k, b^{-k})P_i(y^k, b^{-k})$. We denote by P_i^k the marginal preference induced by P_i over component k. The existence of marginal preference orderings is guaranteed by separability. We let $\tau(P_i)$ and $\tau(P_i^k)$ denote the top-ranked alternative in P_i and the top-ranked alternative in the k^{th} component according to the marginal ordering P_k^i. In general, $r_t(P_i)$ the t-th ranked alternative in P_i where $t \in \{1, 2, \ldots, 2^m\}$. The upper contour set of an alternative a at preference P_i denoted by $U(a, P_i)$ is defined as follows: $U(a, P_i) = \{b \mid bP_ia\} \cup a$. Let $\mathcal{D}$ denote the set of all separable preferences over A. An element P_N of $\mathcal{D}^n$ is called a (preference) profile.

A random social choice function (RSCF) ϕ is a mapping $\phi : \mathcal{D}^n \to \Delta A$ where ΔA denotes the set of probability distributions over A. We define some important properties of an RSCF most of which are familiar from the literature.

Definition 4.2.1 *An RSCF $\phi : \mathcal{D}^n \to \Delta A$ is unanimous if for all P_N and all $a \in A$,*

$$[\tau(P_i) = a \text{ for all } i \in N] \implies [\phi_a(P_N) = 1].$$

If all agents have a common top-ranked committee at a profile, a unanimous RCSF picks that committee at that profile. It is clearly a weak form of efficiency.

Definition 4.2.2 *An RSCF $\phi : \mathcal{D}^n \to \Delta A$ is strategy-proof if for all $i \in N$, all $P_i, P'_i \in \mathcal{D}$, and all $P_{-i} \in \mathcal{D}^{n-1}$, $\phi(P_i, P_{-i})$ first order stochastically dominates $\phi(P'_i, P_{-i})$ according to P_i, that is,*

$$\sum_{t=1}^{j} \phi_{r_t(P_i)}(P_i, P_{-i}) \geq \sum_{t=1}^{j} \phi_{r_t(P_i)}(P'_i, P_{-i}) \text{ for all } j = 1, \ldots, 2^m.$$

Our notion of strategy-proofness for RSCFs is the standard one of first-order stochastic dominance introduced in [57]. No agent can strictly increase the aggregate probability over any upper contour set according to her true preferences. If it were possible to do, there would exist a utility representation of her true preferences with the property that the expected utility from misrepresentation strictly exceeds that from truth-telling.

4.3 Formation of Arbitrary Committees

In this section, we consider the problem of forming a committee by random voting rules. We assume that there are no restrictions on the committee that is to be formed. [3] A few additional concepts are required for the analysis.

Let $\mathcal{N}$ denote the set of all subsets (power set) of N. For any $K \subseteq M$, S^K denotes a collection $(S^k)_{k \in K}$, where $S^k \subseteq N$ for all $k \in K$. Also $\mathcal{N}^K$ denotes the set of all such collections. Note that the cardinality of $\mathcal{N}^K$ is $(2^n)^{|K|}$. We illustrate these notions by means of an example.

Example 4.3.1 *Suppose $N = \{1, 2, 3, 4\}$, $M = \{1, 2, 3\}$ and $K = \{2, 3\}$. An example of $S^{\{2,3\}}$ is (S^2, S^3) where $S^2 = \{1, 2, 4\}$ and $S^3 = \{2, 3\}$. Also, $\mathcal{N}^{\{2,3\}}$ is the collection of all (S^2, S^3) where S^2 and S^3 are arbitrary subsets of $\{1, 2, 3, 4\}$.*

[3] We will consider one such problem in the next section.

Consider an arbitrary $K \subseteq M$ and profile $P_N \in \mathcal{D}^n$. Then $S^K(P_N)$ denotes an element $(S^k)_{k \in K}$ of $\mathcal{N}^K$ such that for all $k \in K$, we have $i \in S^k$ if and only if $\tau(P_i^k) = 1$. In other words S^k consists of the agents who have 1 as the top-ranked element in component k at the profile P_N. Hence S^k consists of exactly those agents who approve candidate k for the committee at the profile P_N.

Example 4.3.2 *Suppose $N = \{1, 2, 3, 4\}$ and $M = \{1, 2, 3\}$. Consider the profile P_N where the top-ranked alternatives of the agents are as follows: $((1, 0, 1), (0, 0, 1), (1, 1, 0))$. Let $K = \{1, 3\}$ or $\{1, 2, 3\}$ Then, $S^{\{1,3\}}(P_N) = (\{1, 3\}, \{1, 2\})$ and $S^{\{1,2,3\}}(P_N) = (\{1, 3\}, \{3\}, \{1, 2\})$.*

For $K \subseteq M$, $a^K \in A^K$ and $P_N \in \mathcal{D}^n$, we define $\phi_{a^K}(P_N) = \sum_{\{b \in A | b^K = a^K\}} \phi_b(P_N)$. Thus $\phi_{a^K}(P_N)$ is the total probability of realizing outcomes whose k^{th} component agrees with the k^{th} component of a^K for all $k \in K$, in the probability distribution $\phi(P_N)$.

4.3.1 Characterization

In this section, we identify properties that characterize unanimous and strategy-proof RSCFs in our model. The first property is marginal decomposability. Roughly speaking, it says that the marginal probability distribution generated by the RSCF over an arbitrary set of components depends only on the preferences of the agents over those components. In particular, it does not change if agents change their preferences over the other components.

Definition 4.3.3 *An RSCF $\phi : \mathcal{D}^n \to \Delta A$ is marginally decomposable if for all $K \subseteq M$, $P_N, \bar{P}_N \in \mathcal{D}^n$ with $S^K(P_N) = S^K(\bar{P}_N)$, and all $a^K \in A^K$, we have*

$$\phi_{a^K}(P_N) = \phi_{a^K}(\bar{P}_N).$$

Marginal decomposability is weaker than decomposability as defined in [26]. As mentioned earlier, marginal decomposability requires the marginal probability distribution over a set of components at a profile to be completely determined by the marginal preference profile over those components. Importantly, it does not say anything about the joint probability distribution. Clearly, a marginally decomposable RSCF is decomposable if the joint probability distribution is given by the product of marginal probability distributions, i.e. if the joint probability distribution is independent over components. In our model, unanimity and strategy-proofness imply marginal decomposability; however they do not imply independence over components.

We illustrate the notion of marginal decomposability by means of the following example.

Example 4.3.4 *Let $N = \{1, 2\}$ and $M = \{1, 2\}$. Consider the RSCF $\phi : \mathcal{D}^n \to \Delta A$ given in Table 4.3.1. Here, rows are indexed by the $S^1(P_N)$ and columns are by $S^2(P_N)$. The matrix, say X, corresponding to row $\hat{S}^1$ and column $\hat{S}^2$ gives the value of $\phi(P_N)$, where $S^1(P_N) = \hat{S}^1$, $S^2(P_N) = \hat{S}^2$, and $\phi_{(0,0)}(P_N) = X_{11}$, $\phi_{(0,1)}(P_N) = X_{12}$, $\phi_{(1,0)}(P_N) = X_{21}$, and $\phi_{(1,1)}(P_N) = X_{22}$. For instance, $\phi_{(0,1)}((0,1),(1,0)) = 0.55$, where $((0,1),(1,0))$ denotes the profile P_N with $r_1(P_1) = (0,1)$ and $r_1(P_2) = (1,0)$.*

We argue that ϕ satisfies marginal decomposability. Consider for instance, the row corresponding to the set $\{2\}$. Note that for each matrix X in this row, $X_{21} + X_{22} = 0.3$, that is, the marginal probability that candidate 1 is elected is 0.3, as required by marginal decomposability. It can be readily verified that ϕ satisfies this constant marginal property for other rows and columns. Consequently the RSCF is marginally decomposable.

$1\backslash 2$	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
$\emptyset$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0.3 & 0.7 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0.5 & 0.5 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
$\{1\}$	$\begin{pmatrix} 0.4 & 0 \\ 0.6 & 0 \end{pmatrix}$	$\begin{pmatrix} 0.2 & 0.2 \\ 0.1 & 0.5 \end{pmatrix}$	$\begin{pmatrix} 0.3 & 0.1 \\ 0.2 & 0.4 \end{pmatrix}$	$\begin{pmatrix} 0 & 0.4 \\ 0 & 0.6 \end{pmatrix}$
$\{2\}$	$\begin{pmatrix} 0.7 & 0 \\ 0.3 & 0 \end{pmatrix}$	$\begin{pmatrix} 0.15 & 0.55 \\ 0.15 & 0.15 \end{pmatrix}$	$\begin{pmatrix} 0.25 & 0.45 \\ 0.25 & 0.05 \end{pmatrix}$	$\begin{pmatrix} 0 & 0.7 \\ 0 & 0.3 \end{pmatrix}$
$\{1,2\}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0.3 & 0.7 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0.5 & 0.5 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Table 4.3.1: Outcomes of ϕ

We now argue that the ϕ is not decomposable in the sense of [26]. For $k \in \{1, 2\}$, let ϕ^k be the marginal RSCF on the k-th component that is induced by ϕ by means of marginal decomposability. In Tables 4.3.2 and 4.3.3, we present ϕ^1 and ϕ^2, respectively.

1	ϕ^1_1
$\emptyset$	0
$\{1\}$	0.6
$\{2\}$	0.3
$\{1,2\}$	1

Table 4.3.2: Outcomes of ϕ^1_1

2	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
ϕ^2_1	0	0.7	0.5	1

Table 4.3.3: Outcomes of ϕ^2_1

Consider a profile P_N with $r_1(P_1) = (0, 1)$ and $r_1(P_2) = (1, 0)$, that is, $S^1(P_N) = \{2\}$ and $S^2(P_N) = \{1\}$. If ϕ were decomposable, then $\phi_{(1,0)}(P_N)$ must be $0.3 \times 0.3 = 0.09$. However, as given in Table 4.3.1, $\phi_{(1,0)}(P_N) = 0.15$, which means ϕ is not decomposable.

Next, we define a monotonicity property for an RSCF. This is a standard property in the literature on strategy-proof social choice functions which says that the likelihood of an outcome increases as agents become more "favourable" to that outcome.

Definition 4.3.5 *An RSCF $\phi : \mathcal{D}^n \to \Delta A$ satisfies the monotonicity property if for all $k \in M$, all $a^{-k} \in A^{M-k}$ and all $P_N, \bar{P}_N \in \mathcal{D}^n$ such that $S^l(P_N) = S^l(\bar{P}_N)$ for all $l \in M \setminus k$ and $S^k(P_N) \subseteq S^k(\bar{P}_N)$, we have*

(i) $\phi_{(1,a^{-k})}(P_N) \leq \phi_{(1,a^{-k})}(\bar{P}_N)$, *and*

(ii) *if $S^k(P_N) = \emptyset$ and $S^k(\bar{P}_N) = N$, then $\phi_{(1,a^{-k})}(P_N) = 0$ and $\phi_{(1,a^{-k})}(\bar{P}_N) = 1$.*

Suppose that some agents change preferences in favour of some candidate while maintaining their position on all other candidates. According to (i) of the monotonicity property, the probability of *each* committee including that candidate, must increase. According to (ii) a candidate not approved by any agent is not selected with certainty a candidate approved by all agents is selected with probability one. The monotonicity property is illustrated below.

Example 4.3.6 *Consider the RSCF ϕ given in Table 4.3.1. We argue that it satisfies monotonicity properties. To see this, take, for instance, the profiles indexed by $(\{1\}, \{2\})$ and $(\{1, 2\}, \{2\})$. Note that agent 2 has joined agent 1 in approving candidate 1 from the former profile to the latter, while keeping his/her stand unchanged for candidate 2. By monotonicity, the probability of each committee that includes candidate 1 must increase (weakly). This is indeed the case here since $\phi_{(1,0)}(\{1\}, \{2\}) = 0.2 < \phi_{(1,0)}(\{1, 2\}, \{2\}) = 0.5$ and $\phi_{(1,1)}(\{1\}, \{2\}) = 0.4 < \phi_{(1,1)}(\{1, 2\}, \{2\}) = 0.5$. It can be directly verifies that ϕ satisfies this conditions for other relevant cases,. Hence it is monotonic.*

Now, we present our characterization result for unanimous and strategy-proof RSCFs. It is shown in [32] that unanimity and strategy-proofness imply tops-onlyness. We use this fact in our proof.

Theorem 4.3.7 *An RSCF $\phi : \mathcal{D}^n \to \Delta A$ is unanimous and strategy-proof if and only if it is monotone and marginally decomposable.*

Proof: (If part) Let $\phi : \mathcal{D}^n \to \Delta A$ be monotone and marginally decomposable. We show ϕ is unanimous and strategy-proof. Unanimity follows from (ii) in Definition 4.3.5. We proceed to show that ϕ is strategy-proof.

Take $b \in A$ and let P_i and $\bar{P}_i$ be two arbitrary preferences of some agent i. It is enough to show that

$$\phi_{U(b,P_i)}(P_N) \geq \phi_{U(b,P_i)}(\bar{P}_i, P_{-i}). \tag{4.1}$$

We assume without loss of generality that there exists $\hat{m} < m$ such that $r_1(P_i^k) = 1$ and $r_1(\bar{P}_i^k) = 0$ for all $k \in \{1, \ldots, \hat{m}\}$ and $r_1(P_i^k) = r_1(\bar{P}_i^k)$ for all $k \in \{\hat{m}+1, \ldots, m\}$. For $t = 0, 1, \ldots, \hat{m}$, let $P_i(t) \in \mathcal{D}$ be such that $r_1(P_i^l(t)) = 1$ if $l \leq t$, $r_1(P_i^l(t)) = 0$ if $t < l \leq \hat{m}$, and $r_1(P_i^l(t)) = r_1(P_i) = r_1(\bar{P}_i)$ if $\hat{m} < l$. Note that $P_i(\hat{m}) = P_i$ and $P_i(0) = \bar{P}_i$.

Claim 4.3.1 $\phi_{U(b,P_i)}(P_i(k), P_{-i}) \geq \phi_{U(b,P_i)}(P_i(k-1), P_{-i})$ *for all* $k = 1, \ldots, \hat{m}$.

For all $a^{-k} \in A^{-k}$, marginal decomposability implies

$$\phi_{a^{-k}}(P_i(k), P_{-i}) = \phi_{a^{-k}}(P_i(k-1), P_{-i}), \tag{4.2}$$

while monotonicity implies

$$\phi_{(1,a^{-k})}(P_i(k), P_{-i}) \geq \phi_{(1,a^{-k})}(P_i(k-1), P_{-i}). \tag{4.3}$$

Pick $k \in \{1, \ldots, \hat{m}\}$. Since $r_1(P_i^l) = 1$ for all $l \in \{1, \ldots, \hat{m}\}$, it must be true that $(1, a^{-k}) P_i (0, a^{-k})$ for all $a^{-k} \in A^{-k}$. This means $(0, a^{-k}) \in U(b, P_i)$ implies $(1, a^{-k}) \in U(b, P_i)$. In view of this, we can write $U(b, P_i) = B \cup C$, where B consists of a collection of pairs of alternatives of the form $(1, a^{-k}), (0, a^{-k})$ for some $a^{-k} \in A^{-k}$ and C consists of alternatives of the form $(1, a^{-k})$ for some $a^{-k} \in A^{-k}$ such that $(0, a^{-k})$ is not in $U(b, P_i)$. More formally, $B = \{(0, a^{-k}), (1, a^{-k}) \mid (0, a^{-k}) \in U(b, P_i)\}$ and $C = \{(1, a^{-k}) \in U(b, P_i) \mid (0, a^{-k}) \notin U(b, P_i)\}$.

By (4.2),

$$\phi_B(P_i(k), P_{-i}) = \phi_B(P_i(k-1), P_{-i}).$$

Further, by (4.3),

$$\phi_C(P_i(k), P_{-i}) \geq \phi_C(P_i(k-1), P_{-i}).$$

Combining, we have

$$\phi_{U(b,P_i)}(P_i(k), P_{-i}) \geq \phi_{U(b,P_i)}(P_i(k-1), P_{-i}).$$

This completes the proof of Claim 4.3.1.

By applying Claim 4.3.1 sequentially for $k = \hat{m}, \hat{m}-1, \ldots, 1$, we get

$$\phi_{U(a,P_i)}(P_i(\hat{m}), P_{-i}) \geq \phi_{U(b,P_i)}(P_i(\hat{m}-1), P_{-i}) \geq \ldots \geq \phi_{U(b,P_i)}(P_i(0), P_{-i}),$$

which shows (4.1).

(Only-if part) Let $\phi : \mathcal{D}^n \to \Delta A$ be a unanimous and strategy-proof RSCF. It follows from Proposition 2 in [32] that ϕ is *tops-only*, that is, $\phi(P_N) = \phi(\bar{P}_N)$ for all $P_N, \bar{P}_N \in \mathcal{D}^n$ with $r_1(P_i) = r_1(\bar{P}_i)$ for all $i \in N$.

The following claim establishes a crucial property of ϕ.

Claim 4.3.2 *Let $k \in \{1, \ldots, m\}$ and let $P_N, \bar{P}_N \in \mathcal{D}^n$ be such that $S^l(P_N) = S^l(\bar{P}_N)$ for all $l \in M \setminus k$ and $S^k(P_N) \subseteq S^k(\bar{P}_N)$. Then, for all $a^{-k} \in A^{M-k}$, we have*
(i) $\phi_{a^{-k}}(P_N) = \phi_{a^{-k}}(\bar{P}_N)$, and
(ii) $\phi_{(1,a^{-k})}(\bar{P}_N) \geq \phi_{(1,a^{-k})}(P_N)$.

Proof: Let $k \in \{1, \ldots, m\}$. Take $P_N, \bar{P}_N \in \mathcal{D}^n$ such that $S^l(P_N) = S^l(\bar{P}_N)$ for all $l \in M \setminus k$ and $S^k(P_N) \subseteq S^k(\bar{P}_N)$. It is enough to prove the claim for the case where $S^k(\bar{P}_N) = S^k(P_N) \cup i$ for some $i \in N$. Since ϕ is tops-only, we can further assume that
(i) $P_{-i} = \bar{P}_{-i}$, and
(ii) for all $b^{-k} \in A^{M-k}$,
(a) $(1, b^{-k})$ and $(0, b^{-k})$ are consecutively ranked in both $P_i, \bar{P}_i$, and
(b) $(0, b^{-k}) P_i (1, b^{-k})$ and $(1, b^{-k}) \bar{P}_i (0, b^{-k})$.[4]

It is easy to verify that P_i and $\bar{P}_i$ satisfy separability. Take $a^{-k} \in A^{-k}$. By our assumption on P_i and $\bar{P}_i$,

$$U((0, a^{-k}), P_i) \setminus (0, a^{-k}) = U((1, a^{-k}), \bar{P}_i) \setminus (1, a^{-k}).$$

By applying strategy-proofness at (P_i, P_{-i}) via $\bar{P}_i$ and at $(\bar{P}_i, P_{-i})$ via P_i, this means

$$\phi_{U((0,a^{-k}),P_i) \setminus (0,a^{-k})}(P_i, P_{-i}) = \phi_{U((1,a^{-k}),\bar{P}_i) \setminus (1,a^{-k})}(\bar{P}_i, P_{-i}). \tag{4.4}$$

Using a similar argument, we have

$$\phi_{U((1,a^{-k}),P_i)}(P_i, P_{-i}) = \phi_{U((0,a^{-k}),\bar{P}_i)}(\bar{P}_i, P_{-i}). \tag{4.5}$$

Subtracting (5.1) from (5.2), we get

$$\phi_{a^{-k}}(P_N) = \phi_{a^{-k}}(\bar{P}_i, P_{-i}),$$

[4]To see that it is possible to construct such a preference ordering, consider a lexicographic (and hence separable) preference over A where k is the lexicographic worst component (details may be found in [33]).

which proves (i) of Claim 4.3.2.

Since $\phi_{(0,a^{-k})}(P_N) + \phi_{(1,a^{-k})}(P_N) = \phi_{(0,a^{-k})}(\bar{P}_i, P_{-i}) + \phi_{(1,a^{-k})}(\bar{P}_i, P_{-i})$ and $(1, a^{-k})\bar{P}_i(0, a^{-k})$, it follows by an application of strategy-proofness that $\phi_{(1,b^{-k})}(\bar{P}_N) \geq \phi_{(1,b^{-k})}(P_N)$, which proves (ii) of Claim 4.3.2. ■

We return to the proof that ϕ satisfies monotonicity and marginally decomposability. Condition (i) in the definition of monotonicity (Definition 4.3.5) follows from Claim 4.3.2. In what follows, we prove condition (ii) in Definition 4.3.5.

It suffices to show $\sum_{a^{-1}\in A^{-1}} \phi_{(0,a^{-1})}(P_N) = 0$ for all $P_N \in \mathcal{D}^n$ with $S^k(P_N) = \emptyset$. Take P_N such that $S^k(P_N) = \emptyset$. Without loss of generality, assume $k = 1$. Let $\bar{P}_N$ be the profile such that $S^2(\bar{P}_N) = \emptyset$ and $S^l(\bar{P}_N) = S^l(P_N)$ for all $l \neq 2$. By Claim 4.3.2, $\phi_{a^{-2}}(P_N) = \phi_{a^{-2}}(\bar{P}_N)$ for all $a^{-2} \in A^{-2}$. Note that

$$\sum_{a^{-1}\in A^{-1}} \phi_{(0,a^{-1})}(P_N) = \sum_{a^{-\{1,2\}}\in A^{-\{1,2\}}} \phi_{(0,0,a^{-\{1,2\}})}(P_N) + \phi_{(0,1,a^{-\{1,2\}})}(P_N). \tag{4.6}$$

Take $a^{-2} = (0, a^{-\{1,2\}}) \in A^{-2}$. By applying Claim 4.3.2, we have

$$\phi_{(0,a^{-2})}(P_N) + \phi_{(1,a^{-2})}(P_N) = \phi_{(0,a^{-2})}(\bar{P}_N) + \phi_{(1,a^{-2})}(\bar{P}_N), \tag{4.7}$$

Combining (5.5) and (4.7), we have $\sum_{a^{-1}\in A^{-1}} \phi_{(0,a^{-1})}(P_N) = \sum_{a^{-1}\in A^{-1}} \phi_{(0,a^{-1})}(\bar{P}_N)$. Continuing in this manner, it follows that

$$\sum_{a^{-1}\in A^{-1}} \phi_{(0,a^{-1})}(P_N) = \sum_{a^{-1}\in A^{-1}} \phi_{(0,a^{-1})}(\hat{P}_N), \tag{4.8}$$

where $S^l(\hat{P}_N) = \emptyset$ for all $l \in \{1, \ldots, m\}$. By unanimity, $\phi_{(0,a^{-1})}(\hat{P}_N) = 0$ for all $a^{-1} \in A^{-1}$. This, together with (5.3), implies $\sum_{a^{-1}\in A^{-1}} \phi_{(0,a^{-1})}(P_N) = 0$, which shows (ii) in Definition 4.3.5.

Finally we show that ϕ is marginally decomposable. Let $K \subseteq M$ and let P_N and $\bar{P}_N$ be such that $S^K(P_N) = S^K(\bar{P}_N)$. Assume without loss of generality that $K = \{k+1, \ldots, m\}$ for some $k < m$. Take $a^K \in A^K$. Consider a sequence of profiles $\{P_N^l\}_{l=0}^k$ such that $P_N^0 = P_N$, $P_N^k = \bar{P}_N$, and for all $1 \leq l \leq k$, $S^{\{1,\ldots,l\}}(P_N^l) = S^{\{1,\ldots,l\}}(\bar{P}_N)$ and $S^{\{l+1,\ldots,m\}}(P_N^l) = S^{\{l+1,\ldots,m\}}(P_N)$. By (i) of Claim 4.3.2, for all $1 \leq l \leq k$, $\phi_{b^{-l}}(P_N^{l-1}) = \phi_{b^{-l}}(P_N^l)$ for all $b^{-l} \in A^{-l}$. Since $l \notin K = \{k, \ldots, m\}$, an argument similar to the one used in the derivation of (5.5), implies $\phi_{a^K}(P_N^{l-1}) = \phi_{a^K}(P_N^l)$. Therefore, $\phi_{a^K}(P_N) = \phi_{a^K}(\bar{P}_N)$, completing the proof of the only-if part. ■

Theorem 4.3.7 suggests a procedure for constructing *all* unanimous and strategy-proof RSCF on $\mathcal{D}^n$. We can start with marginal probability distributions over all subsets of components that satisfy monotonicity. We can then arbitrarily specify the appropriate joint probabilities of each alternative that generate the chosen marginal distributions.

4.4 Formation of Committees of Fixed Size

In this section, we consider the problem of forming a committee with a predetermined number of members. The size of a committee is defined as the number of members in it. Formally, the size of an alternative $a \in A$ is $|a| = |\{k \mid a^k = 1\}|$. For $l < m$, $A(l)$ is the set of all committees with size l, i.e. $A(l) = \{a \in A \mid |a| = l\}$. In this section, we consider RSCFs $\phi : \mathcal{D}^n \to \Delta A(l)$ for some $l < m$. By definition, these RSCFs give positive probabilities only to the elements of $A(l)$.

Clearly unanimity is incompatible with this range restriction. We therefore need to replace unanimity by the onto property.

Definition 4.4.1 *An RSCF $\phi : \mathcal{D}^n \to \Delta A(l)$ is onto if for all $a \in A(l)$, there is $P_N \in \mathcal{D}^n$ such that $\phi_a(P_N) = 1$.*

Our next theorem characterizes the set of onto strategy-proof RSCFs for selecting a committee with a predetermined size. It says that every such rule is random dictatorial restricted to $A(l)$.

Definition 4.4.2 *A DSCF $f : \mathcal{D}^n \to A(l)$ is $A(l)$-restricted dictatorial if there exists $i \in N$ such that $f(P_N)$ chooses the most preferred alternative of agent i from the set $A(l)$. An RSCF is called random $A(l)$-restricted dictatorial if it is a convex combination of $A(l)$-restricted dictatorial DSCFs.*

Theorem 4.4.3 *Let $l < m$. Then, an RSCF $\phi : \mathcal{D}^n \to \Delta A(l)$ is onto and strategy-proof if and only if it is random $A(l)$-restricted dictatorial.*

Proof: First we prove a claim.

Claim 4.4.1 *Let $P_N, \bar{P}_N$ be such that $P_i|_{A(l)} = \bar{P}_i|_{A(l)}$ for all $i \in N$. Then $\phi(P_N) = \phi(\bar{P}_N)$.*

Proof: We show that $\phi(P_N) = \phi(\bar{P}_i, P_{-i})$ where $P_i|_{A(l)} = \bar{P}_i|_{A(l)}$. Suppose not. Let $b \in A(l)$ be such that $\phi_b(P_N) \neq \phi_b(\bar{P}_i, P_{-i})$ and $\phi_a(P_N) = \phi_a(\bar{P}_i, P_{-i})$ for all $a \in A(l)$ with aP_ib. In other words, b is the maximal element of $A(l)$ according to P_i that violates the assertion of the claim. Without loss of generality, assume that $\phi_b(P_N) < \phi_b(\bar{P}_i, P_{-i})$. However, since $\phi_a(P_N) = \phi_a(\bar{P}_i, P_{-i})$ for all $a \notin A(l)$ with aP_ib, we have $\phi_{U(b,P_i)}(P_N) < \phi_{U(b,P_i)}(\bar{P}_i, P_{-i})$. This means agent i manipulates at P_N via $\bar{P}_i$, which is a contradiction. This completes the proof of the claim. ■

Consider an RSCF $\phi : \mathcal{D}^n \to \Delta A(l)$. For $P \in \mathcal{D}$, define $P|_{A(l)} \in \mathbb{L}(A(l))$ as follows: for all $a, b \in A(l)$, $aP|_{A(l)}b$ if and only if aPb. Let $\mathcal{D}|_{A(l)} = \{P|_{A(l)} \mid P \in \mathcal{D}\}$. Construct the RSCF $\hat{\phi} : (\mathcal{D}|_{A(l)})^n \to \Delta A(l)$ as follows: for all $\hat{P}_N \in \mathcal{D}|_{A(l)}$, $\hat{\phi}(\hat{P}_N) = \phi(P_N)$ where $P_N \in \mathcal{D}^n$ is such that $P_i|_{A(l)} = \hat{P}_i$ for all $i \in N$. This is well-defined by Claim 4.4.1. Because ϕ is strategy-proof, $\hat{\phi}$ is also strategy-proof. Moreover, since ϕ is onto with range $A(l)$, strategy-proofness of ϕ implies $\hat{\phi}$ is unanimous. In what follows, we show $\mathcal{D}|_{A(l)}$ is an unrestricted domain.

Claim 4.4.2 *The domain $\mathcal{D}|_{A(l)}$ is unrestricted.*

Proof: Take $P \in \mathcal{D}$ such that $r_1(P^l) = 1$ for all $l \in M$. Consider arbitrary $a, b \in A(l)$ such that $a \neq b$. For $x \in \{a, b\}$, let $I(x) = \{k \in M \mid x^k = 1\}$. By definition, $|I(x)| = l$ for all $x \in \{a, b\}$. Moreover, since a and b are distinct, it must be that $I(a)$ and $I(b)$ are also distinct. This, together with the fact that $|I(a)| = |I(b)| = l$, implies there must be $k, \hat{k} \in M$ such that $k \in I(a) \setminus I(b)$ and $\hat{k} \in I(b) \setminus I(a)$. This means $a^k = r_1(P^k)$ but $a^{\hat{k}} = r_1(P^{\hat{k}})$ and $b^k = r_1(P^k)$ but $b^{\hat{k}} = r_1(P^{\hat{k}})$. Therefore, responsive does not put any restriction on the relative ordering of a and b at P, and consequently, every preference in $\mathcal{D}|_{A(l)}$ can be achieved by considering a suitable preference with the alternative $(1, \ldots, 1)$ as the top-ranked element. This completes the proof of the claim. ■

Since $\mathcal{D}|_{A(l)}$ is unrestricted and $\hat{\phi}$ is unanimous and strategy-proof, it follows from [57] that $\hat{\phi}$ is random dictatorial. By the construction of $\hat{\phi}$, this means ϕ is random dictatorial restricted to $A(l)$. This completes the proof of Theorem 4.4.3. ■

It is known that strategy-proof and onto DSCFs on $A(l)$-restricted domains are dictatorial (for a general version of this result, see [14] and [5]). Unfortunately, there is no escape from this negative result is we consider random rather than deterministic rules.

4.5 Conclusion

In this paper, we have provided a characterization of random unanimous and strategy-proof rules in the well-known committee formation model in terms of two properties: marginal decomposability and monotonicity. We also show that if committees of a predetermined size have to be chosen, an onto and strategy-proof rule must be an appropriate random dictatorship.

5

A unified characterization of the randomized strategy-proof rules

5.1 Introduction

5.1.1 Background of the problem

We analyze the classical social choice problem of choosing an alternative from a set of feasible alternatives based on preferences of individuals in a society. Such a procedure is known as a *deterministic social choice function* (DSCF). Some desirable properties of a DSCF are *unanimity* and *strategy-proofness*. The classic [56]-[96] impossibility theorem states that if there are at least three alternatives and the preferences of the individuals are *unrestricted*, then every unanimous and strategy-proof DSCF is *dictatorial*.

Although unanimity and strategy-proofness are desirable properties of a DSCF, the assumption of an unrestricted domain made in Gibbard-Satterthwaite Theorem is quite strong. Not only do there exist many political and economic scenarios where preferences of individuals satisfy natural restrictions such as *single-peakedness*, *single-dippedness*, *single-crossingness*, *Euclidean*, etc., but also the conclusion of Gibbard-Satterthwaite Theorem does not apply to such restricted domains.

The study of single-peaked domains can be traced back to [20] where he shows that a *Condorcet winner* exists on such domains. Later, [72] shows that a DSCF on a single-peaked domain is unanimous and strategy-proof if and only if it is a *min-max* rule. [79] show that a DSCF on such a domain is unanimous and strategy-proof if and only if it is a *monotone rule* between the left-most and the right-most alternatives. [94] shows that a DSCF on a single-crossing domain is unanimous and strategy-proof if and only if it is an *augmented representative voter scheme*. A domain is Euclidean if its alternatives are elements of Euclidean space and its preferences are based on Euclidean distances. [65] and [78] characterize the unanimous and strategy-proof DSCFs on Euclidean domains.

The horizon of social choice theory has been expanded by the concept of *random social choice functions* (RSCF). An RSCF assigns a probability distribution over the alternatives at every preference profile. The importance of RSCFs over DSCFs is well-established in the literature (see, for example, [46], [81]).

The study of RSCFs dates back to [57] where he shows that an RSCF on the unrestricted domain is unanimous and strategy-proof if and only if it is a *random dictatorial* rule. For the case of continuous alternatives, [46] characterise unanimous and strategy-proof RSCFs on maximal single-peaked domains, and [24] and [43] characterise unanimous and strategy-proof DSCFs and RSCFs, respectively, on multi-dimensional single-peaked domains. [8] characterise efficient and strategy-proof DSCFs on multi-dimensional single-peaked domains with cardinal preferences when the range is one-dimensional. Later, [81] show that every unanimous and strategy-proof RSCF on maximal single-peaked domain is a convex combination of min-max rules. [87] establish a similar result by using the theory of totally unimodular matrices from combinatorial integer programming. Recently, [82] and [91] characterize unanimous and strategy-proof RSCFs on single-dipped domains and Euclidean domains, respectively. However, to the best of our knowledge, unanimous and strategy-proof RSCFs on domains such as single-crossing, multi-peaked, intermediate ([58]), and single-peaked on trees with top-set along a path have not yet been characterized in the literature.

5.1.2 Our motivation and contribution

Our main motivation of this paper is to present one unified characterization of unanimous and strategy-proof RSCFs that summarizes all existing results for *both DSCFs and RSCFs* and allows for new ones. We intend to do this under minimal assumption on the domains.

We show that a large class of restricted domains can be modelled by using the concept of *betweenness* ([74], [75]). Given a prior order over the alternatives, a preference satisfies the betweenness property with respect to an alternative a if, whenever a lies in-between (with respect to the prior order) the top-ranked alternative of the preference and some other alternative b, a is preferred to b. A domain satisfies the betweenness property with respect to an alternative if each preference in it satisfies the property with respect to that alternative. Consider the set of alternatives that appear as top-ranked for some preference in the domain. Assume the betweenness property is satisfied for each such alternatives. Then, the domain is called *generalized intermediate*.

We show that in case of finitely many alternatives, an RSCF is unanimous and strategy-proof on a *minimally rich* generalized intermediate domain if and only if it is a convex combination of the tops-restricted min-max rules. A min-max rule is tops-restricted if all its parameters belong to the top-set of the domain. We also consider the case of infinitely many alternatives and provide a direct characterization of unanimous and strategy-proof RSCFs on the generalized intermediate domains. It is worth mentioning that both the formulation of generalized intermediate domains and the proof techniques required to characterize the RSCFs on those are completely different in the case of infinite number of alternatives. Finally, we establish that all restricted domains that we have discussed so far, namely single-peaked, single-crossing, single-dipped, tree-single-peaked with top-set along a path, Euclidean, multi-peaked, and intermediate are special cases of generalized intermediate domains.

Our result strengthens existing results for DSCFs by dropping the maximality assumption to minimal richness. Note that in a social choice problem with m alternatives, the number of preferences in the maximal single-peaked or single-dipped domain is 2^{m-1} and in a maximal single-crossing domain is $(m(m-1)/2)+1$, whereas that number can range from $2m-2$ to 2^{m-1} in a minimally rich single-peaked domain, from 2 to 2^{m-1} in a minimally rich single-dipped domain, and from $2m^*-2$ to $(m(m-1)/2)+1$ in a minimally rich single-crossing domain, where m^* is the cardinality of the top-set of the domain.

It follows from our results that minimally-rich generalized intermediate domains satisfy both tops-only property and deterministic extreme point property. [31] provide a sufficient condition on a domain that guarantees tops-onlyness for the unanimous and strategy-proof RSCFs on it, however minimally-rich generalized intermediate domains do not satisfy their condition. A domain is said to satisfy the *deterministic extreme point* (DEP) property if every unanimous and strategy-proof RSCF on the domain is a convex combination of unanimous and strategy-proof DSCFs on it. This property can be utilized in finding the *optimal* RSCFs for a society. [55] characterize the optimal DSCFs on single-crossing domains. Therefore, by means of the DEP property of single-crossing domains, one can extend their result to the case of RSCFs.

5.1.3 Organization of the paper

The rest of the paper is organized as follows: Section 7.2 introduces the model and basic definitions. Section 7.3 presents our main result for finitely many alternatives characterizing unanimous and strategy-proof RSCFs on minimally rich generalized intermediate domains. Section 7.6 introduces the concept of generalized intermediate domains for infinitely many alternatives and presents a characterization of unanimous and strategy-proof RSCFs on those. Section 7.5 contains some applications of our results. Finally, Section 5.6 concludes the paper. The Appendix gathers all omitted proofs.

5.2 Preliminaries

Let $N = \{1, \ldots, n\}$ be a finite set of agents. Except where otherwise mentioned, $n \geq 2$. Let $A = \{a_1, \ldots, a_m\}$ be a finite set of alternatives with a prior ordering $\prec$ given by $a_1 \prec \cdots \prec a_m$. Whenever we write minimum or maximum of a subset of A, we mean it with respect to the ordering $\prec$. By $a \preceq b$, we mean $a = b$ or $a \prec b$. For $a, b \in A$, we define $[a, b] = \{c \mid \text{ either } a \preceq c \preceq b \text{ or } b \preceq c \preceq a\}$ as the set of alternatives that lie in-between a and b, and for $B \subseteq A$, we define $[a, b]_B = [a, b] \cap B$ as the alternatives in B that lie in the interval $[a, b]$. For notational convenience, whenever it is clear from the context, we do not use braces for singleton sets, for instance we denote a set $\{i\}$ by i.

5.2.1 Domain of preferences

A complete, antisymmetric, and transitive binary relation over A (also called a linear order) is called a *preference*. We denote by $\mathbb{L}(A)$ the set of all preferences over A. For $P \in \mathbb{L}(A)$ and $a, b \in A$, aPb is interpreted as "a is strictly preferred to b according to P". For $P \in \mathbb{L}(A)$ and $1 \leq k \leq m$, by $r_k(P)$ we denote the k-th ranked alternative in P, i.e., $r_k(P) = a$ if and only if $|\{b \in A \mid bPa\}| = k - 1$. Since we refer to the top-ranked alternative of a preference P very frequently, we use a simpler notation, $\tau(P)$, for that. For $P \in \mathcal{D}$ and $a \in A$, the *upper contour set* of a at P, denoted by $U(a, P)$, is defined as the set of alternatives that are as good as a in P, i.e., $U(a, P) = \{b \in A \mid bPa\} \cup a$. By P^a, we denote a preference with a as the top-ranked alternative, that is, P^a is such that $\tau(P^a) = a$. Similarly, by $P^{a,b}$, we denote a preference with a as the top-ranked and b as the second-top-ranked alternatives, that is, $P^{a,b}$ is such that $\tau(P^{a,b}) = a$ and $r_2(P^{a,b}) = b$. For ease of presentation, sometimes we write $P \equiv P^{a,b}$ to mean $\tau(P) = a$ and $r_2(P) = b$.

We denote by $\mathcal{D} \subseteq \mathbb{L}(A)$ a set of admissible preferences (henceforth, will be called a domain). For $a \in A$, let $\mathcal{D}^a = \{P \in \mathcal{D} \mid \tau(P) = a\}$ denote the preferences in $\mathcal{D}$ that have a as the top-ranked alternative. For a domain $\mathcal{D}$, the top-set of $\mathcal{D}$, denoted by $\tau(\mathcal{D})$, is the set of alternatives that appear as a top-ranked alternative in some preference in $\mathcal{D}$, that is, $\tau(\mathcal{D}) = \cup_{P \in \mathcal{D}} \tau(P)$. Whenever we write $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$, we assume without loss of generality that the indexation is such that $b_1 \prec \cdots \prec b_k$. A domain $\mathcal{D}$ is *regular* if $\tau(\mathcal{D}) = A$.

A preference profile, denoted by $P_N = (P_1, \ldots, P_n)$, is an element of $\mathcal{D}^n = \mathcal{D} \times \cdots \times \mathcal{D}$ that represents a collection of preferences one for each agent.

For $P \in \mathbb{L}(A)$ and $B \subseteq A$, the restriction of P to B, $P|_B \in \mathbb{L}(B)$ is defined as follows: for all $a, b \in B$,

$aP|_B b$ if and only if aPb. For $\mathcal{D} \subseteq \mathbb{L}(A)$, $P_N \in \mathcal{D}^n$, and $B \subseteq A$, we define the restriction of the domain $\mathcal{D}$ to B as $\mathcal{D}|_B = \{P|_B \mid P \in \mathcal{D}\}$, and the restriction of the profile P_N to B as $P_N|_B = (P_1|_B, \ldots, P_n|_B)$.

Properties of a domain

In this section, we introduce a few properties of a domain. First, we introduce the concept of a single-peaked domain. A preference is single-peaked if it decreases as one goes far away (with respect to the ordering $\prec$) in any particular direction from its peak (top-ranked alternative). More formally, a preference P is *single-peaked* if for all $a, b \in A$, $[\tau(P) \preceq a \prec b$ or $b \prec a \preceq \tau(P)]$ implies aPb. A domain is *single-peaked* if each preference in it is single-peaked, and is *maximal single-peaked* if it contains all single-peaked preferences. For $B \subseteq A$, a domain $\mathcal{D}$ of preferences is a single-peaked domain restricted to B if $\mathcal{D}|_B$ is a single-peaked domain.

A preference P satisfies the **betweenness property** with respect to an alternative a if for all $b \in A \setminus a$, $a \in [\tau(P), b]$ implies aPb. A domain $\mathcal{D}$ satisfies the betweenness property with respect to an alternative a if each preference $P \in \mathcal{D}$ satisfies the property with respect to a.

Note that the betweenness property of a preference with respect to an alternative a does *not* put any restriction on the relative ordering of two alternatives if both of them are different from a, or if one of them lies in-between the top-ranked alternative of that preference and a, and the other one is a itself. A domain $\mathcal{D}$ is **generalized intermediate** if it satisfies the betweenness property with respect to each alternative in $\tau(\mathcal{D})$.

Remark 5.2.1 *Note that the generalized intermediate property does not impose any restriction on the relative ordering of the alternatives outside the top-set of a domain. Furthermore, if a domain $\mathcal{D}$ satisfies this property, then $\mathcal{D}|_{\tau(\mathcal{D})}$ is single-peaked, which in particular implies that a regular domain is single-peaked if and only if it is generalized intermediate.*

Note that a maximal generalized intermediate domain requires quite a few preferences to be present in the domain. In view of this, we require a *minimal* set of preferences to be present in a generalized intermediate domain. Our minimal requirement ensures that for two alternatives that are consecutive in the top-set of a domain,[1] there are two different preferences which (i) rank those two alternatives in the top-two positions, and (ii) agree on the ranking of the other alternatives.[2]

To ease our presentation, for two preferences P and P' in $\mathcal{D}$, we write $P \sim P'$ if $\tau(P) = r_2(P')$, $r_2(P) = \tau(P')$, and $r_l(P) = r_l(P')$ for all $l \geq 3$, that is, P and P' differ only on the ranking of the top two

[1] We say two alternatives are "consecutive in the top-set" if (i) they are in the top-set of the domain, and (ii) there is no alternative in the top-set of the domain that lies strictly in-between (with respect to the prior order $\prec$) those two alternatives.

[2] This property is known as top-connectedness in the literature ([71], [95], [38]).

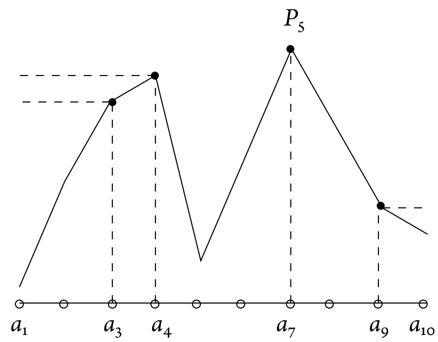


Figure 5.2.1: A graphic illustration of the preference P_5 given in Table 5.2.1

alternatives. Recall that throughout this paper, whenever we write $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$ for a domain $\mathcal{D}$, we assume $b_1 \prec \cdots \prec b_k$.

A domain $\mathcal{D}$ with $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$ satisfies the **minimal richness** property if for all $b_j, b_{j+1} \in \tau(\mathcal{D})$, there are $P \in \mathcal{D}^{b_j}$ and $P' \in \mathcal{D}^{b_{j+1}}$ such that $P \sim P'$. Below, we provide an example of a generalized intermediate domain satisfying the minimal richness property.

Example 5.2.2 *Let the set of alternatives be $A = \{a_1, \ldots, a_{10}\}$ with prior order $a_1 \prec \cdots \prec a_{10}$. Consider the domain $\mathcal{D} = \{P_1, \ldots, P_8\}$ given in Table 5.2.1.*

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
a_3	a_3	a_4	a_4	a_7	a_7	a_9	a_9
a_1	a_4	a_3	a_7	a_4	a_9	a_7	a_{10}
a_4	a_1	a_1	a_3	a_3	a_{10}	a_{10}	a_7
a_2	a_6	a_6	a_8	a_8	a_4	a_4	a_8
a_6	a_7	a_7	a_6	a_6	a_3	a_3	a_6
a_7	a_5	a_5	a_2	a_2	a_1	a_1	a_4
a_5	a_9	a_9	a_9	a_9	a_2	a_2	a_3
a_8	a_2	a_2	a_{10}	a_{10}	a_5	a_5	a_5
a_9	a_{10}	a_{10}	a_5	a_5	a_6	a_6	a_1
a_{10}	a_8	a_8	a_1	a_1	a_8	a_8	a_2

Table 5.2.1

Note that $\tau(\mathcal{D}) = \{a_3, a_4, a_7, a_9\}$. To see that $\mathcal{D}$ is a generalized intermediate domain, consider, for instance, the preference P_3. We show that P_3 satisfies the betweenness property with respect to each alternative in $\{a_3, a_4, a_7, a_9\}$. Consider a_7. Observe that $\tau(P_3) = a_4$ and $a_7 P_3 a_j$ for all $j \in \{8, 9, 10\}$. So, P_3 satisfies the betweenness property with respect to a_7. Similarly, it can be checked that P_3 satisfies the betweenness property with respect to a_3 and a_9. It is left to the reader to verify that the other preferences in $\mathcal{D}$ satisfy the betweenness *property with respect to $\{a_3, a_4, a_7, a_9\}$ and that it is minimally rich. In Figure 6.3.1, we present a pictorial description of the preference $P_5 \in \mathcal{D}$.* □

5.2.2 Social choice functions and their properties

In this section, we define social choice functions and discuss a few properties of those. By ΔA, we denote the set of probability distributions over A. A **random social choice function (RSCF)** is a function $\phi : \mathcal{D}^n \to \Delta A$ that assigns a probability distribution over A at every preference profile. For $a \in A$ and $P_N \in \mathcal{D}^n$, we denote by $\phi_a(P_N)$ the probability of a at the outcome $\phi(P_N)$, and for $B \subseteq A$, we define $\phi_B(P_N) = \sum_{a \in B} \phi_a(P_N)$ as the total probability of the alternatives in B at $\phi(P_N)$.

An RSCF is a **deterministic social choice function (DSCF)** if it selects a degenerate probability distribution at every preference profile. More formally, an RSCF $\phi : \mathcal{D}^n \to \Delta A$ is a DSCF if $\phi_a(P_N) \in \{0, 1\}$ for all $a \in A$ and all $P_N \in \mathcal{D}^n$.

For later reference we include the following (trivial) observation.

Remark 5.2.3 *For all $L, L' \in \Delta A$ and all $P \in \mathbb{L}(A)$, if $L_{U(x,P)} \geq L'_{U(x,P)}$ and $L'_{U(x,P)} \geq L_{U(x,P)}$ for all $x \in A$, then $L = L'$.*

We now introduce some properties of an RSCF that are standard in the literature. An RSCF $\phi : \mathcal{D}^n \to \Delta A$ is **unanimous** if for all $a \in A$ and all $P_N \in \mathcal{D}^n$, $[\tau(P_i) = a \text{ for all } i \in N] \Rightarrow [\phi_a(P_N) = 1]$. An RSCF $\phi : \mathcal{D}^n \to \Delta A$ is **strategy-proof** if for all $i \in N$, all $P_N \in \mathcal{D}^n$, all $P'_i \in \mathcal{D}$, and all $x \in A$, $\phi_{U(x,P_i)}(P_i, P_{-i}) \geq \phi_{U(x,P_i)}(P'_i, P_{-i})$.[3] The concepts of unanimity and strategy-proofness for DSCFs are special cases of the corresponding ones for RSCFs. Two profiles $P_N, P'_N \in \mathcal{D}^n$ are *tops-equivalent* if each agent has the same top-ranked alternative in those two profiles, that is, $\tau(P_i) = \tau(P'_i)$ for all $i \in N$. An RSCF $\phi : \mathcal{D}^n \to \Delta A$ is **tops-only** if $\phi(P_N) = \phi(P'_N)$ for all tops-equivalent $P_N, P'_N \in \mathcal{D}^n$. An RSCF $\phi : \mathcal{D}^n \to \Delta A$ is **uncompromising** if $\phi_B(P_N) = \phi_B(P'_i, P_{-i})$ for all $i \in N$, all $P_N \in \mathcal{D}^n$, all $P'_i \in \mathcal{D}$, and all $B \subseteq A$ such that $B \cap [\tau(P_i), \tau(P'_i)] = \emptyset$. In words, uncompromisingness says that if an agent moves his peak (top-ranked alternative) from an alternative a to another alternative b, then the probability assigned by an RSCF to each alternative outside the interval $[a, b]$ will remain unchanged. Note that an uncompromising RSCF is tops-only by definition.

A class of social choice functions

[72] introduces the concept of min-max rules with respect to a collection of parameters. Tops-restricted

[3] Our notion of strategy-proofness (which is introduced in [57]) is based on first order stochastic dominance. Informally speaking, strategy-proofness ensures that if an agent misreports his/her preference, he/she cannot obtain an outcome that first order stochastically dominates the original one.

min-max rules are special cases of these rules where the parameters must come from the top-set of the domain.

A DSCF $f : \mathcal{D}^n \to A$ is a **tops-restricted min-max (TM)** rule if for all $S \subseteq N$, there exists $\beta_S \in \tau(\mathcal{D})$ satisfying the conditions that $\beta_\emptyset = \max(\tau(\mathcal{D})), \beta_N = \min(\tau(\mathcal{D})),$ and $\beta_T \preceq \beta_S$ for all $S \subseteq T$ such that

$$f(P_N) = \min_{S \subseteq N} \left[\max_{i \in S} \{ \tau(P_i), \beta_S \} \right].$$

If $\tau(\mathcal{D}) = A$, then a TM rule is called a **min-max** rule. In what follows, we present an example of a TM rule.

Example 5.2.4 *Let $A = \{a_1, \ldots, a_{10}\}$ and $N = \{1, 2, 3\}$. Consider a domain $\mathcal{D}$ with $\tau(\mathcal{D}) = \{a_2, a_3, a_4, a_5, a_7, a_8, a_9\}$. Consider the TM rule, say f, with respect to the parameters given in Table 5.2.2.*

β	β_1	β_2	β_3	$\beta_{\{1,2\}}$	$\beta_{\{1,3\}}$	$\beta_{\{2,3\}}$
	a_8	a_9	a_7	a_4	a_5	a_2

Table 5.2.2

Let (a_5, a_3, a_8) denote a profile where a_5, a_3 and a_8 are the top-ranked alternatives of agents 1, 2 and 3, respectively. The outcome of f at this profile is determined as follows.

$$\begin{aligned}
f(P_N) &= \min_{S \subseteq \{1,2,3\}} \left[\max_{i \in S} \{ \tau(P_i), \beta_S \} \right] \\
&= \min \big[\max\{\beta_\emptyset\}, \max\{\tau(P_1), \beta_1\}, \max\{\tau(P_2), \beta_2\}, \max\{\tau(P_3), \beta_3\}, \\
&\qquad \max\{\tau(P_1), \tau(P_2), \beta_{\{1,2\}}\}, \max\{\tau(P_1), \tau(P_3), \beta_{\{1,3\}}\}, \max\{\tau(P_2), \tau(P_3), \beta_{\{2,3\}}\}, \\
&\qquad \max\{\tau(P_1), \tau(P_2), \tau(P_3)\beta_{\{1,2,3\}}\} \big] \\
&= \min \left[a_{10}, a_8, a_9, a_8, a_5, a_8, a_8, a_8 \right] \\
&= a_5.
\end{aligned}$$

□

Note that the outcome of a TM rule f always lies in the top-set of the corresponding domain, i.e., $f(P_N) \in \tau(\mathcal{D})$ for all $P_N \in \mathcal{D}^n$. Our next remark says that a TM rule on a domain can be seen as a min-max rule on the domain obtained by restricting it to its top-set. It further says that the former is strategy-proof if and only if latter is.

Remark 5.2.5 *Let $f : \mathcal{D}^n \to A$ be a TM rule. Define $\hat{f} : (\mathcal{D}|_{\tau(\mathcal{D})})^n \to \tau(\mathcal{D})$ such that $\hat{f}(P_N|_{\tau(\mathcal{D})}) = f(P_N)$.*[4] *Then, f is strategy-proof if and only if $\hat{f}$ is strategy-proof.*

For DSCFs f^j, $j = 1, \ldots, k$ and nonnegative numbers λ^j, $j = 1, \ldots, k$, summing to 1, we define the RSCF $\phi = \sum_{j=1}^{k} \lambda^j f^j$ as $\phi_a(P_N) = \sum_{j=1}^{k} \lambda^j f^j_a(P_N)$ for all $P_N \in \mathcal{D}^n$ and all $a \in A$. We call ϕ a *convex combination* of the DSCFs f^j. So, at every profile, ϕ assigns probability λ^j to the outcome of f^j for all $j = 1, \ldots, k$.

An RSCF $\phi : \mathcal{D}^n \to \Delta A$ is a **tops-restricted random min-max (TRM)** rule if ϕ can be written as a convex combination of some TM rules on $\mathcal{D}^n$. If $\tau(\mathcal{D}) = A$, then a TRM rule $\phi : \mathcal{D}^n \to \Delta A$ is a **random min-max** rule.

5.3 Results

5.3.1 Unanimous and strategy-proof RSCFs on generalized intermediate domains

In this subsection, we present our main result characterizing the unanimous and strategy-proof RSCFs on the minimally rich generalized intermediate domains.

Theorem 5.3.1 *Let $\mathcal{D}$ be a minimally rich generalized intermediate domain. Then, an RSCF $\phi : \mathcal{D}^n \to \Delta A$ is unanimous and strategy-proof if and only if it is a TRM rule.*

The proof of this theorem is relegated to Appendix 5.7. We provide a brief sketch of it here. The if part of the theorem follows from [72]. To see this, first note the following two facts: (i) every minimally rich generalized intermediate domain $\mathcal{D}$ restricted to its top-set $\tau(\mathcal{D})$ is a subset of the maximal single-peaked domain over $\tau(\mathcal{D})$, and (ii) every TRM rule on $\mathcal{D}^n$ is a random min-max rule on $\mathcal{D}^n|_{\tau(\mathcal{D})}$. In view of these observations, it is enough to show that every random min-max rule is unanimous and strategy-proof on $\mathcal{D}|_{\tau(\mathcal{D})}$. From [72], every min-max rule on $\mathcal{D}|_{\tau(\mathcal{D})}$ is unanimous and strategy-proof, and since every random min-max rule is a convex combination of min-max rules, such rules are also unanimous and strategy-proof on $\mathcal{D}|_{\tau(\mathcal{D})}$.

We prove the only-if part of the theorem in the following two steps. In the first step, we prove a proposition that states that every unanimous and strategy-proof RSCF on a minimally rich generalized intermediate domain is uncompromising and assigns probability 1 to the top-set of the domain. We prove this proposition by using the method of induction on the number of agents. We start with the base case $n = 1$. The proposition follows trivially for this case. Assuming that the proposition holds for all cases where the number of agents is less than n, we proceed to prove it for n agents. First, we consider the set of

[4] This is well-defined since by the definition of a TM rule, f is tops-only and $f(P_N) \in \tau(\mathcal{D})$ for all $P_N \in \mathcal{D}^n$.

profiles where agents 1 and 2 have the same preferences. We show that the restriction of ϕ to this set induces a unanimous and strategy-proof RSCF on $\mathcal{D}^{n-1}$, and claim by means of the induction hypothesis that the proposition holds (in a suitable sense) on this set of profiles. Next, we show that the same holds for the profiles where agents 1 and 2 have the same top-ranked alternatives (instead of having the same preferences). Finally, in order to prove the proposition for profiles where agents 1 and 2 have arbitrary top-ranked alternatives, we use another level of induction on the "distance" between the top-ranked alternatives of agents 1 and 2. The distance between two alternatives $b_j, b_{j+l} \in \tau(\mathcal{D})$ is defined as l. Assuming that the proposition holds for the profiles where the said distance is less than some $\hat{l}$, we prove the proposition for the profiles where it is $\hat{l}$. By induction, this completes the proof of the proposition.

For a clearer picture, we explain the first step of the proof by means of an example. Suppose that $N = \{1, 2, 3\}$ and $A = \{a_1, \ldots, a_{10}\}$. Let $\mathcal{D}$ be a minimally rich generalized intermediate domain with $\tau(\mathcal{D}) = \{a_1, a_4, a_5, a_8, a_9\}$. Note that if we had one agent, then trivially every unanimous and strategy-proof RSCF on $\mathcal{D}$ would be uncompromising and would assign probability 1 to the alternatives in $\{a_1, a_4, a_5, a_8, a_{10}\}$ at every profile. Suppose (as the induction hypothesis) that the same holds if we had two agents. Consider all the preference profiles P_N, where agents 1 and 2 have the same preferences. We look at the restriction of a unanimous and strategy-proof RSCF ϕ on these profiles. Since agents 1 and 2 have the same preferences for all these profiles, they can be treated as one agent and ϕ can be seen as an RSCF for two agents. By some elementary arguments, one can show that ϕ, when seen as a two-agent RSCF, is unanimous and strategy-proof. So, by the induction hypothesis, ϕ satisfies uncompromisingness and assigns probability 1 to the set $\{a_1, a_4, a_5, a_8, a_9\}$ for all these profiles. Next, we let the preferences of agents 1 and 2 differ beyond their top-ranked alternatives and extend our proposition to those profiles. We use Remark 5.2.3 to complete this step. Finally, we proceed to prove the proposition when agents 1 and 2 have arbitrary preferences. Here, we use another level of induction. Suppose (as the induction hypothesis) that the proposition holds over the profiles for which the top-ranked alternatives of agents 1 and 2 are at distance 1, that is, over the profiles of the form $(a_1, a_4, \cdot)$ or $(a_4, a_5, \cdot)$ or $(a_5, a_8, \cdot)$ or $(a_8, a_9, \cdot)$. Here, by $(a_1, a_4, \cdot)$ we mean the profiles at which agent 1's top-ranked alternative is a_1, 2's top-ranked alternative is a_4, and 3's top-ranked alternative is arbitrary. We show as the induction step that the same holds over the profiles of the form $(a_1, a_5, \cdot)$ or $(a_4, a_8, \cdot)$ or $(a_5, a_9, \cdot)$. We prove this as a general step of the induction, and thereby cover all profiles in $\mathcal{D}^3$. The details of the arguments needed to show this step is quite technical, so we do not discuss it here.

In the second step, we show that every uncompromising RSCF on $\mathcal{D}^n$ is a random min-max rule. We use results from [46] and [81] to prove this. Finally, we argue that if a random min-max rule assigns positive probability only to the alternatives in the top-set of the domain, then it is a TRM rule. This completes the proof of the only-if part of the theorem.

Remark 5.3.2 *Since every TRM rule is tops-only, it follows from our result that unanimity and strategy-proofness together guarantee tops-onlyness for the RSCFs on minimally rich generalized intermediate domains. [31] provide a sufficient condition for a domain to be tops-only for RSCFs.*[5] *However, minimally rich generalized intermediate domains do not satisfy their condition.*

Remark 5.3.3 *A domain $\mathcal{D}$ satisfies the deterministic extreme point (DEP) property if every unanimous and strategy-proof RSCF on $\mathcal{D}^n$ can be written as a convex combination of unanimous and strategy-proof DSCFs on $\mathcal{D}^n$. It follows from Theorem 5.3.1 that minimally rich generalized intermediate domains satisfy deterministic extreme point property.*

Remark 5.3.4 *[10] introduce the notion of top-monotonicity. It can be verified that if every preference in a domain satisfies the betweenness property, then the corresponding preference profile will satisfy the top-monotonicity property. Therefore, it follows from [10] that generalized intermediateness guarantees the existence of voting equilibria, not only under the majority rule but also for the wide class of voting rules analyzed by [6]. Moreover, these equilibria are closely connected to an extended notion of the median voter.*

Remark 5.3.5 *It can be verified that minimally rich generalized intermediate domains are semilattice single-peaked, and hence by Proposition 3 of [29], it follows that they admit unanimous, anonymous, tops-only, and strategy-proof DSCFs.*

5.4 The case of infinite alternatives

In this section, we assume that the set of alternatives A is an infinite set, for instance, a subset of $\mathbb{R}$.[6] As it is mentioned in [10], such a scenario arises in modelling the decision problem to choose a tax rate to finance a public good ([101]) or a tax rate to finance public schooling in the presence of an option to buy private schooling [49].

A (weak) preference is defined as a weak order (i.e., complete and transitive binary relations) and is denoted by R. The strict part of R is denoted by P. We denote the set of all preferences by $\mathbb{W}(A)$. We assume A to be endowed with a σ-algebra of measurable sets. Only preferences for which the upper contour sets $U(x, R)$, for all $x \in A$, are measurable are considered in $\mathbb{W}(A)$. An RSCF ϕ assigns to an admissible preference profile a probability distribution over the measurable space A, hence a probability to every measurable set. The set of all such probability distributions will still be denoted by ΔA. For a measurable set $B \subseteq A$, $\phi_B(R_N)$ denotes the probability assigned to B at the preference profile R_N. All the introduced properties of an RSCF extend in a straightforward manner to this setting.

[5] A domain is tops-only if every unanimous and strategy-proof RSCF on it is tops-only.

[6] Throughout this paper, $\mathbb{R}$ denotes the set of real numbers.

For all the domains $\mathcal{D}$ we consider in this section, we assume that $\tau(\mathcal{D})$ comprises of a finite union of disjoint closed intervals $I_1, \ldots, I_k$ of $\mathbb{R}$. Here, an interval can also be a singleton set. We further assume that for all $R \in \mathcal{D}$, there exists a unique top-ranked alternative $\tau(R)$ at R, and two alternatives on the same side of $\tau(R)$ cannot be indifferent, that is, for all $x, y \in A$ with $x < y \leq \tau(R)$ or $\tau(R) \leq y < x$, we have either xPy or yPx.

We now introduce the concept of generalized intermediate domains in this setting. A domain $\mathcal{D}$ is **generalized intermediate** if it contains all preferences satisfying the following condition: for all $x, y \in \tau(\mathcal{D})$ and all $R \in \mathcal{D}^x$, if $z < y \leq x$ or $x \leq y < z$ for some $z \in A$, then yPz. In other words, it says that if an alternative in the top-set of the domain lies in-between the top-ranked alternative of a preference and another (arbitrary) alternative, then the former alternative is preferred to the latter. Note that (i) the domain restricted to its top-set is a single-peaked domain, and (ii) there is no restriction on the relative ordering of two alternatives outside the top-set of the domain.

For a profile $R_N \in \mathcal{D}^n$ and $x \in \mathbb{R}$, we define $S(x, R_N) = \{i \in N \mid \tau(R_i) \leq x\}$ as the set of agents whose top-ranked alternatives at R_N are on the (weak) left of x. In what follows, we define the TRM rules in this context.

An RSCF $\phi : \mathcal{D}^n \to \Delta A$ is a **tops-restricted random min-max (TRM)** rule if for each $S \subseteq N$, there exists a probabilistic ballot $\beta_S \in \Delta(\tau(\mathcal{D}))$ such that the following three conditions are satisfied:

(i) $\beta_\emptyset = e_{\max\{\tau(\mathcal{D})\}}$ and $\beta_N = e_{\min\{\tau(\mathcal{D})\}}$.[7]

(ii) For all $T, T' \subseteq N$, we have

$$\beta_{T \cup T'}([\min\{\tau(\mathcal{D})\}, x]) \geq \beta_T([\min\{\tau(\mathcal{D})\}, x]) \text{ for all } x \in [\min\{\tau(\mathcal{D})\}, \max\{\tau(\mathcal{D})\}].$$

(iii) For all $R_N \in \mathcal{D}^n$ and all $x \in [\min\{\tau(\mathcal{D})\}, \max\{\tau(\mathcal{D})\}]$, we have

$$\phi(R_N)([\min\{\tau(\mathcal{D})\}, x]) = \beta_{S(x,R_N)}([\min\{\tau(\mathcal{D})\}, x]).$$

The intuition of the tops-restricted random min-max rules for the case of infinite alternatives is quite similar to that of the tops-restricted min-max rules for the case of finite alternatives. As in the case of finitely many alternatives, here too these are based on their outcomes at boundary profiles. Following our earlier notations, we denote the outcome of a boundary profile, where agents in S are at the left most alternative and the others are at the right most, by β_S. Condition (i) ensures that the rule is unanimous over the boundary profiles. Condition (ii) captures the monotonicity property of the outcomes over the boundary profiles. This monotonicity is a straightforward implication of strategy-proofness. Finally, Condition (iii) presents how the rule works as a function of β's. First note that to find the probabilities of arbitrary intervals at a profile, it is sufficient to find the probabilities of the intervals of the form $[\min\{\tau(\mathcal{D})\}, x]$. Now, to find the probability of such an interval at a profile R_N, construct the boundary

[7] For $x \in \mathbb{R}$, by e_x we denote the degenerate probability distribution at x.

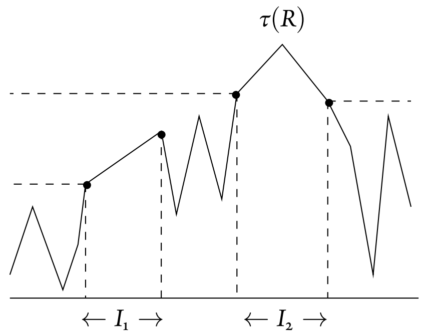


Figure 5.4.1: A graphic illustration of a generalized intermediate preference

profile as follows: move all the agents, whose top-ranked alternatives are on the left of x (that is, less than or equal to x) at R_N, to the left most alternative in $\tau(\mathcal{D})$ (thus, these agents constitute the set S), and move all other agents to the right most alternative in $\tau(\mathcal{D})$. Finally, find the probability of the interval $[\min\{\tau(\mathcal{D})\}, x]$ at R_N by equating it to the probability of the same interval at the boundary profile constructed above, that is, by equating it to the probability $\beta_S([\min\{\tau(\mathcal{D})\}, x])$.

Note that there is a basic difference between how we define the tops-restricted random min-max (TRM) rules for the case of finitely many alternatives and the case of infinitely many alternatives. For the former case, we present them as convex combinations (or, probability mixtures) of top-restricted min-max rules. However, for the latter, we provide a direct description of these rules. We do this for the sake of simplicity as we explain in the following. Observe that there are infinitely many tops-restricted min-max rules in the case of infinitely many alternatives. So, a convex combination has to be presented using integration in place of summation. Furthermore, such a presentation will require us to define a continuous probability distribution over the tops-restricted min-max rules. Such a presentation looks quite technical, as well as makes it hard to comprehend.

Theorem 5.4.1 *Let $\mathcal{D}$ be a generalized intermediate domain. Then, an RSCF $\phi : \mathcal{D}^n \to \Delta A$ is unanimous and strategy-proof if and only if it is a TRM rule.*

The proof of this theorem is relegated to Appendix 5.8. The main challenge in moving from a finite to infinite/continuous set of alternatives is that for the latter case we allow for indifferences, and consequently our earlier proof technique fails. In what follows, we provide a brief sketch of the proof.

First, we prove that a unanimous and strategy-proof RSCF on a generalized intermediate domain (i) assigns total probability 1 at every profile to the alternatives that lie in-between the minimum and the maximum peaks at that profile, that is, at every profile R_N, the interval $[\min(\tau(R_N)), \max(\tau(R_N))]$ gets probability 1, and (ii) the alternatives in the top-set of the domain gets probability 1, that is, the probability of $\tau(\mathcal{D})$ is 1 at every profile. To show this, we use induction on the number of different peaks at a profile.

We consider the case of two different peaks as the base case. For this case, the proof of (i) is more or less straightforward, whereas that of (ii) is somewhat involved. Next, we prove the induction step. Here, we assume that (i) and (ii) hold for all profiles having at most l different peaks for some $l < n$, and continue to prove the same for profiles having $l + 1$ different peaks. To complete this induction step, we use another level of induction on the number of agents whose peaks are the minimum and that whose peaks are the maximum at a profile. Let us call a profile (k_1, k_2)-(min, max) profile if at this profile, there are k_1 agents whose peaks are the minimum of that profile and k_2 agents whose peaks are the maximum of that profile. We treat the case of $(1, 1)$-(min, max) profiles as the base case. As the induction step, we assume that (i) and (ii) hold for all $(k_1 - 1, k_2)$-(min, max) and all $(k_1, k_2 - 1)$-(min, max) profiles and proceed to show that the same holds for all (k_1, k_2)-(min, max) profiles. Let us explain that the induction step is compatible with our base case. Suppose that we have shown (i) and (ii) for all $(1, 1)$-(min, max) profiles and we want to show it for $(2, 1)$-(min, max) profiles. Note that (i) and (ii) trivially hold for all $(2, 0)$-(min, max) profiles. So, by taking $k_1 = 2$ and $k_2 = 1$ in the induction step, we obtain (i) and (ii) for all $(2, 1)$-(min, max) profiles.

5.5 Applications

In this section, we demonstrate the applicability of our results by showing that a class of domains of practical importance are generalized intermediate.

5.5.1 Single-peaked domains

[46] characterize the unanimous and strategy-proof RSCFs on the maximal single-peaked domain as fixed-probabilistic-ballots rules, and [81] show that such an RSCF is a convex combination of the min-max rules. Theorem 5.3.1 improves these results by relaxing the maximality assumption. Note that the number of preferences in the maximal single-peaked domain is 2^{m-1}, whereas that in a minimally rich single-peaked domain can range from $2m - 2$ to 2^{m-1}.

5.5.2 Single-crossing domains

In this subsection, we introduce the concept of single-crossing domains and show that every single-crossing domain is generalized intermediate. [94] characterizes all unanimous and strategy-proof DSCFs on maximal single-crossing domains. [27] considers a slightly more general class of single-crossing domains called successive single-crossing domains in the context of local strategy-proofness with transfers. We show that all these domains are special cases of minimally rich generalized intermediate domains.

A domain $\mathcal{D}$ is *single-crossing* if there is an ordering $\lhd$ over $\mathcal{D}$ such that for all $a, b \in A$ and all $P, P' \in \mathcal{D}$, $[a \prec b, P \lhd P', \text{ and } bPa] \implies bP'a$. In words, a single-crossing domain is one for which the preferences can be ordered in a way such that every pair of alternatives switches their relative ranking at most once along that ordering. A single-crossing domain $\bar{\mathcal{D}}$ is *maximal* if there does not exist another single-crossing domain that is a strict superset of $\bar{\mathcal{D}}$. Note that a maximal single-crossing domain with m alternatives contains $m(m-1)/2+1$ preferences.[8] A domain $\mathcal{D}$ is *successive single-crossing* if there is a maximal single-crossing domain $\bar{\mathcal{D}}$ with respect to some ordering $\lhd$ and two preferences $P', P'' \in \bar{\mathcal{D}}$ with $P' \unlhd P''$ such that $\mathcal{D} = \{P \in \bar{\mathcal{D}} \mid P' \unlhd P \unlhd P''\}$.[9]

In the following example, we present a maximal single-crossing domain and a successive single-crossing domain with 5 alternatives.

Example 5.5.1 *Let the set of alternatives be $A = \{a_1, a_2, a_3, a_4, a_5\}$ with the prior order $a_1 \prec \cdots \prec a_5$. The domain $\bar{\mathcal{D}} = \{a_1a_2a_3a_4a_5,\ a_2a_1a_3a_4a_5,\ a_2a_3a_1a_4a_5,\ a_2a_3a_4a_1a_5,\ a_2a_4a_3a_1a_5,\ a_4a_2a_3a_1a_5,\ a_4a_2a_3a_5a_1,\ a_4a_3a_2a_5a_1,\ a_4a_3a_5a_2a_1,\ a_4a_5a_3a_2a_1,\ a_5a_4a_3a_2a_1\}$ is a maximal single-crossing domain with respect to the ordering $\lhd$ given by $a_1a_2a_3a_4a_5 \lhd a_2a_1a_3a_4a_5 \lhd a_2a_3a_1a_4a_5 \lhd a_2a_3a_4a_1a_5 \lhd a_2a_4a_3a_1a_5 \lhd a_4a_2a_3a_1a_5 \lhd a_4a_2a_3a_5a_1 \lhd a_4a_3a_2a_5a_1 \lhd a_4a_3a_5a_2a_1 \lhd a_4a_5a_3a_2a_1 \lhd a_5a_4a_3a_2a_1$ since every pair of alternatives change their relative ordering at most once along this ordering. Note that the cardinality of A is 5 and that of $\bar{\mathcal{D}}$ is $5(5-1)/2+1=11$. The domain $\mathcal{D} = \{a_1a_2a_3a_4a_5,\ a_2a_1a_3a_4a_5,\ a_2a_3a_1a_4a_5,\ a_2a_3a_4a_1a_5,\ a_2a_4a_3a_1a_5,\ a_4a_2a_3a_1a_5\}$ is a successive single-crossing domain since it contains all the preferences in-between $a_1a_2a_3a_4a_5$ and $a_4a_2a_3a_1a_5$ in the maximal single-crossing domain $\bar{\mathcal{D}}$.* □

In the following lemmas, we show that every single-crossing domain is a generalized intermediate domain, and every successive single-crossing domain is a minimally rich general intermediate domain.

Lemma 5.5.1 *Every single-crossing domain is a generalized intermediate domain.*

Proof: Let $\mathcal{D}$ be a single-crossing domain with an ordering $\lhd$ over the preferences. We show that $\mathcal{D}$ is a generalized intermediate domain. Suppose not and assume without loss of generality that there exist $a \in A$, $b_r, b_s \in \tau(\mathcal{D})$ and $P^{b_r} \in \mathcal{D}$ such that $b_r \prec b_s \prec a$ and $aP^{b_r}b_s$. Consider $P^{b_s} \in \mathcal{D}$. Since $b_rP^{b_r}b_s$, $b_sP^{b_s}b_r$, and $b_r \prec b_s$, it follows from the definition of a single-crossing domain that $P^{b_r} \lhd P^{b_s}$. By means of our assumption that $b_s \prec a$ and $aP^{b_r}b_s$, $P^{b_r} \lhd P^{b_s}$ implies $aP^{b_s}b_s$. However, this is a contradiction since $\tau(P^{b_s}) = b_s$. This completes the proof. ■

Lemma 5.5.2 *Every successive single-crossing domain is a minimally rich single-crossing domain.*

[8] For details see [93].

[9] By $P \unlhd P'$, we mean either $P = P'$ or $P \lhd P'$.

Proof: It is enough to show that every successive single-crossing domain is minimally rich. Let $\mathcal{D}$ be a successive single-crossing domain. Then, by the definition of a successive single-crossing domain, there is a maximal single-crossing domain $\bar{\mathcal{D}}$ with respect to some ordering $\lhd$ such that $\mathcal{D} = \{P \in \bar{\mathcal{D}} \mid \tilde{P} \unlhd P \unlhd \tilde{\tilde{P}}\}$ for some $\tilde{P}, \tilde{\tilde{P}} \in \bar{\mathcal{D}}$ with $\tilde{P} \unlhd \tilde{\tilde{P}}$. Suppose $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$. We show that for all $j = 1, 2, \ldots, k-1$, there are $P \in \mathcal{D}^{b_j}$ and $P' \in \mathcal{D}^{b_{j+1}}$ such that $P \sim P'$. Consider $b_j, b_{j+1} \in \tau(\mathcal{D})$ and consider $\bar{P} \in \mathcal{D}^{b_j}$ and $\hat{P} \in \mathcal{D}^{b_{j+1}}$. Since $b_j\bar{P}b_{j+1}$, $b_{j+1}\hat{P}b_j$, and $b_j \prec b_{j+1}$, it follows from the definition of a single-crossing domain that $\bar{P} \lhd \hat{P}$. Using a similar argument, we obtain $P^{b_l} \lhd \bar{P}$ for all $l < j$, and $P^{b_l} > \hat{P}$ for all $l > j+1$. Therefore, there must be $P \in \mathcal{D}^{b_j}$ and $P' \in \mathcal{D}^{b_{j+1}}$ that are consecutive in the ordering $\lhd$, that is, $P \in \mathcal{D}^{b_j}$ and $P' \in \mathcal{D}^{b_{j+1}}$ are such that there is no $P'' \in \mathcal{D}$ with $P \lhd P'' \lhd P'$. We show $P \sim P'$. Suppose not. Let a be the alternative which is ranked just above b_{j+1} in P, that is, aPb_{j+1} and there is no $x \in A$ with $aPxPb_{j+1}$. Consider the preference P'' that is obtained by switching the alternatives a and b_{j+1} in P. We show $P'' \notin \bar{\mathcal{D}}$. In particular, we show that both $P'' \lhd P$ and $P' \lhd P''$ are impossible. This is sufficient since P and P' are consecutive in the ordering $\lhd$. Suppose $P'' \lhd P$. Since aPb_{j+1}, $P \lhd P'$, and $b_{j+1}P'a$, by the single-crossing property of $\bar{\mathcal{D}}$, it must be that $a \prec b_{j+1}$. However, because $b_{j+1}P''a$ and aPb_{j+1}, this contradicts $P'' \lhd P$. Now, suppose $P' \lhd P''$. Since $P \lhd P'$, there must be a pair of alternatives c, d with $c \prec d$ such that cPd and $dP'c$. Moreover, because P and P' are not top-connected, it must be that $\{c, d\} \neq \{a, b_{j+l}\}$. Since $c \prec d$, $dP'c$, and $P' \lhd P''$, by the single-crossing property of $\bar{\mathcal{D}}$, we have $dP''c$. However, by the construction of P'', we have $cP''d$, which is a contradiction. Thus, we have $P'' \notin \bar{\mathcal{D}}$. This implies $\bar{\mathcal{D}} \cup P''$ is a single-crossing domain with respect to the ordering $\lhd'$ over $\bar{\mathcal{D}} \cup P''$, where $\lhd'$ is obtained by placing P'' in-between P and P' in the ordering $\lhd$, i.e., $\lhd'$ coincides with $\lhd$ over $\bar{\mathcal{D}}$ and $P \lhd' P'' \lhd' P'$. This contradicts the fact that $\bar{\mathcal{D}}$ is a maximal single-crossing domain. Therefore, $P \sim P'$ and $\mathcal{D}$ is minimally rich. This completes the proof of the lemma. ■

5.5.3 Single-dipped domains

In this subsection, we introduce the concept of single-dipped domains and show that they are generalized intermediate. A preference P is *single-dipped* if it has a unique minimal element $d(P)$, the *dip* of P, such that for all $a, b \in A$, $[d(P) \preceq a \prec b$ or $b \prec a \preceq d(P)] \Rightarrow bPa$. A domain is single-dipped if each preference in it is single-dipped.

It is straightforward that a minimally rich single-dipped domain is a minimally rich generalized intermediate domain. Note that the number of preferences in the maximal single-dipped domain is 2^{m-1}, while that in a minimally rich single-dipped domain can range from 2 to 2^{m-1}.

It is worth mentioning that any unanimous and strategy-proof RSCF on a minimally rich single-dipped domain can give positive probability to two particular (the boundary ones) alternatives.

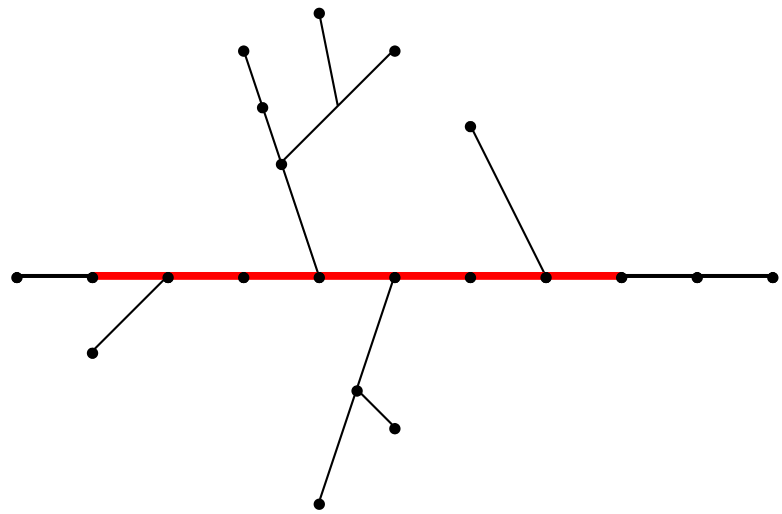


Figure 5.5.1: A graphic illustration of a tree

5.5.4 Single-peaked domains on trees with top-set along a path

A domain is tree-single-peaked if the alternatives are located on a tree and agents' preferences fall as one moves away from his/her top-ranked alternative along any path. [97] characterize the tops-only, unanimous and strategy-proof DSCFs on tree-single-peaked domains. Under the additional restriction that the top-set of the domain lie along a path, our result improves their one in two ways: first, by allowing for random rules, and second, by relaxing tops-onlyness.

We introduce a graph structure over the set of alternatives. A collection $G \subseteq \{\{a, b\} \mid a, b \in A,\ a \neq b\}$ is an *undirected graph over A*. The elements of G are *edges*. A *path* in G from a node a_1 to another a_k is a sequence of distinct nodes $\langle a_1, \ldots, a_k \rangle$ such that $\{a_i, a_{i+1}\} \in G$ for all $i = 1, \ldots, k-1$. Note that a path cannot have a cycle by definition.

A graph over A is a tree, denoted by T, if for all $a, b \in A$, there exists a unique path from a to b. Since such a path is unique in a tree, for ease of presentation we denote it by $[a, b]$. A preference P is *single-peaked on T* if for all distinct $x, y \in A$ with $y \neq \tau(P)$, $x \in [\tau(P), y] \implies xPy$. A domain is *single-peaked on T* if each preference in it is single-peaked on T.

Let T be a tree over A and let $\mathcal{D}$ be a single-peaked domain on T. Suppose $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$. We call $\mathcal{D}$ a single-peaked domain with top-set along a path if $\langle b_1, \ldots, b_k \rangle$ is a path in T. In Figure 5.5.1, we present a tree in which a path is marked with red. A single-peaked domain with respect to this tree with top-set along the red path can be constructed by taking those single-peaked preferences that have top-ranked alternatives in that path.

The following lemma says that a single-peaked domain on a tree with top-set along a path is a minimally rich generalized intermediate domain.

Lemma 5.5.3 *Let $\mathcal{D}$ be a single-peaked domain on a tree T with top-set along a path in T. Then, $\mathcal{D}$ is a minimally rich generalized intermediate domain.*

Proof: Let T be a tree and let $\pi = \langle b_1, \ldots, b_k \rangle$ be a path in it. Let $\mathcal{D}$ be a single-peaked domain on T with $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$. Consider a linear order $\prec$ on A such that

- $b_1 \prec \cdots \prec b_k$, and
- for all $a \in A \setminus \{b_1, \ldots, b_k\}$, $a \prec b_l$ if and only if the projection of a on π is b_j for some $j \leq l$.[10]

Note that the linear order $\prec$ defined above is not unique since it does not specify the relative ordering of two alternatives that are outside the path π but have the same projection. We show that $\mathcal{D}$ is a minimally rich generalized intermediate domain with respect to $\prec$. Since $\mathcal{D}$ is single-peaked on T and $\{b_l, b_{l+1}\}$ is an edge in T for all $l \in \{1, \ldots, k-1\}$, we can always find two preferences P and P' such that $\tau(P) = r_2(P') = b_l$, $r_2(P) = \tau(P') = b_{l+1}$, and $r_l(P) = r_l(P')$ for all $l \geq 3$. Therefore, $\mathcal{D}$ is minimally rich.

Now, we show that $\mathcal{D}$ is generalized intermediate. Consider b_r and b_s with $b_r \prec b_s$. To show $\mathcal{D}$ is generalized intermediate, it is enough to show that for all P with $\tau(P) = b_r$, we have b_sPa for all a with $b_s \prec a$. Assume for contradiction that there exist $P \in \mathcal{D}$ and $a \in A$ with $\tau(P) = b_r$ and $b_s \prec a$ such that aPb_s. If $a \in \{b_{s+1}, \ldots, b_k\}$, then by means of the fact that T is a tree, we have $b_s \in [b_r, a]$. However, by single-peakedness of P, this implies b_sPa, which is a contradiction to aPb_s. Now, suppose $a \in A \setminus \{b_{s+1}, \ldots, b_k\}$. Since $b_s \prec a$, by the definition of $\prec$, there exists $b_l \in \{b_{s+1}, \ldots, b_k\}$ such that the projection of a on π is b_l. By the definition of projection, this implies $b_l \in [b_r, b_s]$, and hence by single-peakedness of P, we have b_lPa. Using a similar argument, it follows that b_sPb_l, which in turn implies b_sPa. However, this is a contradiction to aPb_s. Thus, for all P with $\tau(P) = b_r$, we have b_sPa for all a with $b_s \prec a$. This proves $\mathcal{D}$ is a generalized intermediate domain. ■

5.5.5 Multi-peaked domains

In many practical scenarios in Economics and Political Science, preferences of individuals often exhibit *multi-peakedness* as opposed to single-peakedness. As the name suggests, multi-peaked preferences admit multiple (local) ideal points in a unidimensional policy space. We discuss a few settings where it is plausible to assume that individuals have multi-peaked preferences.

[10] By the projection of an alternative $a \in A$ on a path π in a tree T, we mean the alternative $b \in \pi$ that is closest (with respect to graph distance) to a, i.e., $b \in \pi$ is such that $|\pi(a, b)| \leq |\pi(a, c)|$ for all $c \in \pi$. Here, by $\pi(a, c)$, we mean the unique path in T from a to c.

- *Preference for "Do Something" in Politics*: [39] and [45] consider policy (decision) problems such as choosing alternate tax regimes, lowering health care costs, responding to foreign competition, reducing national debt, etc. They show that such a problem is perceived to be poorly addressed by the status-quo policy, and consequently some individuals prefer both liberal and conservative policies to the moderate status quo one. Clearly, such a preference will have two peaks, one on the left of the status quo and another one on the right of it.

- *Multi-stage Voting System*: [99], [42], [47] deal with multi-stage voting system where individuals vote on a set of issues where each issue can be thought of as a unidimensional spectrum and voting is distributed over several stages considering one issue at a time. In such a model, preference of an individual over the present issue can be affected by his/her prediction of the outcome of future issues. In other words, such a preference is not separable across issues. They show that preferences of individuals in such scenarios exhibit multi-peaked property.

- *Provision of Public Goods with Outside Options*: [17], [101], and [18] consider the problem of setting the level of tax rates to provide public funding in the education sector, and [63] and [50] consider the same problem in the health insurance market. They show that preferences of individuals exhibit multi-peaked property due to the presence of outside options (i.e., the public good is also available in a competitive market as a private good).

- *Provision of Excludable Public Goods*: [53] and [4] consider public good provision models such as health insurance, educational subsidies, pensions, etc., where a government provides the public good to a particular section of individuals and show that individuals' preferences in such scenarios exhibit multi-peaked property.

We now present a formal definition of multi-peaked domains and show that they are special cases of generalized intermediate domains. To ease our presentation, for two alternatives a and b, we denote by (a, b) the set $[a, b] \setminus \{a, b\}$.

Let $b_1 \prec \cdots \prec b_k$ be such that $(b_l, b_{l+1}) \neq \emptyset$ for all $1 \leq l < k$. Then, a preference P is *multi-peaked* with peak-set $\{b_1, \ldots, b_k\}$ if (i) $P|_{[a_1,b_1]}$ and $P|_{[b_k,a_m]}$ are single-dipped with dips at a_1 and a_m, respectively, (ii) for all $1 \leq l < k$, $P|_{[b_l,b_{l+1}]}$ is single-dipped with a dip in (b_l, b_{l+1}), and (iii) $P|_{\{b_1,\ldots,b_k\}}$ is single-peaked. A domain $\mathcal{D}$ is multi-peaked if it contains all multi-peaked preferences with peak-set $\tau(\mathcal{D})$.

In words, for a multi-peaked preference there are several (local) peaks such that the preference behaves like a single-dipped one between every two consecutive peaks and like a single-peaked one over the peaks. In Figure 5.5.2, we present a pictorial description of a multi-peaked preference.

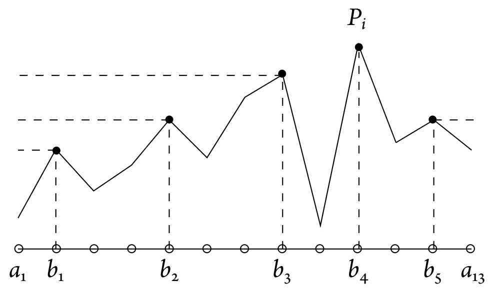


Figure 5.5.2: A graphic illustration of a multi-peaked preference

Lemma 5.5.4 *Every multi-peaked domain is a minimally rich generalized intermediate domain.*

Proof: Let $\mathcal{D}$ be a multi-peaked domain. Suppose $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$ with $b_1 \prec \ldots \prec b_k$. By the definition of $\mathcal{D}$, for all $b_l, b_{l+1} \in \tau(\mathcal{D})$, there are preferences $P, P' \in \mathcal{D}$ such that $\tau(P) = b_l$, $\tau(P') = b_{l+1}$ and $P \sim P'$. This shows $\mathcal{D}$ is minimally rich. Now, we prove $\mathcal{D}$ is a generalized intermediate domain. Consider b_r and b_s where $b_r \prec b_s$. We show that for all P with $\tau(P) = b_r$, we have b_sPa for all $a \in A$ with $b_s \prec a$. Consider $P \in \mathcal{D}$ with $\tau(P) = b_r$ and consider $a \in A$ with $b_s \prec a$. If $a \in [b_r, b_{r+1}]$, then by the definition of multi-peaked preferences, we have b_sPa. Suppose $a \in [b_l, b_{l+1}]$ for some b_l with $b_s \prec b_l$. By the definition of multi-peaked domains, we have b_sPb_l and b_lPa, which implies b_sPa. This proves that $\mathcal{D}$ is a generalized intermediate domain. ■

Remark 5.5.2 *Note that for both applications 5.5.4 and 5.5.5, the top-set of the domain is (exogenously) known to the designer. Domains with exogenously given characteristics are not new to the literature, for instance [3] consider domains where the top-ranked alternative of each agent is known to the designer and [85] consider domains where the indifference classes are known to the designer.*

5.5.6 Euclidean domains

[91] consider Euclidean domains and show that every unanimous and strategy-proof RSCF on such domains is a random minmax rule.

For ease of presentation, we assume that the set of alternatives are (finitely many) elements of the interval $[0, 1]$.[11] In particular, we assume $0 = a_1 < \cdots < a_m = 1$. Suppose that the individuals are located at arbitrary locations in $[0, 1]$ and they derive their preferences using Euclidean distances of the alternatives from their own locations. We call such preferences Euclidean. Below, we provide formal definitions of these preferences.

[11] With abuse of notation, we denote by $[0, 1]$ the set of all real numbers in-between 0 and 1.

Definition 5.5.3 *A preference P is Euclidean if there is $x \in [0, 1]$, called the location of P, such that for all alternatives $a, b \in A$, $|x - a| < |x - b|$ implies aPb. A domain is Euclidean if it contains* all *Euclidean preferences.*

Lemma 5.5.5 *Every Euclidean domain is a minimally rich generalized intermediate domain.*

Proof: Let $\mathcal{D}$ be a Euclidean domain. Then, by definition, it is regular single-peaked, and by Remark 5.2.1, it is generalized intermediate. It remains to show that $\mathcal{D}$ is minimally rich. Consider a_r and a_{r+1} for some $r \in \{1, \ldots, m-1\}$. By the definition of Euclidean domain, there are two preferences P and P' in $\mathcal{D}$ with location $\frac{a_r + a_{r+1}}{2}$ such that $\tau(P) = r_2(P') = a_r$, $r_2(P) = \tau(P') = a_{r+1}$, and $r_l(P) = r_l(P')$ for $l \geq 3$. This completes the proof of the lemma. ■

5.5.7 Intermediate domain

[58] introduces the concept of intermediate domains and shows that under some conditions on the distribution of voters over preferences, majority rule is transitive on these domains. However, to the best of our knowledge, no characterization of unanimous and strategy-proof RSCFs on these domains is available in the literature. Under a mild condition on these domains (mainly to avoid non-transitive preferences), we show that these domains are special cases of generalized intermediate domains, and consequently, we provide a characterization of unanimous and strategy-proof RSCFs on those.

Throughout this section, we denote by X an open convex subset of the Euclidean space E^2, and whenever we refer to a line, we mean a line in X (that is, a collection of points in X that constitute a line).

A preference P is *between* two preferences P_1 and P_2, denoted by $P \in (P_1, P_2)$, if for all $a, b \in A$, aP_1b and aP_2b imply aPb. A domain $\{P_x\}_{x \in X}$ satisfies the *intermediate property* if for every x' and $x'' \in X$, $x \in (x', x'')$ implies $P_x \in (P_{x'}, P_{x''})$.[12]

[58] provides a characterization of the intermediate domains where preferences are allowed to be weak (i.e., can have indifferences) and non-transitive. In the following lemma, we modify his result for the situation where preferences are strict and transitive (i.e., linear orders).

Lemma 5.5.6 *Let a domain $\{P_x\}_{x \in X}$ satisfy the intermediate property. Then, for every pair of alternatives (a, b), exactly one of the following statements must hold:*

(i) *aP_xb for all $x \in X$.*

(ii) *bP_xa for all $x \in X$.*

[12] With slight abuse of notation, by $x \in (x', x'')$, we mean $x = \lambda x' + (1 - \lambda)x''$ for some real number $\lambda \in (0, 1)$.

(iii) *There exist* $q = (q_1, q_2) \in E^2$*;* $(q_1, q_2) \neq (0, 0)$ *and* $\kappa \in \mathbb{R}$ *such that for all* $(x_1, x_2) \in X$*,* aP_xb *implies* $q_1x_1 + q_2x_2 \geq \kappa$ *and* bP_xa *implies* $q_1x_1 + q_2x_2 \leq \kappa$*.*

Proof: Suppose that both (i) and (ii) do not hold. We show that then (iii) must hold. Consider $a, b \in A$. Let $A_1 = \{x \in X \mid aP_xb\}$ and $A_2 = \{x \in X \mid bP_xa\}$. By our assumption that both (i) and (ii) do not hold, it follows that both A_1 and A_2 are non-empty. Moreover, by definition, A_1 and A_2 are disjoint, and by the intermediate property, both A_1 and A_2 are convex. Therefore, by Hyperplane separation theorem ([90], Theorem 11.3), there exist $q = (q_1, q_2) \in E^2$; $(q_1, q_2) \neq (0, 0)$ and $\kappa \in \mathbb{R}$ such that for all $(x_1, x_2) \in X$, aP_xb implies $q_1x_1 + q_2x_2 \geq \kappa$ and bP_xa implies $q_1x_1 + q_2x_2 \leq \kappa$. This completes the proof of the lemma. ■

Note that for a domain satisfying the intermediate property and for a pair of alternatives (a, b) that satisfies (iii) in Lemma 5.5.6, the object $((q_1, q_2), \kappa)$ identifies the line: $q_1x_1 + q_2x_2 = \kappa$. We denote such a line by $l(a, b)$. Lemma 5.5.6 implies that a is preferred to b on one side of this line, and b is preferred to a on the other side.[13] Since such a line separates the preferences with respect to a and b, we call it the separating line for a and b. In what follows, we introduce the concept of strict intermediate property.

Definition 5.5.4 *A domain* $\{P_x\}_{x \in X}$ *satisfies the strict intermediate property if*

(i) *there are no three distinct separating lines of the domain that pass through a common point, that is, for all three distinct (unordered) pairs* (x_1, y_1)*,* (x_2, y_2)*, and* (x_3, y_3)*, we have* $l(x_1, y_1) \cap l(x_2, y_2) \cap l(x_3, y_3) = \emptyset$*,*[14] *and*

(ii) *there exists a line* l *that intersects with all the separating lines of the domain, that is, for all pairs* $x, y \in A$ *satisfying (iii) in Lemma 5.5.6, we have* $l \cap l(x, y) \neq \emptyset$*.*

We provide an example of a domain that satisfies the strict intermediate property. It is worth noting from this example that (i) strictness is indeed a mild condition, and (ii) the strict intermediate property does *not* imply the single-crossing property.

[13] There is no restriction on the relative preference over a and b for the preferences P_x when x lies on this line.

[14] By distinct (unordered pairs), we mean that $\{x_i, y_i\} \neq \{x_j, y_j\}$ for all $i, j \in \{1, 2, 3\}$ with $i \neq j$.

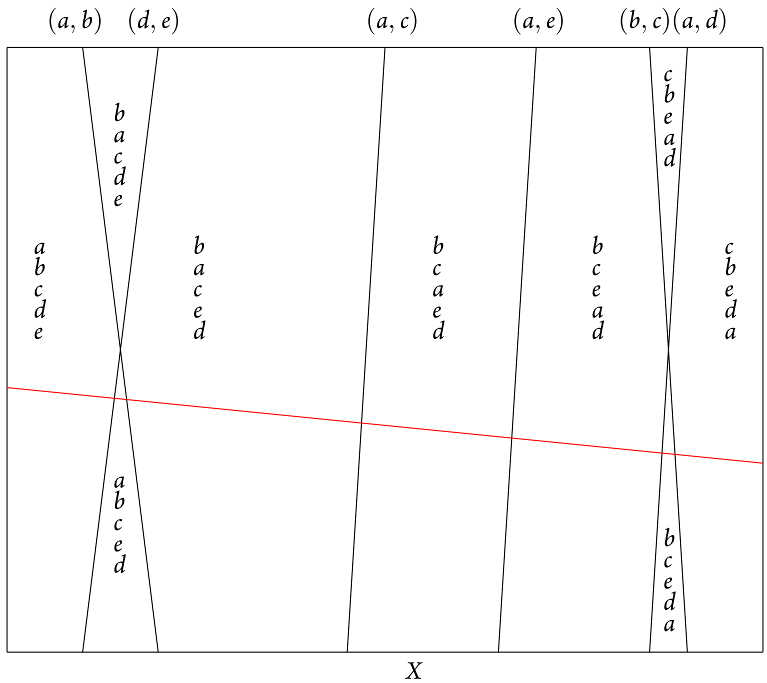


Figure 5.5.3: A graphic illustration of the separating lines for each pair of alternatives

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
a	a	b	b	b	b	b	c	c
b	b	a	a	c	c	c	b	b
c	c	c	c	a	e	e	e	e
d	e	d	e	e	a	d	a	d
e	d	e	d	d	d	a	d	a

Table 5.5.1

Example 5.5.5 *Let X be the open set in Figure 5.5.3 and let* $\{P_x\}_{x \in X} = \{abcde, abced, bacde, baced, bcaed, bcead, bceda, cbead, cbeda\}$ *be a domain satisfying intermediate property. For each pair of alternatives, the separating line is indicated in the figure. Note that for the pairs (b, d), (b, c), etc., there are no separating lines. Further note that P_x is constant over all points x that are enclosed by some separating lines of the domain (this follows from Lemma 5.5.6). Such P_xs are mentioned in the respective region in Figure 5.5.3.*

Clearly, the domain $\{P_x\}_{x \in X}$ satisfies strict intermediate property since no three separating lines pass through a common point and the line l (marked with red) intersects with all these lines. It is left to the reader to verify that the domain $\{P_x\}_{x \in X}$ is not a single-crossing domain. □

It is worth noting that the domain in Example 5.5.5 is a minimally rich generalized intermediate domain. Our next lemma shows that this fact is true in general.

Lemma 5.5.7 *Every domain* $\{P_x\}_{x \in X}$ *satisfying strict intermediate property is a generalized intermediate domain.*

The proof of this lemma is relegated to Appendix 5.9.

5.6 Conclusion

In this paper, we have shown that in case of finitely many alternatives, an RSCF on a minimally rich generalized intermediate domain is unanimous and strategy-proof if and only if it can be written as a convex combination of the tops-restricted min-max rules. We have further demonstrated by means of examples that one cannot go too far from the minimally rich generalized intermediate domains ensuring that the unanimous and strategy-proof RSCFs on it are convex combinations of the tops-restricted min-max rules. We have also provided a characterization of the unanimous and strategy-proof RSCFs in the setting with infinite number of alternatives. However, we do not assume any type of minimal richness in that case. In fact, minimal richness cannot be defined in this setting as there is no notion of "consecutive alternatives" here. As applications of our result, we have obtained a characterization of the unanimous and strategy-proof RSCFs on restricted domains such as single-peaked, single-crossing, single-dipped, single-peaked on a tree with top-set along a path, Euclidean, multi-peaked, and intermediate domain ([58]).

To our understanding, our results apply to all well-known restricted domains in one dimension. An interesting problem would be to see to what extent one can enlarge a generalized intermediate domain ensuring the existence of a non-random-dictatorial, unanimous, and strategy-proof (not necessarily tops-restricted random min-max) random rule. This will give some idea of the robustness of the generalized intermediate domains as possibility domains. Another interesting problem would be to explore the generalized intermediate domains for multiple dimensions. We leave all these problems for future research.

Appendix

5.7 Proof of Theorem 5.3.1

First, we prove a proposition that constitutes a major step in this proof.

Proposition 5.7.1 *Let $\mathcal{D}$ be a minimally rich generalized intermediate domain and let $\phi : \mathcal{D}^n \to \Delta A$ be a unanimous and strategy-proof RSCF. Then,*

(i) $\phi_{\tau(\mathcal{D})}(P_N) = 1$ *for all* $P_N \in \mathcal{D}^n$, *and*

(ii) ϕ *is uncompromising.*

We prove a sequence of lemmas which we will use in the proof of Proposition 5.7.1. The following lemma establishes that a generalized intermediate domain restricted to its top-set is single-peaked.

Lemma 5.7.1 *Let $\mathcal{D}$ be a generalized intermediate domain. Then, $\mathcal{D}|_{\tau(\mathcal{D})}$ is single-peaked.*

Proof: Let $\mathcal{D}$ be a generalized intermediate domain with $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$. We show that $\mathcal{D}|_{\tau(\mathcal{D})}$ is single-peaked. Without loss of generality, assume by contradiction that there exists $P \in \mathcal{D}$ such that $\tau(P) = b_j$ and $b_{l'} P b_l$ for some l, l' with $l' < l < j$. This means P violates the betweenness property with respect to b_l, which is a contradiction since $\mathcal{D}$ is a generalized intermediate domain and $b_l \in \tau(\mathcal{D})$. This completes the proof of the lemma. ■

In what follows, we prove a technical lemma that we use repeatedly in the proof of Proposition 5.7.1. We use the following notation in this lemma: for $X, Y \subseteq A$ and a preference P, XPY means xPy for all $x \in X$ and $y \in Y$.

Lemma 5.7.2 *Let $\mathcal{D}$ be a domain and let $\phi : \mathcal{D}^n \to \Delta A$ be a strategy-proof RSCF. Let $P_N \in \mathcal{D}^n$, $P'_i \in \mathcal{D}$, and $B, C \subseteq A$ be such that BP_iC, BP'_iC, and $P_i|_C = P'_i|_C$. Suppose $\phi_C(P_N) = \phi_C(P'_i, P_{-i})$ and $\phi_a(P_N) = \phi_a(P'_i, P_{-i})$ for all $a \notin B \cup C$. Then, $\phi_a(P_N) = \phi_a(P'_i, P_{-i})$ for all $a \in C$.*

Proof: First note that since $\phi_C(P_N) = \phi_C(P'_i, P_{-i})$ and $\phi_a(P_N) = \phi_a(P'_i, P_{-i})$ for all $a \notin B \cup C$, $\phi_B(P_N) = \phi_B(P'_i, P_{-i})$. Suppose $b \in C$ is such that $\phi_b(P_N) \neq \phi_b(P'_i, P_{-i})$ and $\phi_a(P_N) = \phi_a(P'_i, P_{-i})$ for all $a \in C$ with aP_ib. In other words, b is the maximal element of C according to P_i that violates the assertion of the lemma. Without loss of generality, assume that $\phi_b(P_N) < \phi_b(P'_i, P_{-i})$. Since BP_iC, $\phi_B(P_N) = \phi_B(P'_i, P_{-i})$, and $\phi_a(P_N) = \phi_a(P'_i, P_{-i})$ for all $a \notin B$ with aP_ib, it follows that $\phi_{U(b,P_i)}(P_N) < \phi_{U(b,P_i)}(P'_i, P_{-i})$. This implies agent i manipulates at P_N via P'_i, which is a contradiction. This completes the proof of the lemma. ■

Proof of Proposition 5.7.1

Now, we are ready to complete the proof of Proposition 5.7.1.

Proof:

We prove this proposition by using induction on the number of agents. Let $\mathcal{D}$ be a generalized intermediate domain with $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$.

Let $|N| = 1$ and let $\phi : \mathcal{D} \to \Delta A$ be a unanimous and strategy-proof RSCF. Then, by unanimity, $\phi_{\tau(\mathcal{D})}(P_N) = 1$ for all $P_N \in \mathcal{D}$, and hence ϕ satisfies uncompromisingness.

Assume that the proposition holds for all sets with $k < n$ agents. We prove it for n agents. Let $|N| = n$ and let $\phi : \mathcal{D}^n \to \Delta A$ be a unanimous and strategy-proof RSCF. Suppose $N^* = N \setminus \{1\}$. Define the RSCF $g : \mathcal{D}^{n-1} \to \Delta A$ for the set of voters N^* as follows: for all $P_{N^*} = (P_2, P_3, \ldots, P_n) \in \mathcal{D}^{n-1}$,

$$g(P_2, P_3, \ldots, P_n) = \phi(P_2, P_2, P_3, P_4, \ldots, P_n).$$

Evidently, g is a well-defined RSCF satisfying unanimity and strategy-proofness (See Lemma 3 in [98] for a detailed argument). Hence, by the induction hypothesis, $g_{\tau(\mathcal{D})}(P_{N^*}) = 1$ for all $P_{N^*} \in \mathcal{D}^{n-1}$ and g satisfies uncompromisingness. In terms of ϕ, this implies $\phi_{\tau(\mathcal{D})}(P_N) = 1$ for all $P_N \in \mathcal{D}^n$ with $P_1 = P_2$.

We complete the proof of Proposition 5.7.1 by using the following lemmas. In the next lemma, we show that $\phi_{\tau(\mathcal{D})}(P_N) = 1$ and ϕ is tops-only over all profiles P_N where agents 1 and 2 have the same top alternative.

Lemma 5.7.3 *Let $P_N, P'_N \in \mathcal{D}^n$ be two tops-equivalent profiles such that $P_1, P_2 \in \mathcal{D}^{b_j}$ for some $b_j \in \tau(\mathcal{D})$. Then, $\phi_{\tau(\mathcal{D})}(P_N) = 1$ and $\phi(P_N) = \phi(P'_N)$.*

Proof: Note that since g is uncompromising, g satisfies tops-onlyness. Because g is tops-only and $P_1, P_2 \in \mathcal{D}^{b_j}$, we have $g(P_1, P_{-\{1,2\}}) = g(P_2, P_{-\{1,2\}})$, and hence $\phi(P_1, P_1, P_{-\{1,2\}}) = \phi(P_2, P_2, P_{-\{1,2\}})$. We show $\phi(P_1, P_2, P_{-\{1,2\}}) = \phi(P_1, P_1, P_{-\{1,2\}})$. Using strategy-proofness of ϕ for agent 2, we have $\phi_{U(x,P_1)}(P_1, P_1, P_{-\{1,2\}}) \geq \phi_{U(x,P_1)}(P_1, P_2, P_{-\{1,2\}})$ for all $x \in A$, and using that for agent 1, we have $\phi_{U(x,P_1)}(P_1, P_2, P_{-\{1,2\}}) \geq \phi_{U(x,P_1)}(P_2, P_2, P_{-\{1,2\}})$ for all $x \in A$. Since $\phi(P_1, P_1, P_{-\{1,2\}}) = \phi(P_2, P_2, P_{-\{1,2\}})$, it follows from Remark 5.2.3 that $\phi(P_1, P_1, P_{-\{1,2\}}) = \phi(P_1, P_2, P_{-\{1,2\}})$. Using a similar logic, we have $\phi(P'_1, P'_1, P'_{-\{1,2\}}) = \phi(P'_1, P'_2, P'_{-\{1,2\}})$. Because g is tops-only and P_N, P'_N are tops-equivalent, we have $g(P_1, P_{-\{1,2\}}) = g(P'_1, P'_{-\{1,2\}})$. This implies $\phi(P_1, P_1, P_{-\{1,2\}}) = \phi(P'_1, P'_1, P'_{-\{1,2\}})$, and hence $\phi(P_1, P_2, P_{-\{1,2\}}) = \phi(P'_1, P'_2, P'_{-\{1,2\}})$. Moreover, as $\phi_{\tau(\mathcal{D})}(P_1, P_1, P_{-\{1,2\}}) = 1$, it follows that $\phi_{\tau(\mathcal{D})}(P_1, P_2, P_{-\{1,2\}}) = 1$. This completes the proof of the lemma. ■

Lemma 5.7.4 *Let $1 \leq j \leq j + l \leq k$ and let $P_N, P'_N \in \mathcal{D}^n$ be such that $P_1, P_2 \in \mathcal{D}^{b_j}$ and $P'_1, P'_2 \in \mathcal{D}^{b_{j+l}}$, and $\tau(P_i) = \tau(P'_i)$ for all $i \neq 1, 2$. Then, $\phi_b(P_N) = \phi_b(P'_N)$ for all $b \notin [b_j, b_{j+l}]_{\tau(\mathcal{D})}$.*

Proof: By uncompromisingness of g and the fact that $g_{\tau(\mathcal{D})}(P_{N^*}) = 1$ for all $P_{N^*} \in \mathcal{D}^{n-1}$, we have $g_b(P_1, P_{-\{1,2\}}) = g_b(P'_1, P_{-\{1,2\}})$ for all $b \notin [b_j, b_{j+l}]_{\tau(\mathcal{D})}$. Moreover, since g is tops-only and $\tau(P_i) = \tau(P'_i)$

for all $i \in \{3, 4, \dots, n\}$, we have $g(P'_1, P_{-\{1,2\}}) = g(P'_1, P'_{-\{1,2\}})$. By the definition of g, $g(P_1, P_{-\{1,2\}}) = \phi(P_1, P_1, P_{-\{1,2\}})$ and $g(P'_1, P_{-\{1,2\}}) = \phi(P'_1, P'_1, P_{-\{1,2\}})$. As $\tau(P_1) = \tau(P_2)$ and $\tau(P'_1) = \tau(P'_2)$, Lemma 5.7.3 implies $\phi(P_1, P_2, P_{-\{1,2\}}) = \phi(P_1, P_1, P_{-\{1,2\}})$ and $\phi(P'_1, P'_2, P'_{-\{1,2\}}) = \phi(P'_1, P'_1, P'_{-\{1,2\}})$. Combining all these observations, we have $\phi_b(P_1, P_2, P_{-\{1,2\}}) = \phi_b(P'_1, P'_2, P'_{-\{1,2\}})$ for all $b \notin [b_j, b_{j+l}]_{\tau(\mathcal{D})}$. This completes the proof of the lemma. ■

Lemma 5.7.5 *Let $1 \leq j \leq j + l \leq k$ and let $P_N, P'_N \in \mathcal{D}^n$ be such that $P_1, P_2, P'_1 \in \mathcal{D}^{b_j}$ and $P'_2 \in \mathcal{D}^{b_{j+l}}$, and $\tau(P_i) = \tau(P'_i)$ for all $i \neq 1, 2$. Then, $\phi_c(P_N) = \phi_c(P'_N)$ for all $c \notin U(b_{j+l}, P'_1) \cap U(b_j, P'_2)$.*

Proof: By Lemma 5.7.3, $\phi(P_1, P_2, P_{-\{1,2\}}) = \phi(P'_1, P'_1, P'_{-\{1,2\}})$. Hence, it suffices to show that $\phi_c(P'_1, P'_1, P'_{-\{1,2\}}) = \phi_c(P'_1, P'_2, P'_{-\{1,2\}})$ for $c \notin U(b_{j+l}, P'_1) \cap U(b_j, P'_2)$. We prove this for $c \notin U(b_{j+l}, P'_1)$, the proof of the same when $c \notin U(b_j, P'_2)$ follows from symmetric argument.

Consider $c \notin U(b_{j+l}, P'_1)$. By strategy-proofness of ϕ,

$$\phi_{U(c,P'_1)}(P'_1, P'_1, P'_{-\{1,2\}}) \geq \phi_{U(c,P'_1)}(P'_1, P'_2, P'_{-\{1,2\}}) \geq \phi_{U(c,P'_1)}(P'_2, P'_2, P'_{-\{1,2\}}).$$

Moreover, by Lemma 5.7.4, $\phi_b(P'_1, P'_1, P'_{-\{1,2\}}) = \phi_b(P'_2, P'_2, P'_{-\{1,2\}})$ for all $b \notin [b_j, b_{j+l}]_{\tau(\mathcal{D})}$, and hence $\phi_B(P'_1, P'_1, P'_{-\{1,2\}}) = \phi_B(P'_2, P'_2, P'_{-\{1,2\}})$ for all $B \subseteq A$ such that $[b_j, b_{j+l}]_{\tau(\mathcal{D})} \subseteq B$. Since $c \notin U(b_{j+l}, P'_1)$ and $\tau(P'_1) = b_j$, by the definition of a generalized intermediate domain, we have $[b_j, b_{j+l}]_{\tau(\mathcal{D})} \subseteq U(c, P'_1)$, and hence $\phi_{U(c,P'_1)}(P'_1, P'_1, P'_{-\{1,2\}}) = \phi_{U(c,P'_1)}(P'_2, P'_2, P'_{-\{1,2\}})$. Thus, we have

$$\phi_{U(c,P'_1)}(P'_1, P'_1, P'_{-\{1,2\}}) = \phi_{U(c,P'_1)}(P'_1, P'_2, P'_{-\{1,2\}}). \tag{5.1}$$

Suppose that $d \in A$ is ranked just above c in P'_1. Then, $[b_j, b_{j+l}]_{\tau(\mathcal{D})} \subseteq U(d, P'_1)$, and hence

$$\phi_{U(d,P'_1)}(P'_1, P'_1, P'_{-\{1,2\}}) = \phi_{U(d,P'_1)}(P'_1, P'_2, P'_{-\{1,2\}}). \tag{5.2}$$

Subtracting (5.2) from (5.1), we have $\phi_c(P'_1, P'_1, P'_{-\{1,2\}}) = \phi_c(P'_1, P'_2, P'_{-\{1,2\}})$, which completes the proof of the lemma. ■

Recall that for two preferences P and P', we write $P \sim P'$ to mean $\tau(P) = r_2(P')$, $r_2(P) = \tau(P')$, and $r_l(P) = r_l(P')$ for all $l > 2$.

Lemma 5.7.6 *Let $P^{b_j,b_{j+1}}, P^{b_{j+1},b_j} \in \mathcal{D}$ be such that $P^{b_j,b_{j+1}} \sim P^{b_{j+1},b_j}$. Then, for all $i \in N$ and all $P_{-i} \in \mathcal{D}^{n-1}$,*

$$[\phi_{\tau(\mathcal{D})}(P^{b_j,b_{j+1}}, P_{-i}) = 1] \implies [\phi_{\tau(\mathcal{D})}(P^{b_{j+1},b_j}, P_{-i}) = 1].$$

Proof: As $P^{b_j,b_{j+1}} \sim P^{b_{j+1},b_j}$, by strategy-proofness, $\phi_a(P^{b_j,b_{j+1}}, P_{-i}) = \phi_a(P^{b_{j+1},b_j}, P_{-i})$ for all $a \notin \{b_j, b_{j+1}\}$. Thus $\phi_{\tau(\mathcal{D})}(P^{b_j,b_{j+1}}, P_{-i}) = 1$ implies $\phi_{\tau(\mathcal{D})}(P^{b_{j+1},b_j}, P_{-i}) = 1$. This completes the proof of the lemma. ■

To simplify notations for the following lemma, for $j < l$, we define the distance from b_l to b_j, denoted by $b_l - b_j$, as $l - j$.

Lemma 5.7.7 *The RSCF ϕ is tops-only and $\phi_{\tau(\mathcal{D})}(P_N) = 1$ for all $P_N \in \mathcal{D}^n$.*[15]

Proof: We prove this lemma by using induction on the distance between the top-ranked alternatives of agents 1 and 2.

Consider l such that $0 \leq l \leq k-1$. Suppose $\phi_{\tau(\mathcal{D})}(P_N) = 1$ and $\phi(P_N) = \phi(\tilde{P}_N)$ for all tops-equivalent profiles $P_N, \tilde{P}_N \in \mathcal{D}^n$ with $|\tau(P_2) - \tau(P_1)| \leq l$. We show $\phi_{\tau(\mathcal{D})}(P'_N) = 1$ and $\phi(P'_N) = \phi(\tilde{P}'_N)$ for all tops-equivalent profiles $P'_N, \tilde{P}'_N \in \mathcal{D}^n$ with $|\tau(P'_2) - \tau(P'_1)| = l+1$.

Let P_N and P'_N be such that $P_1, P'_1 \in \mathcal{D}^{b_j}$, $P_2 \in \mathcal{D}^{b_{j+l}}$, $P'_2 \in \mathcal{D}^{b_{j+l+1}}$, and $\tau(P_i) = \tau(P'_i)$ for all $i \neq 1, 2$. Further, let $\bar{P}_1 \equiv P^{b_j,b_{j+1}}$, $\hat{P}_1 \equiv P^{b_{j+1},b_j}$, $\hat{P}_2 \equiv P^{b_{j+l},b_{j+l+1}}$, and $\bar{P}_2 \equiv P^{b_{j+l+1},b_{j+l}}$ be such that $\bar{P}_u \sim \hat{P}_u$ for all $u = 1, 2$. Note that such preferences exist by the definition of a minimally rich generalized intermediate domain. By the induction hypothesis, $\phi(P_N) = \phi(P'_1, \hat{P}_2, P'_{-\{1,2\}})$. We prove the following claims.

Claim 1. $\phi_{\tau(\mathcal{D})}(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = 1$ and $\phi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi(\bar{P}_1, P'_2, P'_{-\{1,2\}})$.

By the induction hypothesis, $\phi_{\tau(\mathcal{D})}(P'_1, \hat{P}_2, P'_{-\{1,2\}}) = 1$ and $\phi(P_N) = \phi(\bar{P}_1, \hat{P}_2, P'_{-\{1,2\}}) = \phi(P'_1, \hat{P}_2, P'_{-\{1,2\}})$. Let $P''_1 \in \{P'_1, \bar{P}_1\}$. By Lemma 5.7.5,

$$\phi_c(P''_1, P''_1, P'_{-\{1,2\}}) = \phi_c(P''_1, \hat{P}_2, P'_{-\{1,2\}}) \quad \text{for all } c \notin U(b_{j+l}, P''_1) \cap U(b_j, \hat{P}_2), \tag{5.3}$$

and

$$\phi_c(P''_1, P''_1, P'_{-\{1,2\}}) = \phi_c(P''_1, \bar{P}_2, P'_{-\{1,2\}}) \quad \text{for all } c \notin U(b_{j+l+1}, P''_1) \cap U(b_j, \bar{P}_2). \tag{5.4}$$

As $\tau(\hat{P}_2) - \tau(P''_1) \leq l$, it follows from the induction hypothesis that $\phi_{\tau(\mathcal{D})}(P''_1, P''_1, P'_{-\{1,2\}}) = \phi_{\tau(\mathcal{D})}(P''_1, \hat{P}_2, P'_{-\{1,2\}}) = 1$. Since $U(b_{j+l}, P''_1) \cap U(b_j, \hat{P}_2) \cap \tau(\mathcal{D}) = [b_j, b_{j+l}]_{\tau(\mathcal{D})}$, (5.3) implies

$$\phi_b(P''_1, P''_1, P'_{-\{1,2\}}) = \phi_b(P''_1, \hat{P}_2, P'_{-\{1,2\}}) \quad \text{for all } b \notin [b_j, b_{j+l}]_{\tau(\mathcal{D})}. \tag{5.5}$$

Moreover, since $\hat{P}_2 \equiv P^{b_{j+l},b_{j+l+1}}$, $\bar{P}_2 \equiv P^{b_{j+l+1},b_{j+l}}$, and $\phi_{\tau(\mathcal{D})}(P''_1, \hat{P}_2, P'_{-\{1,2\}}) = 1$, by Lemma 5.7.6, $\phi_{\tau(\mathcal{D})}(P''_1, \bar{P}_2, P'_{-\{1,2\}}) = 1$. This, in particular, implies $\phi_{\tau(\mathcal{D})}(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = 1$. Because

[15][31] provide a sufficient condition for a domain to be tops-only for RSCFs. However, generalized intermediate domains do not satisfy their condition.

$U(b_{j+l+1}, P_1'') \cap U(b_j, \bar{P}_2) \cap \tau(\mathcal{D}) = [b_j, b_{j+l+1}]_{\tau(\mathcal{D})}$, (5.4) implies

$$\phi_b(P_1'', P_1'', P'_{-\{1,2\}}) = \phi_b(P_1'', \bar{P}_2, P'_{-\{1,2\}}) \quad \text{for all } b \notin [b_j, b_{j+l+1}]_{\tau(\mathcal{D})}. \tag{5.6}$$

Combining (5.5) and (5.6), $\phi_b(P_1'', \hat{P}_2, P'_{-\{1,2\}}) = \phi_b(P_1'', \bar{P}_2, P'_{-\{1,2\}})$ for all $b \notin [b_j, b_{j+l+1}]_{\tau(\mathcal{D})}$. Since $\hat{P}_2 \equiv P^{b_{j+l}, b_{j+l+1}}$ and $\bar{P}_2 \equiv P^{b_{j+l+1}, b_{j+l}}$, we have by strategy-proofness that $\phi_{\{b_{j+l}, b_{j+l+1}\}}(P_1'', \hat{P}_2, P'_{-\{1,2\}}) = \phi_{\{b_{j+l}, b_{j+l+1}\}}(P_1'', \bar{P}_2, P'_{-\{1,2\}})$. Let $B' = [b_j, b_{j+l+1}]_{\tau(\mathcal{D})} \setminus \{b_{j+l}, b_{j+l+1}\}$. Then, $\phi_{B'}(P_1'', \hat{P}_2, P'_{-\{1,2\}}) = \phi_{B'}(P_1'', \bar{P}_2, P'_{-\{1,2\}})$. Note that by Lemma 5.7.1, $\hat{P}_2|_{B'} = \bar{P}_2|_{B'}$. Therefore, by applying Lemma 5.7.2 with $B = \{b_{j+l}, b_{j+l+1}\}$ and $C = B'$, we have

$$\phi_b(P_1'', \hat{P}_2, P'_{-\{1,2\}}) = \phi_b(P_1'', \bar{P}_2, P'_{-\{1,2\}}) \quad \text{for all } b \neq b_{j+l}, b_{j+l+1}. \tag{5.7}$$

By the induction hypothesis, $\phi(\bar{P}_1, \hat{P}_2, P'_{-\{1,2\}}) = \phi(P_1', \hat{P}_2, P'_{-\{1,2\}})$. Again, by Lemma 5.7.1, $b_{j+l}\bar{P}_1 b_{j+l+1}$ and $b_{j+l}P_1' b_{j+l+1}$, which implies $\phi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi(P_1', \bar{P}_2, P'_{-\{1,2\}})$. Using a similar logic, $\phi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi(\bar{P}_1, P_2', P'_{-\{1,2\}})$. This completes the proof of Claim 1. □

Claim 2. $\phi_c(P_1', \bar{P}_2, P'_{-\{1,2\}}) = \phi_c(P_N')$ for all $c \notin U(b_{j+l+1}, P_1') \cap U(b_j, P_2')$.

By (5.6), $\phi_b(P_1', P_1', P'_{-\{1,2\}}) = \phi_b(P_1', \bar{P}_2, P'_{-\{1,2\}})$ for all $b \notin [b_j, b_{j+l+1}]_{\tau(\mathcal{D})}$. Since $[b_j, b_{j+l+1}]_{\tau(\mathcal{D})} \subseteq U(b_{j+l+1}, P_1') \cap U(b_j, P_2')$, we have $\phi_c(P_1', P_1', P'_{-\{1,2\}}) = \phi_c(P_1', \bar{P}_2, P'_{-\{1,2\}})$ for all $c \notin U(b_{j+l+1}, P_1') \cap U(b_j, P_2')$. Moreover, by Lemma 5.7.5, $\phi_c(P_1', P_1', P'_{-\{1,2\}}) = \phi_c(P_N')$ for all $c \notin U(b_{j+l+1}, P_1') \cap U(b_j, P_2')$. Hence, $\phi_c(P_1', \bar{P}_2, P'_{-\{1,2\}}) = \phi_c(P_N')$ for all $c \notin U(b_{j+l+1}, P_1') \cap U(b_j, P_2')$. This completes the proof of Claim 2. □

Claim 3. $\phi_b(P_1', \bar{P}_2, P'_{-\{1,2\}}) = \phi_b(P_N')$ for all $b \in [b_j, b_{j+l+1}]_{\tau(\mathcal{D})}$.

First, we show $\phi_{b_j}(P_1', \bar{P}_2, P'_{-\{1,2\}}) = \phi_{b_j}(P_N')$. By Claim 1, $\phi(P_1', \bar{P}_2, P'_{-\{1,2\}}) = \phi(\bar{P}_1, P_2', P'_{-\{1,2\}})$. Moreover, as $\tau(\bar{P}_1) = \tau(P_1') = b_j$, by strategy-proofness, $\phi_{b_j}(\bar{P}_1, P_2', P'_{-\{1,2\}}) = \phi_{b_j}(P_N')$. Combining, we have $\phi_{b_j}(P_1', \bar{P}_2, P'_{-\{1,2\}}) = \phi_{b_j}(P_N')$.

Now, we complete the proof of Claim 3 by induction. Consider $s < l + 1$. Suppose $\phi_{b_{j+r}}(P_1', \bar{P}_2, P'_{-\{1,2\}}) = \phi_{b_{j+r}}(P_N')$ for all $0 \leq r \leq s$. We show $\phi_{b_{j+s+1}}(P_1', \bar{P}_2, P'_{-\{1,2\}}) = \phi_{b_{j+s+1}}(P_N')$. We show this in two steps. In Step 1, we show that if an alternative outside $\tau(\mathcal{D})$ appears above b_{j+s+1} in the preference P_1', then it receives zero probability at $\phi(P_N')$. In Step 2, we use this fact to complete the proof of the claim.

Step 1. Consider $c \in A \setminus \tau(\mathcal{D})$ such that $cP_1'b_{j+s+1}$. We show $\phi_c(P_N') = 0$. Assume for contradiction that $\phi_c(P_N') > 0$. Since $cP_1'b_{j+s+1}$, by the definition of a generalized intermediate domain, we have $b_{j+s+1}P_2'c$. Let $t \in \{2, \ldots, k - j - l\}$ be such that $U(b_{j+s+1}, P_2') \cap \tau(\mathcal{D}) = [b_{j+s+1}, b_{j+l+1}]_{\tau(\mathcal{D})} \cup [b_{j+l+2}, b_{j+l+t}]_{\tau(\mathcal{D})}$.

By Claim 1, $\phi_{\tau(\mathcal{D})}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = 1$, and hence

$$\begin{aligned}\phi_{U(b_{j+s+1},P'_2)}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) &= \phi_{[b_{j+s+1},b_{j+l+1}]_{\tau(\mathcal{D})}}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) + \phi_{[b_{j+l+2},b_{j+l+t}]_{\tau(\mathcal{D})}}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) \\ &= 1 - \phi_{[b_1,b_{j+s}]_{\tau(\mathcal{D})}}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) - \phi_{[b_{j+l+t+1},b_k]_{\tau(\mathcal{D})}}(P'_1, \bar{P}_2, P'_{-\{1,2\}}). \quad (5.8)\end{aligned}$$

By Claim 2, $\phi_{b_i}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi_{b_i}(P'_N)$ for all $i \in [1, j-1] \cup [j+l+t+1, k]$, and by the assumption of Claim 3, $\phi_{b_i}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi_{b_i}(P'_N)$ for all $i \in [j, j+s]$. Combining all these observations, we have $\phi_{[b_1,b_{j+s}]_{\tau(\mathcal{D})}}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi_{[b_1,b_{j+s}]_{\tau(\mathcal{D})}}(P'_N)$ and $\phi_{[b_{j+l+t+1},b_k]_{\tau(\mathcal{D})}}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi_{[b_{j+l+t+1},b_k]_{\tau(\mathcal{D})}}(P'_N)$. Note that the sets $[b_1, b_{j+s}]_{\tau(\mathcal{D})}$, $U(b_{j+s+1}, P'_2)$, $[b_{j+l+t+1}, b_k]_{\tau(\mathcal{D})}$, and $\{c\}$ are pairwise disjoint. Therefore, $\phi_{[b_1,b_{j+s}]_{\tau(\mathcal{D})}}(P'_N) + \phi_{U(b_{j+s+1},P'_2)}(P'_N) + \phi_{[b_{j+l+t+1},b_k]_{\tau(\mathcal{D})}}(P'_N) + \phi_c(P'_N) \leq 1$, and hence

$$\begin{aligned}\phi_{U(b_{j+s+1},P'_2)}(P'_N) &\leq 1 - \phi_{[b_1,b_{j+s}]_{\tau(\mathcal{D})}}(P'_N) - \phi_{[b_{j+l+t+1},b_k]_{\tau(\mathcal{D})}}(P'_N) - \phi_c(P'_N) \\ &= 1 - \phi_{[b_1,b_{j+s}]_{\tau(\mathcal{D})}}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) - \phi_{[b_{j+l+t+1},b_k]_{\tau(\mathcal{D})}}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) - \phi_c(P'_N). \quad (5.9)\end{aligned}$$

As $\phi_c(P'_N) > 0$, (5.8) and (5.9) imply $\phi_{U(b_{j+s+1},P'_2)}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) > \phi_{U(b_{j+s+1},P'_2)}(P'_N)$, which implies agent 2 manipulates at P'_N via $\bar{P}_2$, a contradiction. This completes Step 1.

Step 2. In this step, we complete the proof of Claim 3. By Claim 1, it is sufficient to show that $\phi_{b_{j+s+1}}(\bar{P}_1, P'_2, P'_{-\{1,2\}}) = \phi_{b_{j+s+1}}(P'_N)$.

Suppose $\phi_{b_{j+s+1}}(\bar{P}_1, P'_2, P'_{-\{1,2\}}) > \phi_{b_{j+s+1}}(P'_N)$. Consider $d \in U(b_{j+s+1}, P'_1) \setminus \tau(\mathcal{D})$. By Step 1, $\phi_d(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi_d(P'_N)$, and by Claim 1, $\phi_d(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi_d(\bar{P}_1, P'_2, P'_{-\{1,2\}})$. Now, consider $d \in U(b_{j+s+1}, P'_1) \cap \tau(\mathcal{D})$ such that $d \neq b_{j+s+1}$. This implies $d = b_{j'}$ for some $j' \leq j+s$. By Claim 2 and the assumption of Claim 3, $\phi_d(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi_d(P'_N)$. By Claim 1, $\phi(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi(\bar{P}_1, P'_2, P'_{-\{1,2\}})$. Combining all these observations, we have $\phi_d(\bar{P}_1, P'_2, P'_{-\{1,2\}}) = \phi_d(P'_N)$ for all $d \in U(b_{j+s+1}, P'_1) \setminus b_{j+s+1}$. Therefore, $\phi_{b_{j+s+1}}(\bar{P}_1, P'_2, P'_{-\{1,2\}}) > \phi_{b_{j+s+1}}(P'_N)$ implies $\phi_{U(b_{j+s+1},P'_1)}(\bar{P}_1, P'_2, P'_{-\{1,2\}}) > \phi_{U(b_{j+s+1},P'_1)}(P'_N)$, which implies agent 1 manipulates at P'_N via $\bar{P}_1$.

Now, suppose $\phi_{b_{j+s+1}}(\bar{P}_1, P'_2, P'_{-\{1,2\}}) < \phi_{b_{j+s+1}}(P'_N)$. By Claim 1, $\phi_{\tau(\mathcal{D})}(\bar{P}_1, P'_2, P'_{-\{1,2\}}) = 1$. Let $u \leq j$ be such that $U(b_{j+s+1}, \bar{P}_1) \cap \tau(\mathcal{D}) = [b_u, b_{j+s+1}]_{\tau(\mathcal{D})}$. Then, by the assumption of Claim 3, $\phi_b(\bar{P}_1, P'_2, P'_{-\{1,2\}}) = \phi_b(P'_N)$ for all $b \in [b_j, b_{j+s}]_{\tau(\mathcal{D})}$, and by Claim 2, $\phi_b(\bar{P}_1, P'_2, P'_{-\{1,2\}}) = \phi_b(P'_N)$ for all $b \in [b_u, b_{j-1}]_{\tau(\mathcal{D})}$. Therefore, $\phi_{b_{j+s+1}}(\bar{P}_1, P'_2, P'_{-\{1,2\}}) < \phi_{b_{j+s+1}}(P'_N)$ implies $\phi_{U(b_{j+s+1},\bar{P}_1)}(\bar{P}_1, P'_2, P'_{-\{1,2\}}) < \phi_{U(b_{j+s+1},\bar{P}_1)}(P'_N)$, which implies agent 1 manipulates at $(\bar{P}_1, P'_2, P'_{-\{1,2\}})$ via P'_1. This completes the proof of Claim 3. □

We are now ready to complete the proof of Lemma 5.7.7. First, we show $\phi_{\tau(\mathcal{D})}(P'_N) = 1$. By Claim 3,

$\phi_b(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi_b(P'_N)$ for all $b \in [b_j, b_{j+l+1}]_{\tau(\mathcal{D})}$. By Claim 2, $\phi_b(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi_b(P'_N)$ for all $b \in [b_1, b_{j-1}]_{\tau(\mathcal{D})} \cup [b_{j+l+2}, b_k]_{\tau(\mathcal{D})}$. Combining all these observations, we have $\phi_{\tau(\mathcal{D})}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi_{\tau(\mathcal{D})}(P'_N)$. Moreover, by Claim 1, $\phi_{\tau(\mathcal{D})}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = 1$, and hence $\phi_{\tau(\mathcal{D})}(P'_N) = 1$.

Now, we show $\phi(P'_N) = \phi(\tilde{P}'_N)$ for all tops-equivalent profiles $P'_N, \tilde{P}'_N \in \mathcal{D}^n$. By claims 1, 2, and 3, we have $\phi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi(P'_N)$. Moreover, as $\tilde{P}'_1 \in \mathcal{D}^{b_j}$ and $\tilde{P}'_2 \in \mathcal{D}^{b_{j+l+1}}$, applying claims 1, 2, and 3 to $\tilde{P}'_N$, we have $\phi(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}}) = \phi(\tilde{P}'_N)$. Hence, to show $\phi(P'_N) = \phi(\tilde{P}'_N)$, it is enough to show $\phi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}})$. Recall that $\hat{P}_2 \equiv P^{b_{j+l}, b_{j+l+1}}$. Since $\tau(\hat{P}_2) - \tau(P'_1) = l$ and $\tau(P'_i) = \tau(\tilde{P}'_i)$ for all $i \neq 1, 2$, by the assumption of Lemma 5.7.7, we have $\phi(\bar{P}_1, \hat{P}_2, P'_{-\{1,2\}}) = \phi(\bar{P}_1, \hat{P}_2, \tilde{P}'_{-\{1,2\}})$. Also, by (5.7), $\phi_b(\bar{P}_1, \hat{P}_2, P'_{-\{1,2\}}) = \phi_b(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}})$ for all $b \neq b_{j+l}, b_{j+l+1}$, which implies $\phi_b(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi_b(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}})$ for all $b \neq b_{j+l}, b_{j+l+1}$. Using similar arguments as for the proof of (5.7), it follows that $\phi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi(\hat{P}_1, \bar{P}_2, P'_{-\{1,2\}})$ for all $b \neq b_j, b_{j+1}$, and hence $\phi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}})$ for all $b \neq b_j, b_{j+1}$. Note that if $l \geq 1$, then $\phi_b(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi_b(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}})$ for all $b \in A$. Now suppose $l = 0$. We show $\phi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}})$ for $\tau(\bar{P}_1) = b_j$ and $\tau(\bar{P}_2) = b_{j+1}$. Because $\phi_b(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi_b(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}})$ for all $b \neq b_j, b_{j+1}$ and all tops-equivalent $P'_{-\{1,2\}}, \tilde{P}'_{-\{1,2\}} \in \mathcal{D}^{n-2}$, we have $\phi_b(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi_b(\bar{P}_1, \bar{P}_2, \tilde{P}'_3, P'_{-\{1,2,3\}})$ for all $b \neq b_j, b_{j+1}$. As $\tau(P'_3) = \tau(\tilde{P}'_3)$, by Lemma 5.7.1, $b_j P'_3 b_{j+1}$ if and only if $b_j \tilde{P}'_3 b_{j+1}$. Therefore, if $\phi_{b_j}(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) \neq \phi_{b_j}(\bar{P}_1, \bar{P}_2, \tilde{P}'_3, P'_{-\{1,2,3\}})$, then agent 3 manipulates either at $(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}})$ via $\tilde{P}'_3$ or at $(\bar{P}_1, \bar{P}_2, \tilde{P}'_3, P'_{-\{1,2,3\}})$ via P'_3. Hence, $\phi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi(\bar{P}_1, \bar{P}_2, \tilde{P}'_3, P'_{-\{1,2,3\}})$. Continuing in this manner, we have $\phi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \phi(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}})$. Therefore, $\phi(P'_N) = \phi(\tilde{P}'_N)$ for all tops-equivalent profiles $P'_N, \tilde{P}'_N \in \mathcal{D}^n$. This completes the proof of the lemma. ■

Lemma 5.7.8 *The RSCF ϕ satisfies uncompromisingness.*

Proof: We prove this in two steps. In Step 1, we provide a sufficient condition for uncompromisingness, and in Step 2, we use that to prove the lemma.

STEP 1. In this step, we show that ϕ is uncompromising if the following happens: for all $j < k$, all $P_i \equiv P^{b_j, b_{j+1}} \in \mathcal{D}$, all $P'_i \equiv P^{b_{j+1}, b_j} \in \mathcal{D}$, and all $P_{-i} \in \mathcal{D}^{n-1}$,

$$\phi_b(P_i, P_{-i}) = \phi_b(P'_i, P_{-i}) \ \ \forall b \notin [\tau(P_i), \tau(P'_i)]. \tag{5.10}$$

Suppose (5.10) holds. Since ϕ is tops-only, (5.10) implies that for all $P_i \in \mathcal{D}^{b_j}$, all $P'_i \in \mathcal{D}^{b_{j+1}}$, all P_{-i}, and all $b \notin [\tau(P_i), \tau(P'_i)]$,

$$\phi_b(P_i, P_{-i}) = \phi_b(P'_i, P_{-i}). \tag{5.11}$$

Similarly, for all $\bar{P}_i \in \mathcal{D}^{b_{j+1}}$, all $\bar{P}'_i \in \mathcal{D}^{b_{j+2}}$, all P_{-i}, and all $b \notin [\tau(\bar{P}_i), \tau(\bar{P}'_i)]$, we have

$$\phi_b(\bar{P}_i, P_{-i}) = \phi_b(\bar{P}'_i, P_{-i}). \tag{5.12}$$

Combining (5.11) and (5.12), we have $\phi_b(P_i, P_{-i}) = \phi_b(\bar{P}'_i, P_{-i})$ for all $P_i \in \mathcal{D}^{b_j}$, all $\bar{P}'_i \in \mathcal{D}^{b_{j+2}}$, all P_{-i}, and all $b \notin [\tau(P_i), \tau(\bar{P}'_i)]$. Continuing in this manner, we have $\phi_b(P_i, P_{-i}) = \phi_b(P'_i, P_{-i})$ for all $P_i, P'_i \in \mathcal{D}$, all P_{-i}, and all $b \notin [\tau(P_i), \tau(P'_i)]$, which implies ϕ is uncompromising.

STEP 2. In this step, we show that ϕ satisfies (5.10). We do this in two further steps. In Step 2.a., we show (5.10) for agents 1 and 2, and in Step 2.b., we show this for other agents.

STEP 2.a. It is enough to show (5.10) for agent 1, the proof of the same for agent 2 follows from symmetric argument. Without loss of generality, assume $\tau(P_2) = b_{j+l}$. Note that by Lemma 5.7.7, $\phi_{\tau(\mathcal{D})}(P_N) = 1$. Therefore, by Lemma 5.7.5, $\phi_b(P_1, P_2, P_{-\{1,2\}}) = \phi_b(P_2, P_2, P_{-\{1,2\}})$ for all $b \notin [b_j, b_{j+l}]_{\tau(\mathcal{D})}$ and $\phi_b(P'_1, P_2, P_{-\{1,2\}}) = \phi_b(P_2, P_2, P_{-\{1,2\}})$ for all $b \notin [b_{j+1}, b_{j+l}]_{\tau(\mathcal{D})}$. This implies $\phi_b(P_1, P_2, P_{-\{1,2\}}) = \phi_b(P'_1, P_2, P_{-\{1,2\}})$ for all $b \notin [b_j, b_{j+l}]_{\tau(\mathcal{D})}$. By strategy-proofness, $\phi_{\{b_j, b_{j+1}\}}(P_1, P_2, P_{-\{1,2\}}) = \phi_{\{b_j, b_{j+1}\}}(P'_1, P_2, P_{-\{1,2\}})$. Let $B' = [b_j, b_{j+l}]_{\tau(\mathcal{D})} \setminus \{b_j, b_{j+1}\}$. Since $P_1|_{B'} = P'_1|_{B'}$, by applying Lemma 5.7.2 with $B = \{b_j, b_{j+1}\}$ and $C = B'$, we have $\phi_b(P_1, P_2, P_{-\{1,2\}}) = \phi_b(P'_1, P_2, P_{-\{1,2\}})$ for all $b \neq b_j, b_{j+l}$. This proves (5.10) for agent 1. Therefore, by Step 1, we have for all $i \in \{1, 2\}$, all $P_i \in \mathcal{D}$, all $P'_i \in \mathcal{D}$, and all $P_{-i} \in \mathcal{D}^{n-1}$,

$$\phi_b(P_i, P_{-i}) = \phi_b(P'_i, P_{-i}) \;\; \forall b \notin [\tau(P_i), \tau(P'_i)]. \tag{5.13}$$

This completes Step 2.a.

STEP 2.b. In this step, we show (5.10) for agents $i \in \{3, \ldots, n\}$. It is enough to show this for $i = 3$. If $P_1 = P_2$, then by the induction hypothesis, $\phi_b(P_3, P_{-3}) = g_b(P_1, P_3, P_{-\{1,2,3\}}) = g_b(P_1, P'_3, P_{-\{1,2,3\}}) = \phi_b(P'_3, P_{-3})$ for all $P_3, P'_3 \in \mathcal{D}$ and all $b \notin [\tau(P_3), \tau(P'_3)]$. Let $\tau(P_1) = b_p$ and $\tau(P_2) = b_q$. Since $\phi_{\tau(\mathcal{D})}(P_N) = 1$ for all $P_N \in \mathcal{D}^n$, it follows from Lemma 5.7.5 that $\phi_b(P_1, P_1, P_3, P_{-\{1,2,3\}}) = \phi_b(P_1, P_2, P_3, P_{-\{1,2,3\}})$ for all $b \notin [b_p, b_q]_{\tau(\mathcal{D})}$ and $\phi_b(P_1, P_1, P'_3, P_{-\{1,2,3\}}) = \phi_b(P_1, P_2, P'_3, P_{-\{1,2,3\}})$ for all $b \notin [b_p, b_q]_{\tau(\mathcal{D})}$. Combining all these observations, we have

$$\phi_b(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \phi_b(P_1, P_2, P'_3, P_{-\{1,2,3\}}) \text{ for all } b \notin [b_p, b_q]_{\tau(\mathcal{D})} \cup [b_j, b_{j+1}]_{\tau(\mathcal{D})}. \tag{5.14}$$

Also, by strategy-proofness,

$$\phi_{\{b_j, b_{j+1}\}}(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \phi_{\{b_j, b_{j+1}\}}(P_1, P_2, P'_3, P_{-\{1,2,3\}}). \tag{5.15}$$

Now, we distinguish two cases.

Case 1. Suppose $p, q \leq j+1$ or $p, q \geq j$.

Let $B' = [b_p, b_q]_{\tau(\mathcal{D})} \setminus [b_j, b_{j+1}]_{\tau(\mathcal{D})}$. Then, by (5.14) and (5.15), $\phi_{B'}(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \phi_{B'}(P_1, P_2, P'_3, P_{-\{1,2,3\}})$. Since $P_3|_{B'} = P'_3|_{B'}$, by applying Lemma 5.7.2 with $B = \{b_j, b_{j+1}\}$ and $C = B'$, $\phi_b(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \phi_b(P_1, P_2, P'_3, P_{-\{1,2,3\}})$ for all $b \in B'$. Therefore,

$$\phi_b(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \phi_b(P_1, P_2, P'_3, P_{-\{1,2,3\}}) \text{ for all } b \notin \{b_j, b_{j+1}\}. \tag{5.16}$$

This completes Step 2.b. for Case 1.

Case 2. Suppose $p < j \leq j+1 < q$ or $q < j \leq j+1 < p$.

We prove the lemma for the case $p < j \leq j+1 < q$, the proof of the same for the case $q < j \leq j+1 < p$ follows from symmetric arguments. By (5.13), for all $b \notin [b_j, b_q]_{\tau(\mathcal{D})}$, we have $\phi_b(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \phi_b(P_1, P_3, P_3, P_{-\{1,2,3\}})$ and $\phi_b(P_1, P_2, P'_3, P_{-\{1,2,3\}}) = \phi_b(P_1, P_3, P'_3, P_{-\{1,2,3\}})$. Moreover, since $\tau(P_1) \leq b_{j+1}$, $\tau(P_3) = b_j$ and $\tau(P'_3) = b_{j+1}$, it follows from (5.16) that $\phi_b(P_1, P_3, P_3, P_{-\{1,2,3\}}) = \phi_b(P_1, P_3, P'_3, P_{-\{1,2,3\}})$ for all $b \notin [b_j, b_{j+1}]_{\tau(\mathcal{D})}$. Combining all these observations, $\phi_b(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \phi_b(P_1, P_2, P'_3, P_{-\{1,2,3\}})$ for all $b \notin [b_j, b_q]_{\tau(\mathcal{D})}$. By strategy-proofness, $\phi_{\{b_j, b_{j+1}\}}(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \phi_{\{b_j, b_{j+1}\}}(P_1, P_2, P'_3, P_{-\{1,2,3\}})$. Let $B' = [b_j, b_q]_{\tau(\mathcal{D})} \setminus \{b_j, b_{j+1}\}$. Since $P_3|_{B'} = P'_3|_{B'}$, by applying Lemma 5.7.2 with $B = \{b_j, b_{j+1}\}$ and $C = B'$, we have $\phi_b(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \phi_b(P_1, P_2, P'_3, P_{-\{1,2,3\}})$ for all $b \in B'$. Hence,

$$\phi_b(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \phi_b(P_1, P_2, P'_3, P_{-\{1,2,3\}}) \text{ for all } b \notin \{b_j, b_{j+1}\},$$

which completes Step 2.b. for Case 2.

Since cases 1 and 2 are exhaustive, this completes Step 2, and consequently the proof of Lemma 5.7.8.
■ Proposition 5.7.1 now follows from Lemma 5.7.7 and Lemma 5.7.8. ■

Now, we come back to the proof of Theorem 5.3.1. Our proof uses the following theorem which is taken from [81].

Theorem 5.7.1 (Theorem 3(a) in [81]) *Let $\mathcal{D}$ be the maximal single-peaked domain. Then, every tops-only and strategy-proof RSCF $\phi : \mathcal{D}^n \to \Delta A$ is a convex combination of some tops-only and strategy-proof DSCFs $f : \mathcal{D}^n \to A$.*

Our next lemma presents the structure of an uncompromising and strategy-proof RSCF on a regular single-peaked domain.

Lemma 5.7.9 *Let $\mathcal{D}$ be a regular single-peaked domain and let $\phi : \mathcal{D}^n \to \Delta A$ be uncompromising and strategy-proof. Then, ϕ is a convex combination of the generalized min-max rules on $\mathcal{D}^n$.*[16]

Proof: Note that since ϕ is uncompromising, ϕ is tops-only. Let $\hat{\mathcal{D}}$ be the maximal single-peaked domain. Let $\hat{\phi} : \hat{\mathcal{D}}^n \to \Delta A$ be the tops-only extension of ϕ on $\hat{\mathcal{D}}$. More formally, for all $\hat{P}_N \in \hat{\mathcal{D}}^n$, $\hat{\phi}(\hat{P}_N) = \phi(P_N)$, where $P_N \in \mathcal{D}^n$ is such that P_N and $\hat{P}_N$ are tops-equivalent. This is well-defined as ϕ is tops-only and $\mathcal{D}$ is regular. Since $\hat{\mathcal{D}}$ is single-peaked and ϕ is strategy-proof, $\hat{\phi}$ is also strategy-proof. Hence, by Theorem 5.7.1, $\hat{\phi}$ is a convex combination of the generalized min-max rules on $\hat{\mathcal{D}}^n$. By the definition of $\hat{\phi}$, this implies ϕ is a convex combination of the generalized min-max rules on $\mathcal{D}^n$, which completes the proof. ■

Finally, we are ready to complete the proof of Theorem 5.3.1. *Proof:* (If Part) Let $\mathcal{D}$ be a generalized intermediate domain with $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$ and let $\phi : \mathcal{D}^n \to \Delta A$ be a TRM rule. Since ϕ is a TRM rule, it is unanimous by definition. We show that ϕ is strategy-proof. Let $\phi = \sum_{l=1}^{t} \lambda_l f_l$, where λ_ls are non-negative numbers summing to 1 and f_ls are TM rules. To show ϕ is strategy-proof, it is enough to show that f_ls are strategy-proof. For all $l \in \{1, \ldots, t\}$, define $\hat{f}_l : (\mathcal{D}|_{\tau(\mathcal{D})})^n \to \tau(\mathcal{D})$ as $\hat{f}_l(P_N|_{\tau(\mathcal{D})}) = f_l(P_N)$. Note that by Lemma 5.7.1, $\mathcal{D}|_{\tau(\mathcal{D})}$ is a single-peaked domain. Therefore, it follows from [72] that $\hat{f}_l$ is strategy-proof for all $l = 1, \ldots, t$. By Remark 5.2.5, this implies f_l is strategy-proof for all $l = 1, \ldots, t$. This completes the proof of the if part.

(Only-if Part) Let $\mathcal{D}$ be a generalized intermediate domain with $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$ and let $\phi : \mathcal{D}^n \to \Delta A$ be a unanimous and strategy-proof RSCF. Define $\hat{\phi} : (\mathcal{D}|_{\tau(\mathcal{D})})^n \to \Delta\tau(\mathcal{D})$ as $\hat{\phi}_b(P_N|_{\tau(\mathcal{D})}) = \phi_b(P_N)$ for all $b \in \tau(\mathcal{D})$. This is well-defined as by Proposition 5.7.1, $\phi_{\tau(\mathcal{D})}(P_N) = 1$ for all $P_N \in \mathcal{D}^n$ and ϕ is tops-only. Because ϕ satisfies uncompromisingness, $\hat{\phi}$ also satisfies uncompromisingness. Hence, by Lemma 5.7.9, $\hat{\phi}$ is convex combination of generalized min-max rules on $(\mathcal{D}|_{\tau(\mathcal{D})})^n$. Moreover, since ϕ is unanimous, $\hat{\phi}$ is a also unanimous. This implies $\hat{\phi}$ is a convex combination of the min-max rules on $(\mathcal{D}|_{\tau(\mathcal{D})})^n$. By the definition of $\hat{\phi}$, this implies ϕ is a TRM rule. This completes the proof of the only-if part. ■

5.8 Proof of Theorem 5.4.1

Proof:

Let $\mathcal{D}$ be a generalized intermediate domain and let ϕ be a unanimous and strategy-proof RSCF. We introduce a piece of notation to facilitate the presentation of our next lemma. For $R_N \in \mathcal{D}^n$, by $I(R_N)$ we denote the interval $[\min_{i \in N} \tau(R_i), \max_{i \in N} \tau(R_i)]$, and by $p(R_N)$ we denote the number of different peaks at R_N, that is, $p(R_N) = |\{\tau(R_i) \mid i \in N\}|$. Further, for a preference R and an alternative $x \in A$, the lower

[16] If the set of alternatives is an interval of real numbers, then every uncompromising RSCF on the maximal single-peaked domain is strategy-proof (see Lemma 3.2 in [46]). However, the same does not hold for the case of finitely many alternatives.

contour set of x at R is defined as $L(x, R) = \{y \in A \mid xRy\}$. Our next proposition says that at every profile R_N, the interval $I(R_N)$ will receive the full probability (i.e., probability 1) at $\phi(R_N)$. It further says that the top-set of the domain $\mathcal{D}$ will always (i.e., at any profile) receive probability 1 by ϕ.

Proposition 5.8.1 *For all $R_N \in \mathcal{D}^n$, $\phi(R_N)(I(R_N)) = 1$ and $\phi(R_N)(\tau(\mathcal{D})) = 1$.*

Proof: Consider $R_N \in \mathcal{D}^n$. We prove the proposition on the basis of the number of different peaks $p(R_N)$ at R_N. The proposition follows trivially by unanimity when $p(R_N) = 1$. To prove the proposition for the cases where $p(R_N) > 1$, we use induction on $p(R_N)$. Here, we consider the case $p(R_N) = 2$ as the base case.

Base case for the proof of Proposition 5.8.1: Suppose $p(R_N) = 2$.
Let $\{\tau(R_i) \mid i \in N\} = \{a, b\}$, where $a < b$. We use induction on the number of agents having a as the top-ranked alternative.
Base case for the proof of the base case of Proposition 5.8.1: We first prove this for the case $\tau(R_1) = a$ and $\tau(R_2) = \cdots \tau(R_n) = b$.

Proof of $\phi(R_N)([a, b]) = 1$:

We claim $\phi(R_N)((b, \infty)) = 0$. Suppose to the contrary that $\phi(R_N)((b, \infty)) > 0$. Let $R' \in \mathcal{D}^b$. By unanimity, $\phi(R', R_{-1})(\{b\}) = 1$, and hence agent 1 manipulates at R_N by misreporting his/her preference as R', a contradiction. Since $\phi(R_N)((b, \infty)) = 0$, to show $\phi(R_N)([a, b]) = 1$, it is enough to show $\phi(R_N)((-\infty, a)) = 0$. Assume to the contrary $\phi(R_N)((-\infty, a)) > 0$. Let $R'_2 \in \mathcal{D}^a$ be a strict preference with the property that (i) there exist $x, y \in A$ such that $U(x, R'_2) = U(a, R_2) \cap [a, b]$ and $L(y, R'_2) = (b, \infty)$, and (ii) for all $w, z \notin U(x, R'_2) \cup L(y, R'_2)$, we have wR'_2z if and only if wR_2z. In other words, the strict preference R'_2 satisfies the following conditions: (i) the alternatives that lie in the interval $[a, b]$ and are preferred to a according to R_2 form an upper contour set at R'_2, and the alternatives in the interval (b, ∞) form a lower contour set, and (ii) all the remaining alternatives maintain the same relative ordering in R'_2 as in R_2. Since the interval (b, ∞) forms a lower contour set at R'_2, by strategy-proofness, $\phi(R'_2, R_{-2})((b, \infty)) = 0$. This, together with the construction of R'_2 and strategy-proofness, implies $\phi(R_N)(U(a, R_2)) = \phi(R'_2, R_{-2})(U(a, R_2))$. As $R_2|_{(-\infty,b)\cap(A\setminus U(a,R_2))} = R'_2|_{(-\infty,b)\cap(A\setminus U(a,R_2))}$, by straightforward application of strategy-proofness for all Borel set $D \subseteq (-\infty, b) \cap (A \setminus U(a, R_2))$, we have

$$\phi(R_N)(D) = \phi(R'_2, R_{-2})(D). \tag{5.17}$$

This, in particular, means $\phi(R'_2, R_{-2})((-\infty, a)) > 0$. We can repeatedly use this argument to move all the agents $i = 2, \ldots, n$ to a preference $R'_i \in \mathcal{D}^a$ and conclude $\phi(R_1, R'_{-1})((-\infty, a)) > 0$. However, by unanimity, $\phi(R_1, R'_{-1})(\{a\}) = 1$, a contradiction. This proves $\phi(R_N)([a, b]) = 1$.

Proof of $\phi(R_N)(\tau(\mathcal{D})) = 1$: Suppose that $1 \leq s < s' \leq k$ are such that $a \in I_s$ and $b \in I_{s'}$. Consider the profile $\hat{R}_N \in \mathcal{D}$ such that $\hat{R}_1 = \hat{R}$ where $\hat{R} \in \mathcal{D}^a$ is a single-peaked preference and $\hat{R}_i = \hat{R}'$, where $\hat{R}' \in \mathcal{D}^b$ is a single-peaked preference for all $i \in \{2, \ldots, n\}$. In Claim 1, we show that $\phi(\hat{R}_N)(\tau(\mathcal{D})) = 1$, and in Claim 2, we show that $\phi(\hat{R}_N) = \phi(R_N)$, which will complete the proof of $\phi(R_N)(\tau(\mathcal{D})) = 1$.

Claim 1. $\phi(\hat{R}_N)(\tau(\mathcal{D})) = 1$.

Proof of Claim 1. Let $r(I)$ and $l(I)$ denote the right end point and the left end point of an interval I. Define $X_j = (r(I_j), l(I_{j+1}))$ for all $j \in \{1, \ldots, k-1\}$. Since $\phi(\hat{R}_N)([a,b]) = 1$, to prove Claim 1, it is sufficient to show that $\phi(\hat{R}_N)(X_j) = 0$ for all $j \in \{s, \ldots, s'-1\}$. Assume for contradiction that there exists $t \in \{s, \ldots, s'-1\}$ such that $\phi(\hat{R}_N)(X_t) > 0$. Without loss of generality assume that $\phi(\hat{R}_N)(X_j) = 0$ for all $j \in \{s, \ldots, t-1\}$. Let $\bar{R} \in \mathcal{D}^a$ and $\bar{\bar{R}} \in \mathcal{D}^b$ be such that for all $x, y \in A$ with $x \in \cup_{q=s}^{s'} I_q$ and $y \in [a,b] \setminus \cup_{q=s}^{s'} I_q$, we have $x\bar{R}y$ and $x\bar{\bar{R}}y$. Further let $R'_N, R''_N \in \mathcal{D}^n$ be such that

- $R'_1 = \bar{R}$ and $R'_i = \hat{R}_i$ for all $i \in \{2, \ldots, n\}$, and
- $R''_i = \bar{\bar{R}}$ for all $i \in \{2, \ldots, n\}$ and $R''_1 = \hat{R}_1$.

Claim 1.1. $\phi(R'_N)(\tau(\mathcal{D})) = \phi(R''_N)(\tau(\mathcal{D})) = 1$.

Proof of Claim 1.1. We show this only for R'_N. For R''_N the similar arguments hold. Let $\tilde{R}_1 = \bar{\bar{R}}$. Note that since $\phi(\bar{R}'_N)([a,b]) = 1$ for all $\bar{R}'_N \in \mathcal{D}^n$ such that $\{\tau(\bar{R}'_i) \mid i \in N\} = \{a, b\}$, we have $\phi(\tilde{R}_1, R'_{-1})([a,b]) = 1$. Again, since $R'_1 = \bar{R}$ and $\tilde{R}_1 = \bar{\bar{R}}$, by strategy-proofness it follows that $\phi(R'_N)([a,b] \cap \tau(\mathcal{D})) = \phi(\tilde{R}_1, R'_{-1})([a,b] \cap \tau(\mathcal{D}))$. Since $\tau(\tilde{R}_1) = \tau(R'_i) = b$ for all $i \in \{2, \ldots, n\}$, by unanimity, $\phi(\tilde{R}_1, R'_{-1})(\{b\}) = 1$. Combining all these observations, we get $\phi(R'_N)([a,b] \cap \tau(\mathcal{D})) = 1$. This completes the proof of the Claim 1.1. □

Claim 1.2. $\phi(R'_N) = \phi(R''_N)$.

Proof of Claim 1.2. Let $\tilde{R}_N \in \mathcal{D}^n$ be such that $\tilde{R}_1 = \bar{R}$ and $\tilde{R}_i = \bar{\bar{R}}$ for all $i \in \{2, \ldots, n\}$. Note that by Claim 1.1, $\phi(R'_N)([a,b] \cap \tau(\mathcal{D})) = \phi(R''_N)([a,b] \cap \tau(\mathcal{D})) = \phi(\tilde{R}_N)([a,b] \cap \tau(\mathcal{D}))$. Since $R'_2|_{[a,b] \cap \tau(\mathcal{D})} = \tilde{R}_2|_{[a,b] \cap \tau(\mathcal{D})}$, by strategy-proofness, $\phi(R'_N) = \phi(\tilde{R}_2, R'_{-2})$. Continuing in the manner, we can show that $\phi(R'_N) = \phi(\tilde{R}_N)$. Using similar arguments we can show that $\phi(R''_N) = \phi(\tilde{R}_N)$, and complete the proof of Claim 1.2. □

Claim 1.3. $\phi(\hat{R}_N)([a, r(I_s)]) = \phi(R'_N)([a, r(I_s)])$ and $\phi(\hat{R}_N)(I_j) = \phi(R'_N)(I_j)$ for all $j \in \{s+1, \ldots, t\}$.

Proof of Claim 1.3. Consider the preference profile $(R'_1, \hat{R}_{-1})$. Note that since $p(\hat{R}_N) = p(R'_1, \hat{R}_{-1}) = 2$, we have $\phi_{[a,b]}(\hat{R}_N) = \phi_{[a,b]}(R'_1, \hat{R}_{-1}) = 1$. Furthermore, because $\hat{R}_1, R'_1 \in \mathcal{D}^a$ and $\hat{R}_1|_{[a, r(I_s)]} = R'_1|_{[a, r(I_s)]}$, by strategy-proofness, we have $\phi(\hat{R}_N)([a, r(I_s)]) = \phi(R'_1, \hat{R}_{-1})([a, r(I_s)])$. By our assumption, $\phi(\hat{R}_N)(X_s) = 0$. We show $\phi(R'_1, \hat{R}_{-1})(X_s) = 0$. Assume to the contrary, $\phi(R'_1, \hat{R}_{-1})(X_s) > 0$. This, together with the fact that $\phi(\hat{R}_N)([a, r(I_s)]) = \phi(R'_1, \hat{R}_{-1})([a, r(I_s)])$, implies

$$\phi(\hat{R}_N)([a, l(I_{s+1})]) < \phi(R'_1, \hat{R}_{-1})([a, l(I_{s+1})]). \tag{5.18}$$

Since $\hat{R}_1$ is a single-peaked preference and $\phi(\hat{R}_N)([a, b]) = \phi(R'_1, \hat{R}_{-1})([a, b]) = 1$, (5.18) implies $\phi(\hat{R}_N)(U(l(I_{s+1}), \hat{R}_1)) < \phi(R'_1, \hat{R}_{-1})(U(l(I_{s+1}), \hat{R}_1))$, which in turn means agent 1 manipulates at $\hat{R}_1$ via R'_1. Therefore, we have $\phi(R'_1, \hat{R}_{-1})(X_s) = 0$. By strategy-proofness, this implies $\phi(\hat{R}_N)(I_{s+1}) = \phi(R'_1, \hat{R}_{-1})(I_{s+1})$. Using similar arguments, we can show $\phi(\hat{R}_N)(I_j) = \phi(R'_1, \hat{R}_{-1})(I_j)$ for all $j \in \{s+2, \ldots, t\}$. Since $R'_N = (R'_1, \hat{R}_{-1})$ this completes the proof of Claim 1.3. □

Now, we complete the proof of Claim 1, that is, $\phi(\hat{R}_N)(\tau(\mathcal{D})) = 1$. By Claim 1.2, we have $\phi(R'_N) = \phi(R''_N)$. On the other hand, by Claim 1.3, we have $\phi(\hat{R}_N)([a, r(I_s)]) = \phi(R'_N)([a, r(I_s)])$, and $\phi(\hat{R}_N)(I_j) = \phi(R'_N)(I_j)$ for all $j \in \{s+1, \ldots, t\}$. Combining these two observations, we get $\phi(\hat{R}_N)([a, r(I_s)]) = \phi(R''_N)([a, r(I_s)])$, and $\phi(\hat{R}_N)(I_j) = \phi(R''_N)(I_j)$ for all $j \in \{s+1, \ldots, t\}$. Note that as $p(R''_N) = 2$, we have $\phi(R''_N)([a, b]) = 1$, and hence by Claim 1.1, $\phi(R''_N)(X_t) = 0$. This, together with our assumption that $\phi(\hat{R}_N)(X_t) > 0$, implies

$$\phi(R''_N)(U(l(I_{t+1}), \hat{R}')) > \phi(\hat{R}_N)(U(l(I_{t+1}), \hat{R}')). \tag{5.19}$$

Since $\hat{R}_i = \hat{R}'$ for all $i \in \{2, \ldots, n\}$, by strategy-proofness, we have

$$\phi(\hat{R}_N)(U(l(I_{t+1}), \hat{R}')) \geq \phi(R''_{r+1}, \hat{R}_{-\{r+2\}})(U(l(I_{t+1}), \hat{R}')) \geq \cdots \geq \phi(R''_N)(U(l(I_{t+1}), \hat{R}')).$$

However, this contradicts (5.19). Hence, $\phi(\hat{R}_N)(\tau(\mathcal{D})) = 1$, which completes the proof of Claim 1. □

Claim 2. $\phi(\hat{R}_N) = \phi(R_N)$.

We first show that $\phi(\hat{R}_N) = \phi(R_1, \hat{R}_{-1})$. Since $p(\hat{R}_N) = p(R_1, \hat{R}_{-1}) = 2$, we have $\phi(\hat{R}_N)([a, b]) = \phi(R_1, \hat{R}_{-1})([a, b]) = 1$. By strategy-proofness, this implies $\phi(\hat{R}_N)([a, r(I_s)]) = \phi(R_1, \hat{R}_{-1})([a, r(I_s)])$. We claim $\phi(R_1, \hat{R}_{-1})(X_s) = 0$. Assume to the contrary, $\phi(R_1, \hat{R}_{-1})(X_s) > 0$. Since $\hat{R}_1$ is a single-peaked preference and $\phi(\hat{R}_N)(X_s) = 0$, this means $\phi(\hat{R}_N)(U(l(I_{s+1}), \hat{R}_1)) < \phi(R_1, \hat{R}_{-1})(U(l(I_{s+1}), \hat{R}_1))$. However, then agent 1 manipulates at $\hat{R}_N$ via R_1, a contradiction. So, $\phi(R_1, \hat{R}_{-1})(X_s) = 0$. Using similar arguments, we can show $\phi(\hat{R}_N)(I_{s+1}) = \phi(R_1, \hat{R}_{-1})(I_{s+1})$, and thereafter $\phi(\hat{R}_N)(X_{s+1}) = \phi(R_1, \hat{R}_{-1})(X_{s+1})$. Continuing in this manner, it follows that $\phi(\hat{R}_N)(I_j) = \phi(R_1, \hat{R}_{-1})(I_j)$ and $\phi(\hat{R}_N)(X_j) = \phi(R_1, \hat{R}_{-1})(X_j)$ for all $j \in \{s, \ldots, s'-1\}$. Finally, using similar arguments as for the proof of $\phi(\hat{R}_N)([a, r(I_s)]) = \phi(R_1, \hat{R}_{-1})([a, r(I_s)])$, we can show $\phi(\hat{R}_N)([l(I_{s'}), b]) = \phi(R_1, \hat{R}_{-1})([l(I_{s'}), b])$. Combining all these observations, we conclude $\phi(\hat{R}_N) = \phi(R_1, \hat{R}_{-1})$.

Now, we proceed to complete the proof of Claim 2. By replicating symmetric arguments as for the same proof, i.e., the proof of $\phi(\hat{R}_N) = \phi(R_1, \hat{R}_{-1})$, we can show $\phi(\hat{R}_N) = \phi(R_1, R_2, \hat{R}_{\{3,\ldots,n\}})$. Here, by symmetric arguments, we mean by using b in place of a, s' in place of s, and by following the sequence $s', s'-1, \ldots, s$ in place of $s, s+1, \ldots, s'$. As before, we can now sequentially move the agents i in $\{3, \ldots, n\}$ from the preference $\hat{R}_i$ to the preference R_i and conclude that $\phi(\hat{R}_N) = \phi(R_N)$. This completes the proof of Claim 2. □

Induction step for the proof of the base case of Proposition 5.8.1: Suppose the proposition holds for the case $p(R_N) = 2$ and $\tau(R_1) = \cdots = \tau(R_{r-1}) = a$ and $\tau(R_r) = \cdots = \tau(R_n) = b$ for some $r < n$. We proceed to show that the proposition holds for the case $\tau(R_1) = \cdots = \tau(R_r) = a$ and $\tau(R_{r+1}) = \cdots = \tau(R_n) = b$

Proof of $\phi(R_N)([a,b]) = 1$:

We claim $\phi(R_N)((b,\infty)) = 0$. Suppose to the contrary that $\phi(R_N)((b,\infty)) > 0$. This means $\phi(R_N)(U(b,R_r)) < 1$. Let $R' \in \mathcal{D}^b$. By the base case, $\phi(R', R_{-r})([a,b] \cap \tau(\mathcal{D})) = 1$. Since $b \in \tau(\mathcal{D})$ by the definition of generalized intermediate domains $\phi(R', R_{-r})(U(b,R_r)) = 1$, and hence agent 1 manipulates at R_N by misreporting his/her preference as R', a contradiction. Since $\phi(R_N)((b,\infty)) = 0$, to show $\phi(R_N)([a,b]) = 1$, it is enough to show $\phi(R_N)((-\infty,a)) = 0$. The proof of this follows by using arguments similar to the proof of $\phi(R_N)([a,b]) = 1$ under "base case for the proof of the base case of Proposition 5.8.1".

Proof of $\phi(R_N)(\tau(\mathcal{D})) = 1$: The proof of this follows by using arguments similar to the proof of $\phi(R_N)(\tau(\mathcal{D})) = 1$ under "base case for the proof of the base case of Proposition 5.8.1".

Induction step for the proof of Proposition 5.8.1: Suppose that the proposition holds when $p(R_N) \leq l$ for some $l < n$. We show that the same holds when $p(R_N) = l + 1$.

Let $\kappa_1(R_N)$ and $\kappa_2(R_N)$ denote the numbers of agents whose top-ranked alternatives are the minimum and the maximum, respectively, at the profile R_N. More formally, $\kappa_1(R_N) = |\{i \mid \tau(R_i) \leq \tau(R_j) \text{ for all } j \in N \setminus i\}|$ and $\kappa_2(R_N) = |\{i \mid \tau(R_i) \geq \tau(R_j) \text{ for all } j \in N \setminus i\}|$. We prove the proposition for this induction step by using another level of induction on the basis of the numbers $\kappa_1(R_N)$ and $\kappa_2(R_N)$. We treat the case $\kappa_1(R_N) = \kappa_2(R_N) = 1$ as the base case.

Base case for the proof of the induction step of Proposition 5.8.1: Suppose $\kappa_1(R_N) = \kappa_2(R_N) = 1$. Without loss of generality assume that agent 1 is the (unique) agent whose top-ranked alternative is the minimum at R_N and agent 2 is the (unique) one whose top-ranked alternative is the maximum at R_N. Suppose $\tau(R_1) = a$ and $\tau(R_2) = b$.

Proof of $\phi(R_N)([a,b]) = 1$:

We only show that $\phi(R_N)((b,\infty)) = 0$, using a similar argument it can be shown that

$\phi(R_N)((-\infty, a)) = 0$, which will complete the proof of $\phi(R_N)([a, b]) = 1$ for the case at hand. Assume for contradiction that $\phi(R_N)((b, \infty)) > 0$. Let R_1' be such that the top-ranked alternative at R_1' is the second minimum among the top-ranked alternatives at R_N, that is, $\tau(R_1') = \min_{i \neq 1}\{\tau(R_i)\}$. Since $p(R_1', R_{-1}) = l$, by means of the induction hypothesis, we have $\phi(R_1', R_{-1})([\tau(R_1'), b]) = 1$ and $\phi(R_1', R_{-1})(\tau(\mathcal{D})) = 1$. This, together with the fact that $[\tau(R_1'), b] \cap \tau(\mathcal{D}) \subseteq U(b, R_1)$, implies $\phi(R_1', R_{-1})(U(b, R_1)) = 1$. On the other hand, because $\phi(R_N)((b, \infty)) > 0$, we have $\phi(R_N)(U(b, R_1)) < 1$. Combining all these observations, it follows that agent 1 manipulates at R_N via R_1', a contradiction.

Proof of $\phi(R_N)(\tau(\mathcal{D})) = 1$:

Let R_1' be such that $\tau(R_1') = \min_{i \neq 1}\{\tau(R_i)\}$, $U(a, R_1') = U(\tau(R_1'), R_1) \cap [a, \tau(R_1')]$, and there exists $x \in A$ such that $L(x, R_1') = (-\infty, a)$. In other words, the top-ranked alternative at R_1' is the second minimum among the top-ranked alternatives at R_N, an alternative is (weakly) preferred to a at R_1' if and only if it lies in-between a and $\tau(R_1')$ as well as is (weakly) preferred to $\tau(R_1')$ at R_1, and finally the alternatives in the interval $(-\infty, a)$ come at the bottom of the preference R_1'. By strategy-proofness, $\phi(R_N)(D) = \phi(R_1', R_{-1})(D)$ for all Borel sets D such that $D \cap [a, \tau(R_1')] = \emptyset$.

Now, consider the preference R_2' of agent 2 such that $\tau(R_2') = \max_{i \neq 2}\{\tau(R_i)\}$, $U(b, R_2') = U(\tau(R_2'), R_2) \cap [\tau(R_2'), b]$, and there exists $y \in A$ such that $L(y, R_2') = (b, \infty)$. Using symmetric arguments as for agent 1 (in the last paragraph), we can show that $\phi(R_N)(D) = \phi(R_2', R_{-2})(D)$ for all Borel sets D such that $D \cap [\tau(R_2'), b] = \emptyset$. Since $p(R_1', R_{-1}) = p(R_2', R_{-2}) = l$, by the induction hypothesis, $\phi(R_1', R_{-1})(\tau(\mathcal{D})) = \phi(R_2', R_{-2})(\tau(\mathcal{D})) = 1$. If $p(R_N) = 3$, then $a < \tau(R_1') = \tau(R_2') < b$, and hence $[a, \tau(R_1')] \cap [\tau(R_2'), b] = \{\tau(R_1')\}$. On the other hand, if $p(R_N) > 3$, then $a < \tau(R_1') < \tau(R_2') < b$, and hence $[a, \tau(R_1')] \cap [\tau(R_2'), b] = \emptyset$. This, together with the fact that $p(R_N) = l \geq 3$, implies $[a, \tau(R_1')] \cap [\tau(R_2'), b] \subseteq \tau(\mathcal{D})$. Combining all these observations, we obtain $\phi(R_N)(\tau(\mathcal{D})) = 1$.

This completes the proof of the base case for the induction step of Proposition 5.8.1.

Induction step for the proof of the induction step of Proposition 5.8.1: Suppose that the proposition holds for all pairs of values of $(\kappa_1(R_N), \kappa_2(R_N))$ of the form $(k_1, k_2 + 1)$ and $(k_1 + 1, k_2)$ for some $k_1, k_2 \in \mathbb{N}$ such that $k_1 + k_2 + 1 < n$. We proceed to show that the proposition holds when $(\kappa_1(R_N), \kappa_2(R_N)) = (k_1 + 1, k_2 + 1)$.

First, we explain how the induction hypothesis is compatible with our base case and how our induction step completes the proof of Proposition 5.8.1. Suppose we want prove the proposition for the case $(\kappa_1(R_N), \kappa_2(R_N)) = (2, 1)$. Then, our induction hypothesis requires that the proposition is already proved for the cases $(\kappa_1(R_N), \kappa_2(R_N)) = (1, 1)$ and $(\kappa_1(R_N), \kappa_2(R_N)) = (2, 1)$. We have already proved the proposition when $(\kappa_1(R_N), \kappa_2(R_N)) = (1, 1)$. Technically speaking, the case $(\kappa_1(R_N), \kappa_2(R_N)) = (2, 0)$ is not defined since it means that there is no agent whose top-ranked alternative is the (hypothetical) maximum of R_N, however practically this case boils down to the case where the number of different peaks at R_N is l. Therefore, the proof of the proposition for this case follows from the induction hypothesis for the proof of Proposition 5.8.1. So, we have the proposition for the case $(\kappa_1(R_N), \kappa_2(R_N)) = (2, 1)$. By similar arguments, it can be proved for the case $(\kappa_1(R_N), \kappa_2(R_N)) = (2, 1)$. Now, to prove it for the case $(\kappa_1(R_N), \kappa_2(R_N)) = (2, 2)$, we require it to be proved for the cases $(\kappa_1(R_N), \kappa_2(R_N)) = (2, 1)$ and $(\kappa_1(R_N), \kappa_2(R_N)) = (1, 2)$, which are already proved in the previous step. Continuing in this manner, our induction step proves the proposition for all values of $(\kappa_1(R_N), \kappa_2(R_N))$.

Let $\min_{i \in N}\{\tau(R_i)\} = a$ and $\max_{i \in N}\{\tau(R_i)\} = b$. Assume without loss of generality that $\tau(R_1) = a$ and $\tau(R_2) = b$.

Proof of $\phi(R_N)([a, b]) = 1$:

We only show $\phi(R_N)((b, \infty)) = 0$. This is sufficient since by a similar argument, we can show that $\phi(R_N)((-\infty, a)) = 0$ and conclude that $\phi(R_N)([a, b]) = 1$. Assume for contradiction that $\phi(R_N)((b, \infty)) > 0$. Let R'_1 be such that $\tau(R'_1) = \min\{\tau(R_2), \ldots, \tau(R_n)\}$. Combining our induction hypothesis with the facts that $p(R'_1, R_{-1}) = l$, $\kappa_1(R'_1, R_{-1}) = k_1$, and $\kappa_2(R'_1, R_{-1}) = k_2 + 1$, we obtain $\phi(R'_1, R_{-1})([a, b]) = 1$ and $\phi(R'_1, R_{-1})(\tau(\mathcal{D})) = 1$. This, together with the fact that $[\tau(R'_1), b] \cap \tau(\mathcal{D}) \subseteq U(b, R_1)$, implies $\phi(R'_1, R_{-1})(U(b, R_1)) = 1$. On the other hand, because $\phi(R_N)((b, \infty)) > 0$, we have $\phi(R_N)(U(b, R_1)) < 1$. Combining all these observations, it follows that agent 1 manipulates at R_N via R'_1, a contradiction.

Proof of $\phi(R_N)(\tau(\mathcal{D})) = 1$:

Consider a preference R'_1 of agent 1 satisfying the following conditions: $\tau(R'_1) = \min\{\tau(R_2, \ldots, \tau(R_n))\}$, $U(a, R'_1) = U(\tau(R'_1), R_1) \cap [a, \tau(R'_1)]$, and $L(x, R'_1) = (-\infty, a)$ for some $x \in A$. By strategy-proofness, $\phi(R_N)(D) = \phi(R'_1, R_{-1})(D)$ for all Borel sets D such that $D \cap [a, \tau(R'_1)] = \emptyset$.

Now, consider a preference R'_2 of agent 2 satisfying the following conditions: $\tau(R'_2) = \max\{\tau(R_1), \tau(R_3), \ldots, \tau(R_n)\}$, $U(b, R'_2) = U(\tau(R'_2), R_2) \cap [\tau(R'_2), b]$, and $L(y, R'_2) = (b, \infty)$ for some $y \in A$. Using symmetric arguments as for agent 1, we can show that $\phi(R_N)(D) = \phi(R'_2, R_{-2})(D)$ for all Borel sets D such that $D \cap [\tau(R'_2), b] = \emptyset$. Since $\kappa_1(R'_1, R_{-1}) = k_1$ and $\kappa_2(R'_1, R_{-1}) = k_2 + 1$, by the induction hypothesis, $\phi(R'_1, R_{-1})(\tau(\mathcal{D})) = 1$. Similarly, since $\kappa_1(R'_2, R_{-2}) = k_1 + 1$ and $\kappa_2(R'_2, R_{-2}) = k_2$, by the induction hypothesis $\phi(R'_2, R_{-2})(\tau(\mathcal{D})) = 1$. If $p(R_N) = 3$, then $a < \tau(R'_1) = \tau(R'_2) < b$, and hence $[a, \tau(R'_1)] \cap [\tau(R'_2), b] = \{\tau(R'_1)\}$. On the other hand, if $p(R_N) > 3$, then $a < \tau(R'_1) < \tau(R'_2) < b$, and hence $[a, \tau(R'_1)] \cap [\tau(R'_2), b] = \emptyset$. This, together with the fact that $p(R_N) = l \geq 3$, implies $[a, \tau(R'_1)] \cap [\tau(R'_2), b] \subseteq \tau(\mathcal{D})$. Combining all these observations, we obtain $\phi(R_N)(\tau(\mathcal{D})) = 1$. This completes the proof of Proposition 5.8.1. ■

Now, we complete the proof of the theorem. Define $\hat{\phi} : (\mathcal{D}|_{\tau(\mathcal{D})})^n \to \Delta\tau(\mathcal{D})$ as $\hat{\phi}_B(R_N|_{\tau(\mathcal{D})}) = \phi_B(R_N)$ for all Borel sets $B \in \tau(\mathcal{D})$. This is well-defined as by Proposition 5.8.1, $\phi_{\tau(\mathcal{D})}(R_N) = 1$ for all $R_N \in \mathcal{D}^n$ and ϕ is tops-only. Since $\mathcal{D}|_{\tau(\mathcal{D})}$ is a single-peaked domain, and hence Theorem 5.4.1 follows from Theorem 4.1 in [46]. ■

5.9 Proof of Lemma 5.5.7

First we prove a lemma which we repeatedly use in the proof of Lemma 5.5.7.

Lemma 5.9.1 *Let $\{P_x\}_{x \in X}$ be a strict intermediate domain. Then for all distinct $a, b, c \in A$, the separating lines of the pairs (a, b) and (b, c) do not intersect.*

Proof: Let $\{P_x\}_{x \in X}$ be a strict intermediate domain. Assume for contradiction that there exist distinct $a, b, c \in A$ such that the separating lines of (a, b) and (b, c) intersect. Since $\{P_x\}_{x \in X}$ is strict, no three separating lines of $\{P_x\}_{x \in X}$ intersect at a common point. Therefore, we can choose an open (see Figure 5.9.1) ball such that no separating line other than those of the pairs (a, b) and (b, c) passes through that open ball. Consider the regions X_1 and X_2 in Figure 5.9.1. Consider $x \in X_1$. Since aP_xb and bP_xc, by transitivity, we have aP_xc. Now, consider $y \in X_2$. Again, since bPa and cPb, by transitivity, we have cPa. Since the relative preference over a and c is changing from X_1 to X_2, it must be that the separating line of (a, c) intersects at least one of these regions. However, this is a contradiction to our assumption that no separating line other than those of (a, b) and (b, c) intersects this open ball. This completes the proof of the lemma.

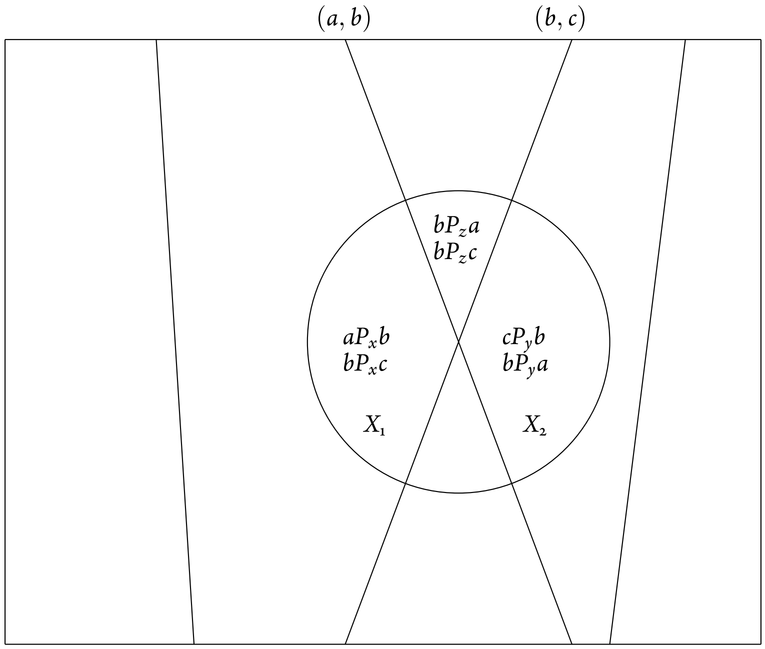


Figure 5.9.1: A graphic illustration

■

Now we prove Lemma 5.5.7. *Proof:* Let $\{P_x\}_{x\in X}$ be a domain satisfying strict intermediate property. Since the number of alternatives is finite, there are finitely many preferences in the domain $\{P_x\}_{x\in X}$. Consider a preference $P \in \{P_x\}_{x\in X}$. Let $X_P = \{x \in X | P_x = P\}$. Since there are finitely many preferences in the domain $\{P_x\}_{x\in X}$, we can find a finite collection of parallel lines $\{l_1, \ldots, l_k\}$ such that for each $P \in \{P_x\}_{x\in X}$, there exists $l \in \{l_1, \ldots, l_k\}$ such that $X_P \cap l \neq \emptyset$. This implies that $\{P_x\}_{x\in X} = \cup_{i=1}^k \{P_x\}_{x\in l_i}$. Since $\{P_x\}_{x\in X}$ satisfies strict intermediate property, there exists a line $\hat{l}$ that intersects all the separating lines (as defined in Lemma 5.5.6). We assume that (i) $\hat{l} \in \{l_1, \ldots, l_k\}$, and (ii) no l_i passes through the point of intersection of any two separating lines. This assumption is without of loss of generality because for (i), we can start with $\hat{l}$ and can consider a collection of parallel lines satisfying the required properties, and for (ii), since we have finitely many separating lines and hence finitely many points of intersection of those, we can always choose the lines $\{l_1, \ldots, l_k\}$ by avoiding those points.

Now we show that $\cup_{i=1}^k \{P_x\}_{x\in l_i}$ is a generalized intermediate domain satisfying minimal richness. We show this using the following three claims.

Claim 1. For each $l \in \{l_1, \ldots, l_k\}$, the family of preferences $\{P_x\}_{x\in l}$ is a generalized intermediate domain satisfying minimal richness.

Consider $l \in \{l_1, \ldots, l_k\}$. Let $x_1, \ldots, x_s$ be the points of intersection of the line l with the separating lines of $\{P_x\}_{x\in X}$. Note that $s \leq k$ since there can be separating lines of $\{P_x\}_{x\in X}$ that do not intersect with l.

Assume without loss of generality that $x_j \in (x_{j-1}, x_{j+1})$ for all $j \in \{2, \ldots, s-1\}$, that is, the points $\{x_1, \ldots, x_s\}$ are ordered in a particular direction. Consider $x \in l$ such that $x_1 \in (x, x_2)$. Such a point x can always be chosen as X is open and $x_1 \in X$. Let $P_x = P_1$. By Lemma 5.5.6, $P_y = P_1$ for all $y \in [x, x_1)$. By our assumption of x_1, there exists a separating line, say for the pair of alternatives (a, b), that intersects l at x_1. This implies there exists $P_2 \in \{P_x\}_{x \in l}$ such that $P_y = P_2$ for all $y \in (x_1, x_2)$. By Lemma 5.5.6, P_1 and P_2 differ only over the ordering of the pair (a, b). Again, by Lemma 5.5.6, the preference P_{x_1} is either P_1 or P_2. Continuing in this manner, we can get hold of a sequence of preferences $\{P_j\}_{j \in \{1,\ldots,s+1\}}$ such that (i) $\{P_x\}_{x \in l} = \{P_1, \ldots, P_{s+1}\}$, and (ii) for all $j = \{2, \ldots, s\}$, P_j and P_{j+1} differ only over the ordering of a particular pair of alternatives. This implies that $\{P_x\}_{x \in l}$ is minimally rich.

Next, we show $\{P_1, \ldots, P_{s+1}\}$ is a generalized intermediate domain with respect to the ordering given by P_1. Assume for contradiction that there exist $c, d, e \in A$ with cP_1dP_1e such that $d, e \in \tau(\{P_1, \ldots, P_{s+1}\})$ and cPd for some $P \in \{P_1, \ldots, P_{s+1}\}$ with $\tau(P) = e$. Let $x_e \in X$ be such that $P_{x_e} = P$. Since $d \in \tau(\{P_1, \ldots, P_{s+1}\})$ and cP_1d, it follows that the separating line of the pair (c, d) intersects with l. Let x_t be this point of intersection. Since cPd by our assumption, $x_e \in (x_1, x_t)$. Consider $x_d \in X$ such that $\tau(P_{x_d}) = d$. Such a point x_d must exist since $d \in \tau(\{P_1, \ldots, P_{s+1}\})$ Then, it must be that $x_t \in (x_1, x_d)$. Also, dP_1e and ePd together imply $x_d \in (x_1, x_e)$. But this contradicts the fact that $x_e \in (x_1, x_t)$. This implies that $\{P_1, \ldots, P_{s+1}\}$ is a generalized intermediate domain completing the proof of Claim 1. □

Recall that by our assumption, $\hat{l} \in \{l_1, \ldots, l_k\}$. Therefore, by applying Claim 1 for $l = \hat{l}$, it follows that $\{P_x\}_{x \in \hat{l}}$ is a minimally rich generalized intermediate domain with respect to some ordering, say $\prec$. Suppose $\tau(\{P_x\}_{x \in \hat{l}}) = \{b_1, \ldots, b_r\}$, where $b_1 \prec b_2 \prec \ldots \prec b_r$.

Claim 2. For all $l \in \{l_1, \ldots, l_k\}$, there exist s and t with $1 \leq s \leq t \leq r$ such that $\{P_x\}_{x \in l}$ is a generalized intermediate domain with $\tau(\{P_x\}_{x \in l}) = \{b_s, \ldots, b_t\}$.

Consider $l \in \{l_1, \ldots, l_k\} \setminus \hat{l}$. Let $y_1, \ldots, y_q$ be the points of intersection of l with the separating lines such that $y_j \in (y_{j-1}, y_{j+1})$ for all $j \in \{2, \ldots, q-1\}$. Similarly, let $x_1, \ldots, x_p$ be the points of intersection of $\hat{l}$ with the separating lines such that $x_j \in (x_{j-1}, x_{j+1})$ for all $j \in \{2, \ldots, p-1\}$. Assume without loss of generality that $x_p x_1 = y_q y_1$, that is, the direction along which the points $x_1, \ldots, x_p$ are counted is the same as that along which the points $y_1, \ldots, y_q$ are counted (see Figure 5.9.2).

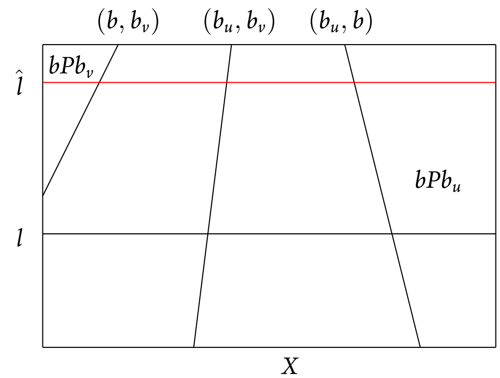


Figure 5.9.3: A graphic illustration

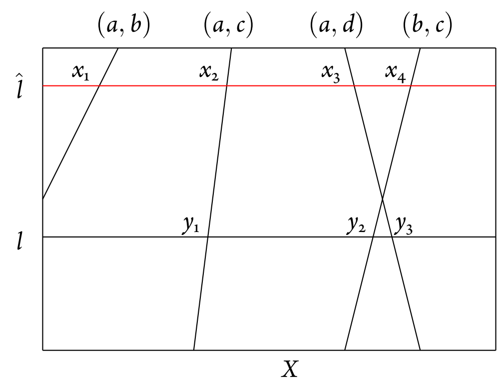


Figure 5.9.2: A graphic illustration

First, we show $\tau(\{P_x\}_{x\in l}) \subseteq \tau(\{P_x\}_{x\in\hat{l}})$. Consider $b \in \tau(\{P_x\}_{x\in l})$. Assume for contradiction that $b \notin \tau(\{P_x\}_{x\in\hat{l}})$. Since $\min_{\prec} \tau(\{P_x\}_{x\in\hat{l}}) = b_1$, this implies $b_1 \prec b$. Suppose $b_r \prec b$. Then, it must be that for all preferences in $\{P_x\}_{x\in\hat{l}}$, b_r is ranked above b, and hence the separating line of the pair (b_r, b) does not intersect with $\hat{l}$. However, since $b \in \tau(\{P_x\}_{x\in l})$, there must be a separating line of the pair (b_r, b). This is a contradiction to our assumption that $\hat{l}$ intersects with all separating lines. This shows $b \prec b_r$. Now, suppose $b_u \prec b \prec b_v$ where b_u and b_v are two consecutive alternatives (with respect to the ordering $\prec$) in the top-set $\tau(\{P_x\}_{x\in\hat{l}})$.[17] Since $b_u \prec b \prec b_v$ and $b \notin \tau(\{P_x\}_{x\in\hat{l}})$, by Lemma 5.5.6, there must be x_e, x_f and x_g with $x_f \in (x_e, x_g)$ such that the separating lines of the pairs (b, b_v), (b_u, b_v), and (b_u, b) intersect $\hat{l}$ at x_e, x_f, and x_g, respectively. By Lemma 5.9.1, no two of these separating lines intersect. Note that $b = \tau(P_z)$ for some $z \in X$ implies that z must be on the left side of the separating line of (b, b_v) and on the right side of the separating line of (b_u, b) (see Figure 5.9.3). However, as it is evident from Figure 5.9.3, there cannot be any such z. Moreover, this is true in general since the separating lines of (b, b_v) and (b_u, b) do not intersect. This shows $b \in \tau(\{P_x\}_{x\in\hat{l}})$, and hence $\tau(\{P_x\}_{x\in l}) \subseteq \tau(\{P_x\}_{x\in\hat{l}})$.

[17] By consecutive in $\tau(\{P_x\}_{x\in\hat{l}})$, we mean $(b_u, b_v) \cap \tau(\{P_x\}_{x\in\hat{l}}) = \emptyset$.

Next, we show that for all b, b_u, b_v such that $b_u, b_v \in \tau(\{P_x\}_{x\in l})$ and $b_u \preceq b \preceq b_v$, we have $b \in \tau(\{P_x\}_{x\in l})$. Suppose not. Assume without loss of generality that b_u and b_v are consecutive in $\tau(\{P_x\}_{x\in l})$, that is, $(b_u, b_v) \cap \tau(\{P_x\}_{x\in l}) = \emptyset$. Recall that by our assumption, all the separating lines of $\{P_x\}_{x\in X}$ intersect $\hat{l}$. Suppose that the separating lines of the pairs (b_u, b), (b_u, b_v), and (b, b_v) intersect $\hat{l}$ at x_e, x_f, and x_g, respectively, where $x_f \in (x_e, x_g)$. By Lemma 5.9.1, no two of those three separating lines intersect each other. This, together with the fact that $b_u, b_v \in \tau(\{P_x\}_{x\in l})$, implies that the separating lines of the pairs (b_u, b), (b_u, b_v), and (b, b_v) intersect l at y_h, y_i, and y_j, respectively, where $y_i \in (y_h, y_j)$ (see Figure 5.9.4). By Lemma 5.9.1, $b_u \preceq \tau(P_{y_i}) \preceq b_v$. However, since $bP_{y_i}b_u$ and $bP_{y_i}b_v$, it must be that $\tau(P_{y_i}) \neq b_u, b_v$. This is a contradiction since $(b_u, b_v) \cap \tau(\{P_x\}_{x\in\hat{l}}) = \emptyset$. This completes the proof of Claim 2. ☐

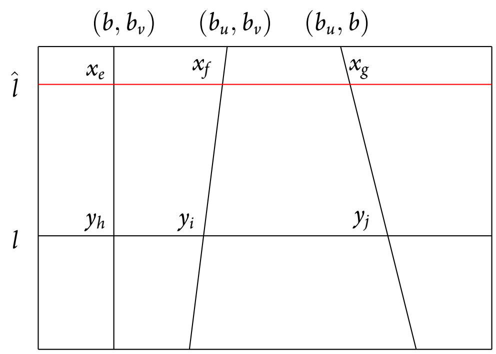


Figure 5.9.4: A graphic illustration

Claim 3. For all $l \in \{l_1, \ldots, l_k\}$, all $\bar{P} \in \{P_x\}_{x\in l}$, and all $b_v \in \{b_1, \ldots, b_r\}$, $\bar{P}$ satisfies the betweenness property with respect to b_v.

If $b_v \in \tau(\{P_x\}_{x\in l})$, then Claim 3 follows from Claim 2. Suppose $b_v \notin \tau(\{P_x\}_{x\in l})$. Without loss of generality, assume $b_v \prec b_s$ where $b_s = \min \tau(\{P_x\}_{x\in l})$. Let $a \prec b_v$. It is enough to show that $b_v\bar{P}a$. Since $b_v \prec b_s$ and b_sPb_v for all $P \in \{P_x\}_{x\in l}$, it must be that the separating line of (b_v, b_s) does not intersect l. Let $b_t = \max_{\prec} \tau(\{P_x\}_{x\in l})$. Suppose that the points of intersection of $\hat{l}$ with the separating lines of (a, b_v), (b_v, b_s), and (b_s, b_t) are x_c, x_d, and x_e, respectively. Because $a \prec b_v \prec b_s$ and $b_v \in \tau(\{P_x\}_{x\in\hat{l}})$, we have $x_d \in (x_c, x_e)$. By Lemma 5.9.1, separating lines of (a, b_v) and (b_v, b_s) cannot intersect each other. This, together with the fact that the separating line of (b_v, b_s) does not intersect l, implies that the separating line of (a, b_v) too does not intersect l (see Figure 5.9.5). This, in particular, implies $b_v\bar{P}a$, which completes the proof of Claim 3. ☐

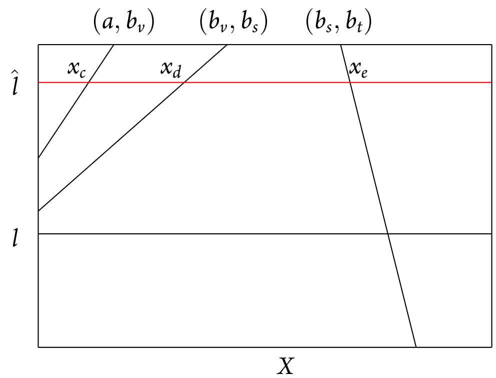


Figure 5.9.5: A graphic illustration

Now, the proof of Lemma 5.5.7 follows from Claim 2 and Claim 3. ■

6

Restricted Probabilistic Fixed Ballot Rules and Hybrid Domains

6.1 Introduction

Two familiar preference domains in the literature on mechanism design in voting environments are the complete domain and the domain of single-peaked preferences. The complete domain arises naturally when there are no a priori restrictions on preferences. The classic results of [56], [96] and [57] apply here. According to them, requiring strategy-proofness forces the mechanism to be a dictatorship in the deterministic case and to be a random dictatorship in the probabilistic case. Single-peaked preferences on the other hand, require more structure on the set of alternatives. However, they arise naturally in a variety of situations such as preference aggregation [19], strategic voting [72], public facility allocation [21], fair division [100] and assignment [**?**]. The single-peaked domain also admits well-behaved strategy-proof social choice functions. In this paper, we propose a flexible preference domain that admits both the complete domain and the single-peaked domain as special cases. We call them *hybrid domains* and completely characterize unanimous and strategy-proof *random social choice functions* (or RSCFs) over the hybrid domains. We refer to these random social choice functions as *Restricted Probabilistic Fixed Ballots*

Rules (or RPFBRs) and analyze their salient properties. Finally, we provide an axiomatic justification of hybrid domains and show that all domains that satisfy some richness properties must be hybrid.

We briefly recall the definition of single-peaked preferences. The set of alternatives is a finite set $A = \{a_1, a_2, \ldots, a_m\}$ which is endowed with the prior order $a_1 \prec a_2 \prec \cdots \prec a_m$. A preference ordering over A is *single-peaked* if there exists a unique top-ranked alternative, say a_k, such that preferences decline when alternatives move "farther away" from a_k. For instance, if "$a_r \prec a_s \prec a_k$ or $a_k \prec a_s \prec a_r$", then a_s is strictly preferred to a_r. A preference is *hybrid* if there exist *threshold* alternatives $a_{\underline{k}}$ and $a_{\overline{k}}$ with $a_{\underline{k}} \prec a_{\overline{k}}$ such that preferences over the alternatives in the interval between $a_{\underline{k}}$ and $a_{\overline{k}}$ are "unrestricted" relative to each other, while preferences over other alternatives retain features of single-peakedness. Thus, the set A can be decomposed into three parts: left interval $L = \{a_1, \ldots, a_{\underline{k}}\}$, right interval $R = \{a_{\overline{k}}, \ldots, a_m\}$ and middle interval $M = \{a_{\underline{k}}, \ldots, a_{\overline{k}}\}$. Formally, a preference is $(\underline{k}, \overline{k})$*-hybrid* if the following holds: (i) for a voter whose best alternative lies in L (respectively in R), preferences over alternatives in the set $L \cup R$ are conventionally single-peaked, while preferences over alternatives in M are arbitrary subject to the restriction that the best alternative in M is the left threshold $a_{\underline{k}}$ (respectively, right threshold $a_{\overline{k}}$), and (ii) for a voter whose peak lies in M, preferences restricted to $L \cup R$ are single-peaked but arbitrary over M. Observe that if $\underline{k} = 1$ and $\overline{k} = m$, then preferences are unrestricted, while the case where $\overline{k} - \underline{k} = 1$ coincides with the case of single-peaked preferences.

A $(\underline{k}, \overline{k})$-hybrid preference is a preference ordering which is single-peaked everywhere except over the alternatives in the middle interval. Consider the location of candidates in the forthcoming Democratic party primary elections in the USA, in the usual political left-right spectrum. It is clear that candidates such as Sanders and Warren belong to the left, while others such as Biden (perhaps) belong to the right. However, there are several candidates who cannot easily be ordered in this manner. The typical reason is that they are left on some issues and right on others. Hybrid preferences treat these candidates as ones belonging to the middle part, and the hybrid domain reflects the reversals in the relative rankings of these alternatives that arise from the underlying multidimensional issues. A more general way to model departures from single-peaked preferences would be to consider several intervals of alternatives where single-peakedness fails. However, as suggested by Theorem 6.7.2, this complicates the analysis significantly without adding substantial new insights.

We study unanimous and strategy-proof RSCFs on hybrid domains. A RSCF associates a lottery over alternatives to each profile of preferences. Randomization is a way to resolve conflicts of interest by ensuring a measure of ex-ante fairness in the collective decision process. More importantly, it has recently been shown that randomization significantly enlarges the scope of designing well-behaved mechanisms, e.g., the compromise RSCF of [35] and the maximal-lottery mechanism of [25].

In order to define the notion of strategy-proofness, we follow the standard approach of [57]. For every voter, truthfully revealing her preference ordering must yield a lottery that stochastically dominates the lottery arising from any unilateral misrepresentation of preferences according to the sincere preference. Unanimity is a weak efficiency requirement which says that the alternative that is unanimously best at a preference profile is selected with probability one.

The main theorem of the paper shows that a RSCF defined on the $(\underline{k}, \overline{k})$-hybrid domain is unanimous and strategy-proof if and only if it is a RPFBR (see Theorem 6.5.1). A RPFBR is a special case of a *Probabilistic Fixed Ballot Rule* (or PFBR) introduced by [46]. A PFBR is specified by a collection of probability distributions β_S, where S is a coalition of voters, over the set of alternatives. We formally call β_S a *probabilistic ballot*. If $\overline{k} - \underline{k} = 1$, then a RPFBR reduces to a PFBR. However, if $\overline{k} - \underline{k} > 1$, then a RPFBR requires an additional restriction on the probabilistic ballots: each voter i has a fixed probability weight ε_i such that the probability of the right interval R according to β_S is the total weight $\sum_{i \in S} \varepsilon_i$ of the voters in S and that of the left interval L is the total weight $\sum_{i \notin S} \varepsilon_i$ of the voters outside S.

We use our characterization result to investigate the the following classical decomposability question on these domains: Can every unanimous and strategy-proof RSCF be decomposed as a mixture of finitely many deterministic unanimous and strategy-proof social choice functions? Decomposability holds on several well-known domains, for instance the complete domain [57] and the single-peaked domains [81, 87]. Thus, decomposability holds for the cases when $\overline{k} - \underline{k} = 1$ or $\overline{k} - \underline{k} = m - 1$. Surprisingly, it does *not* hold for any intermediate values of $\overline{k}$ and $\underline{k}$. In other words, randomization non-trivially expands the scope for designing strategy-proof mechanisms. We identify a necessary and sufficient condition for decomposability under an additional assumption of anonymity, which requires the RSCF be non-sensitive to the identities of voters (see Theorem 6.5.3). We further observe that non-decomposable RPFBRs dominate almost all decomposable RPFBRs in recognizing social compromises.

Finally, we formally demonstrate the salience of hybrid domains. We consider *connected* domains, where connectedness is a property of a graph that is induced by the domain. Essentially, connectedness ensures the existence of a path from one preference to another by a sequence of specific preference switches. Connected domains have been used extensively in the literature on strategic social choice [e.g. 71, 86, 95]. According to Theorem 6.7.2, every connected domain that satisfies the weak no-restoration property of [95] and includes two completely reversed preferences must be a hybrid domain over which the RPFBR characterization still holds. An important feature of this result is that the condition on the domain does not specify an underlying structure of single-peakedness or threshold alternatives. These are derived endogenously from our hypotheses.

The paper is organized as follows. Section 6.1.1 reviews the literature, while Section 6.2 sets out the model and definitions. Section 6.3 and 6.4 introduce hybrid preferences and RPFBRs, respectively. Section 6.5 presents the main characterization result as well as the result on decomposability. Section 6.7

provides an axiomatic justification for hybrid domains.

6.1.1 Relationship with the Literature

The analysis of strategy-proof deterministic social choice functions on single-peaked domains was initiated by [72] and developed further by [12], [37] and [103]. In the deterministic setting, [75], [34], [88], [29], [1] and [23] analyze the structure of unanimous and strategy-proof social choice functions on domains closely related to single-peakedness.

The structure of unanimous and strategy-proof RSCFs on single-peaked domains was first studied by [46]. They considered the case where the set of alternatives is an interval in the real line and characterized the unanimous and strategy-proof RSCFs in terms of probabilistic fixed ballot rules. Recently, [91] strengthen the characterization result on a single-peaked domain which does not require maximal cardinality. Characterizations of unanimous and strategy-proof RSCFs as convex combinations of counterpart deterministic social choice functions were provided by [81] and [87].

Recently, [83] have considered the case where the set of alternatives is endowed with a graph structure. Single-peakedness is defined w.r.t. such graphs as in [40] and [34]. [83] investigate the structure of unanimous and strategy-proof RSCFs. Their characterization result (Theorem 5.6 of [83]) implies our Theorem 6.5.1 for a special graph structure. However, the extension of our result in Theorem 6.7.2 is more general than their result since we do not assume a prespecified graph over the set of alternatives. In particular, our result covers many domains that are excluded by theirs. Finally, we emphasize that the motivation, formulation, and proof techniques in the two papers are completely different.

6.2 Preliminaries

Let $A = \{a_1, a_2, \dots, a_m\}$ be a finite set of alternatives with $m \geq 3$. Let $N = \{1, 2, \dots, n\}$ be a finite set of voters with $n \geq 2$. Each voter i has a preference ordering P_i (i.e., a complete, transitive and antisymmetric binary relation) over the alternatives. We interpret $a_s P_i a_t$ as "a_s is strictly preferred to a_t according to P_i". For each $1 \leq k \leq m$, $r_k(P_i)$ denotes the kth ranked alternative in P_i. We use the following notational convention: $P_i = (a_k\, a_s\, a_t\, \cdots)$ refers to a preference ordering where a_k is first-ranked, a_s is second-ranked, and a_t is third-ranked, while the rest of the rankings in P_i are arbitrary.

We denote the set of all preference orderings by $\mathbb{P}$, which we call **the complete domain**. A domain $\mathbb{D}$ is a subset of $\mathbb{P}$. We say that two distinct preferences $P_i, P'_i \in \mathbb{D}$ are **adjacent**, denoted $P_i \sim P'_i$, if there exist $a_s, a_t \in A$ such that (i) $r_k(P_i) = r_{k+1}(P'_i) = a_s$ and $r_k(P'_i) = r_{k+1}(P_i) = a_t$ for some $1 \leq k \leq m-1$, and (ii) $r_l(P_i) = r_l(P'_i)$ for all $l \notin \{k, k+1\}$. In other words, alternatives a_s and a_t are consecutively ranked in both P_i and P'_i and are swapped between the two preferences, while the ordering of all remaining alternatives is unchanged. In this case, we say alternatives a_s and a_t are *locally switched* between P_i and P'_i. Given distinct $P_i, P'_i \in \mathbb{D}$, a sequence of preferences $\{P_i^k\}_{k=1}^t \subseteq \mathbb{D}$ is called a **path** connecting P_i and P'_i if $P_i^1 = P_i$, $P_i^t = P'_i$ and $P_i^k \sim P_i^{k+1}$ for all $k = 1, \ldots, t-1$. Two preferences P_i, P'_i are **completely reversed** if for all $a_s, a_t \in A$, we have $[a_s P_i a_t] \Leftrightarrow [a_t P'_i a_s]$.

A domain $\mathbb{D}$ is **minimally rich** if for each $a_k \in A$, there exists a preference $P_i \in \mathbb{D}$ such that $r_1(P_i) = a_k$. Throughout the paper, we assume the domain in question is minimally rich. A preference profile is an n-tuple of preferences, i.e., $P = (P_1, P_2, \ldots, P_n) = (P_i, P_{-i}) \in \mathbb{D}^n$.

Let $\Delta(A)$ denote the space of all lotteries over A. An element $\lambda \in \Delta(A)$ is a lottery or a probability distribution over A, where $\lambda(a_k)$ denotes the probability received by alternative a_k. For notational convenience, we let e_{a_k} denote the degenerate lottery where alternative a_k receives probability one. A **Random Social Choice Function** (or RSCF) is a map $\phi : \mathbb{D}^n \to \Delta(A)$ which associates each preference profile to a lottery. Let $\phi_{a_k}(P)$ denote the probability assigned to a_k by ϕ at the preference profile P. If a RSCF selects a degenerate lottery at every preference profile, it is called a **Deterministic Social Choice Function** (or DSCF). More formally, a DSCF is a mapping $f : \mathbb{D}^n \to A$.

In this paper, we impose two basic axioms on RSCFs: unanimity and strategy-proofness. A RSCF $\phi : \mathbb{D}^n \to \Delta(A)$ is **unanimous** if for all $P \in \mathbb{D}^n$ and $a_k \in A$, $[r_1(P_i) = a_k \text{ for all } i \in N] \Rightarrow [\phi(P) = e_{a_k}]$. We adopt the first-order stochastic dominance notion of strategy-proofness proposed by [57]. This requires the lottery from truthtelling stochastically dominate the lottery obtained by any misrepresentation by any voter at any possible profile of other voters' preferences. Formally, a RSCF $\phi : \mathbb{D}^n \to \Delta(A)$ is **strategy-proof** if for all $i \in N$, $P_i, P'_i \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{n-1}$, $\phi(P_i, P_{-i})$ stochastically dominates $\phi(P'_i, P_{-i})$ according to P_i, i.e., $\sum_{t=1}^k \phi_{r_t(P_i)}(P_i, P_{-i}) \geq \sum_{t=1}^k \phi_{r_t(P_i)}(P'_i, P_{-i})$ for all $k = 1, \ldots, m$. In addition, a RSCF $\phi : \mathbb{D}^n \to \Delta(A)$ satisfies the **tops-only property** if for all $P, P' \in \mathbb{D}^n$, we have $[r_1(P_i) = r_1(P'_i) \text{ for all } i \in N] \Rightarrow [\phi(P) = \phi(P')]$. In other words, the tops-only property ensures that the social outcome at each preference profile depends only on the first-ranked alternatives at that preference profile.

An important class of unanimous and strategy-proof RSCFs is the class of random dictatorships. Formally, a RSCF $\phi : \mathbb{D}^n \to \Delta(A)$ is a **random dictatorship** if there exists a "dictatorial coefficient" $\varepsilon_i \geq 0$ for each $i \in N$ with $\sum_{i \in N} \varepsilon_i = 1$ such that $\phi(P) = \sum_{i \in N} \varepsilon_i \, e_{r_1(P_i)}$ for all $P \in \mathbb{D}^n$. In particular, if $\varepsilon_i = 1$ for some $i \in N$, the random dictatorship degenerates to a *dictatorship*. It is evident that every random dictatorship is a *mixture* (equivalently, a convex combination) of dictatorships. [57] showed that every unanimous and strategy-proof RSCF on the complete domain $\mathbb{P}$ is a random dictatorship.

An important restricted domain is the domain of single-peaked preferences [19, 72]. A preference P_i is **single-peaked** w.r.t. a prior order $\prec$ over A if for all $a_s, a_t \in A$, we have $[a_s \prec a_t \prec r_1(P_i) \text{ or } r_1(P_i) \prec a_t \prec a_s] \Rightarrow [a_t P_i a_s]$. Let $\mathbb{D}_{\prec}$ denote **the single-peaked domain** which contains all single-peaked preferences w.r.t. $\prec$. Whenever we do not mention the prior order $\prec$, we assume that it is the *natural order*, $a_{k-1} \prec a_k$ for all $k = 2, \ldots, m$. For notational convenience, let $a_s \preceq a_t$ denote either $a_s \prec a_t$ or $a_s = a_t$, and $[a_s, a_t] = \{a_k \in A : a_s \preceq a_k \preceq a_t\}$ denote the set of alternatives between a_s and a_t on $\prec$, provided $a_s \preceq a_t$. Note that the single-peaked domain $\mathbb{D}_{\prec}$ contains a pair of completely reversed preferences $\underline{P}_i = (a_1 \cdots a_{k-1}\, a_k \cdots a_m)$ and $\overline{P}_i = (a_m \cdots a_k\, a_{k-1} \cdots a_1)$.[1]

6.3 Hybrid Domains

Hybrid domains are supersets of single-peaked domains where single-peakedness may be violated over a subset of alternatives that lie in the "middle" of the alternative set. We use the term "hybrid" to emphasize the coexistence of such violations, with other features of single-peakedness.

Consider the natural order $\prec$ over A. Fix two alternatives $a_{\underline{k}}$ and $a_{\overline{k}}$ with $a_{\underline{k}} \prec a_{\overline{k}}$, which we refer to as the *left threshold* and the *right threshold*, respectively. We define three subsets of A using these two thresholds: **Left Interval** $L = [a_1, a_{\underline{k}}]$, **Right Interval** $R = [a_{\overline{k}}, a_m]$ and **Middle Interval** $M = [a_{\underline{k}}, a_{\overline{k}}]$.[2] In what follows, we present the structure of preference orderings in a hybrid domain.

Consider a preference ordering whose peak belongs to M (see the first diagram of Figure 6.3.1). The ranking of the alternatives in M is completely arbitrary, while the ranking of the alternatives in L and R follows the conventional single-peakedness restriction w.r.t. $\prec$. In other words, the only restriction that the preference ordering satisfies is that preference declines as one moves from $a_{\underline{k}}$ towards a_1, or from $a_{\overline{k}}$ towards a_m. Note that this allows some alternatives in L or R be ranked above some alternatives in M.

Next, consider a preference ordering whose peak belongs to L (see the second diagram of Figure 6.3.1). The ranking of the alternatives in L and R follows single-peakedness w.r.t. $\prec$. In other words, preference declines as one moves from the peak towards a_1 or $a_{\underline{k}}$, or moves from $a_{\overline{k}}$ towards a_m. Furthermore, all alternatives in M are ranked below $a_{\underline{k}}$ in an arbitrary manner. Notice that an alternative in R may be ranked above some alternative in M, but can never be ranked above $a_{\underline{k}}$. For a preference ordering with the peak in R, the restriction is analogous.

[1] The notation $\underline{P}_i = (a_1 \cdots a_{k-1}\, a_k \cdots a_m)$ and $\overline{P}_i = (a_m \cdots a_k\, a_{k-1} \cdots a_1)$ denote the preferences $\underline{P}_i$ and $\overline{P}_i$ where $a_{k-1}\underline{P}_i a_k$ and $a_k \overline{P}_i a_{k-1}$ for all $k = 2, \ldots, m$.

[2] Note that $L \cap M = \{a_{\underline{k}}\}$, $R \cap M = \{a_{\overline{k}}\}$ and $L \cap R = \emptyset$.

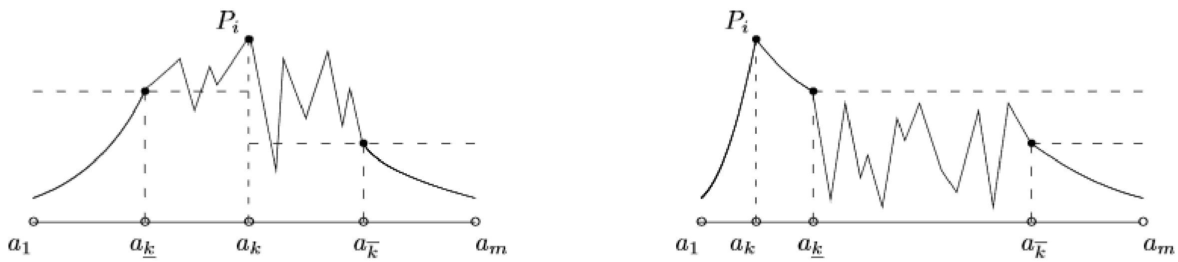


Figure 6.3.1: A graphic illustration of hybrid preference orderings

The formal definition of hybrid domains is given below.

Definition 6.3.1 *Let $\prec$ be the natural order over A and let $1 \leq \underline{k} < \overline{k} \leq m$. A preference P_i is called $(\underline{k}, \overline{k})$-**hybrid** if the following two conditions are satisfied:*

(i) *For all $a_r, a_s \in L$ or $a_r, a_s \in R$, $[a_r \prec a_s \prec r_1(P_i)$ or $r_1(P_i) \prec a_s \prec a_r] \Rightarrow [a_s P_i a_r]$.*

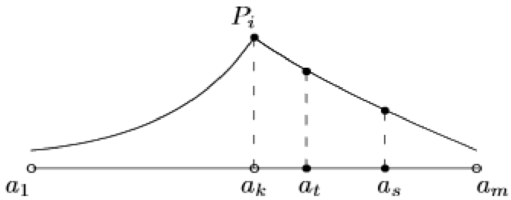
(ii) *$[r_1(P_i) \in L] \Rightarrow [a_{\underline{k}} P_i a_r$ for all $a_r \in M$ with $a_r \neq a_{\underline{k}}]$ and*
$[r_1(P_i) \in R] \Rightarrow [a_{\overline{k}} P_i a_s$ for all $a_s \in M$ with $a_s \neq a_{\overline{k}}]$.

Let $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ denote **the** $(\underline{k}, \overline{k})$-**hybrid domain** which contains *all* $(\underline{k}, \overline{k})$-hybrid preference orderings. Note that $\mathbb{D}_{\prec} \subseteq \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ for all $1 \leq \underline{k} < \overline{k} \leq m$, and $\mathbb{D}_{\mathrm{H}}(\underline{k}', \overline{k}') \subseteq \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ for all $\underline{k} \leq \underline{k}' < \overline{k}' \leq \overline{k}$.

Now, we explain the relation of hybrid domains with five important preference domains studied in the literature.

The single-peaked domain: Consider a hybrid domain $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ with $\overline{k} - \underline{k} = 1$. This means $M = \{a_{\underline{k}}, a_{\overline{k}}\}$ and $L \cup R = A$. Then, conditions (i) and (ii) of Definition 6.3.1 boil down to the single-peakedness restriction (see the first diagram of Figure 6.3.2), and consequently, $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ coincides with the single-peaked domain $\mathbb{D}_{\prec}$.

The complete domain: Consider the hybrid domain $\mathbb{D}(\underline{k}, \overline{k})$ with $\overline{k} - \underline{k} = m - 1$ (equivalently, $\underline{k} = 1$ and $\overline{k} = m$). This means $L = \{a_{\underline{k}}\}$, $R = \{a_{\overline{k}}\}$, and $M = A$. Then, both the conditions of Definition 6.3.1 become vacuous. In other words, no restriction is imposed on the preference orderings (see the second diagram of Figure 6.3.2) in $\mathbb{D}_{\mathrm{H}}(1, m)$, and consequently, $\mathbb{D}_{\mathrm{H}}(1, m)$ becomes the complete domain $\mathbb{P}$.



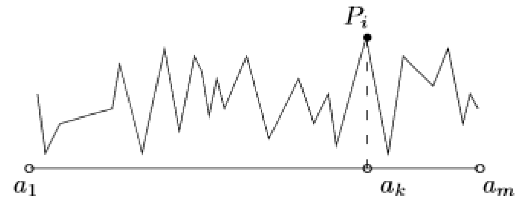


Figure 6.3.2: Two hybrid preferences with $\overline{k} - \underline{k} = 1$ and $\overline{k} - \underline{k} = m - 1$

Multiple single-peaked domains: Hybrid domains generalize the notion of multiple single-peaked domains introduced by [88]. Let $\Omega = \{\prec_r\}_{r=1}^{s}$, $s \geq 2$ be a collection of linear orders over A. For each order $\prec_r$ in Ω, let the single-peaked domain w.r.t. $\prec_r$ be denoted by $\mathbb{D}_{\prec_r}$. Then, the union $\mathbb{D}_\Omega = \cup_{r=1}^{s} \mathbb{D}_{\prec_r}$ is called *the multiple single-peaked domain w.r.t.* Ω.[3]

One can first identify the maximum common left part L_Ω of all orders $\{\prec_r\}_{r=1}^{s}$ over A, and relabel all alternatives of $L_\Omega = \{a_1, \ldots, a_{\underline{k}}\}$ (if $L_\Omega \neq \emptyset$), i.e., for all orders $\prec_r$ in Ω, after relabeling, either $a_1 \prec_r \cdots \prec_r a_{\underline{k}} \prec_r a_p$ for all $a_p \in A \backslash L_\Omega$, or $a_p \prec_r a_{\underline{k}} \prec_r \cdots \prec_r a_1$ for all $a_p \in A \backslash L_\Omega$ holds. Second, one can symmetrically identify and relabel the maximum common right part $R_\Omega = \{a_{\overline{k}}, \ldots, a_m\} \subseteq A \backslash L_\Omega$ of all orders $\{\prec_r\}_{r=1}^{s}$ over A (if $R_\Omega \neq \emptyset$) and finally arbitrarily relabel all remaining alternatives as $a_{\underline{k}+1}, \ldots, a_{\overline{k}+1}$. We correspondingly relabel all alternatives in the preferences of $\mathbb{D}_\Omega$. Then, after setting $a_{\underline{k}}$ and $a_{\overline{k}}$ as two thresholds, it is clear that each preference ordering in $\mathbb{D}_\Omega$ is $(\underline{k}, \overline{k})$-hybrid.[4] Usually, $\mathbb{D}_\Omega$ is "strictly" contained in $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$. This will be illustrated in the following example.

Note that by definition, a multiple single-peaked domain cannot be a single-peaked domain, whereas a hybrid domain can be single-peaked for a suitable choice of thresholds (when $\overline{k} - \underline{k} = 1$).

Multidimensional single-peaked domains in voting under constraints: We provide an example to show that hybrid preferences arise from a model of voting under constraints studied in [13].

Let $X = X_1 \times X_2$, $X_1 = \{1, 2, 3, 4, 5\}$ and $X_2 = \{1, 2, 3\}$, where both X_1 and X_2 are ordered according to the natural order, denoted by $<_1$ and $<_2$. A preference P_i, with $r_1(P_i) = x$, is *multidimensional single-peaked* over X w.r.t. $<_1$ and $<_2$ if for all $y, z \in X$, we have $[z_k \leq_k y_k \leq_k x_k$ or $x_k \leq_k y_k \leq_k z_k$ for both $k = 1, 2] \Rightarrow [y P_i z]$. Meanwhile, let $A = \{a_1, a_2, a_3, a_4, a_5, a_6\} \subset X$ be the set of *feasible* alternatives, which are depicted by the black nodes in Figure 6.3.3 below.

[3] If two orders $\prec_1$ and $\prec_2$ are completely reversed, the two single-peaked domains $\mathbb{D}_{\prec_1}$ and $\mathbb{D}_{\prec_2}$ become identical. Therefore, we assume that there is no pair of orders in Ω that are completely reversed.

[4] As Ω contains at least two orders and no pair of orders are completely reversed, it must be the case that $\overline{k} - \underline{k} > 1$ when $L_\Omega \neq \emptyset$ and $R_\Omega \neq \emptyset$. If $L_\Omega = \emptyset$ and $R_\Omega \neq \emptyset$, then $\mathbb{D}_\Omega$ is $(1, \overline{k})$-hybrid, while if $L_\Omega \neq \emptyset$ and $R_\Omega = \emptyset$, then $\mathbb{D}_\Omega$ is $(\underline{k}, m)$-hybrid. If both L_Ω and R_Ω are empty sets, then $\mathbb{D}_\Omega \subseteq \mathbb{P} = \mathbb{D}_{\mathrm{H}}(1, m)$ and $\mathbb{D}_\Omega \not\subseteq \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ for any other $\underline{k}$ and $\overline{k}$.

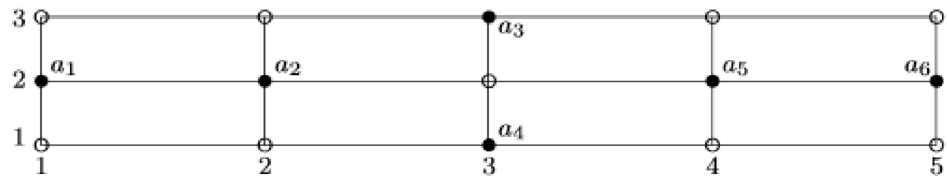


Figure 6.3.3: The Cartesian product of $<_1$ and $<_2$

Note that in a multidimensional single-peaked preference, (i) if a_1 is first-ranked, then a_2 must be second-ranked within A, and a_5 is preferred to a_6; if a_2 is first-ranked, then a_5 is preferred to a_6, and (ii) if a_3 is first-ranked, then a_2 is better than a_1, and a_5 is better than a_6. Analogous preference restrictions over the ranking of feasible alternatives are observed for multidimensional single-peaked preferences with peaks a_6, a_5 and a_4. These two observations coincide with the two preference restrictions in the definition of the $(2, 5)$-hybrid domain $\mathbb{D}_{\mathrm{H}}(2, 5)$ if we rearrange all feasible alternatives according to the natural order $\prec$. In conclusion, when we restrict attention to all multidimensional single-peaked preferences whose peaks are feasible, the domain of induced preferences over the feasible alternatives is identical to $\mathbb{D}_{\mathrm{H}}(2, 5)$.

We may alternatively extract the two linear orders $\prec_1 = (a_1a_2a_3a_4a_5a_6)$ and $\prec_2 = (a_1a_2a_4a_3a_5a_6)$ over feasible alternatives from Figure 6.3.3, and induce the multiple single-peaked domain $\mathbb{D}_{\prec_1} \cup \mathbb{D}_{\prec_2}$. Notice that $\mathbb{D}_{\prec_1} \cup \mathbb{D}_{\prec_2}$ is strictly contained in $\mathbb{D}_{\mathrm{H}}(2, 5)$. For instance, a_3 and a_4 are always ranked above a_5 and a_6 in every preference of $\mathbb{D}_{\prec_1} \cup \mathbb{D}_{\prec_2}$ that has peak a_1, whereas we can identify a particular multidimensional single-peaked preference with peak a_1 that induces the preference ordering over feasible alternatives as $(a_1a_2a_5a_6a_3a_4)$.

This illustrates the additional flexibility that a hybrid domain affords, and may be useful for formulations (for example, political economy or public goods location models) that seek to reduce a model where the underlying issues are multidimensional, to one where the preference restriction is generated via a one dimensional order over alternatives.

Semi-single-peaked domains: The notion of semi-single-peaked domains was introduced by [34]. Consider the natural order $\prec$ and fix *one* threshold alternative. The semi-single-peakedness restriction on a preference requires that (i) the usual single-peakedness restriction prevail in the interval between the peak and the threshold, and (ii) each alternative located beyond the threshold be ranked below the threshold.

One can extend the semi-single-peakedness notion by adding more thresholds and requiring preferences to be semi-single-peaked w.r.t. each threshold alternative. In particular, suppose that there are two distinct thresholds $a_{\underline{k}}$ and $a_{\overline{k}}$ with $a_{\underline{k}} \prec a_{\overline{k}}$. Consider a preference P_i with $a_{\underline{k}} \preceq r_1(P_i) \preceq a_{\overline{k}}$. If P_i is $(\underline{k}, \overline{k})$-hybrid, then the usual single-peakedness restriction prevails on the left and right intervals, and no restriction is imposed on the ranking of the alternatives in the middle interval (see the first diagram of Figure 6.3.4). On the contrary, if P_i is semi-single-peaked w.r.t. both $a_{\underline{k}}$ and $a_{\overline{k}}$, then the single-peakedness restriction prevails on the middle interval but fails on the left and right intervals (see the second diagram of Figure 6.3.4). Thus, the notions of hybrid preferences and semi-single-peaked preferences are not entirely compatible with each other.

[34] show that under a mild domain richness condition, semi-single-peakedness is necessary and sufficient for the existence of a unanimous, anonymous, tops-only and strategy-proof DSCF.[5] This, in particular, implies that when $\overline{k} - \underline{k} > 1$, the $(\underline{k}, \overline{k})$-hybrid domain cannot admit such a well-behaved strategy-proof DSCF.

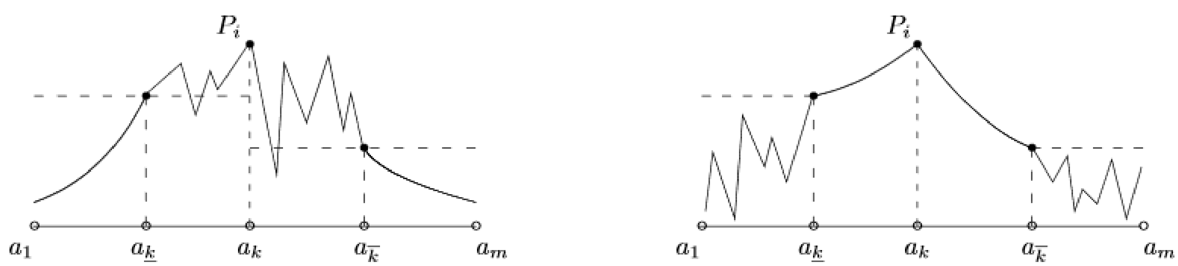


Figure 6.3.4: A hybrid preference *v.s.* a semi-single-peaked preference

6.4 Restricted Probabilistic Fixed Ballot Rules

In this section, we introduce the notion of Restricted Probabilistic Fixed Ballot Rules (or RPFBRs). [46] introduce the notion of Probabilistic Fixed Ballot Rules (or PFBR); RPFBRs are special cases of these rules.

A PFBR ϕ is based on a collection of parameters $(\beta_S)_{S \subseteq N}$, called **probabilistic ballots**. Each probabilistic ballot β_S, which is associated to the coalition $S \subseteq N$, is a probability distribution on A satisfying the following two properties.

- **Ballot unanimity**: β_N assigns probability 1 to a_m, and $\beta_\emptyset$ assigns probability 1 to a_1.

- **Monotonicity**: probabilities according to β_S move towards right as S gets bigger, i.e., $\beta_S([a_k, a_m]) \leq \beta_T([a_k, a_m])$ for all $S \subset T$ and all $a_k \in A$.[6]

[5] Recently, [29] introduce the semilattice single-peaked domain which significantly generalizes semi-single-peakedness, and [23] characterize all unanimous, anonymous, tops-only and strategy-proof DSCFs on the semilattice single-peaked domain.

[6] For a subset B of A, we denote the probability of B according to β_S by $\beta_S(B)$.

For an example, suppose that there are two agents $\{1, 2\}$ and four alternatives $\{a_1, a_2, a_3, a_4\}$. Then, a choice of probabilistic ballots could be $\beta_{\emptyset} = (1, 0, 0, 0)$, $\beta_{\{1\}} = (0.5, 0.2, 0.1, 0.2)$, $\beta_{\{2\}} = (0.4, 0.3, 0.2, 0.1)$ and $\beta_N = (0, 0, 0, 1)$. Here, we denote by (x, y, w, z) a probability distribution where a_1, a_2, a_3 and a_4 receive probabilities x, y, w and z, respectively.

A PFBR ϕ w.r.t. a collection of probabilistic ballots $(\beta_S)_{S \subseteq N}$ works as follows. For each $1 \leq k \leq m$, let $S(k, P) = \{i \in N : a_k \preceq r_1(P_i)\}$ be the set of agents whose peaks are not to the left of a_k. Consider an arbitrary preference profile P and an arbitrary alternative a_k. We induce the probabilities $\beta_{S(k,P)}([a_k, a_m])$ and $\beta_{S(k+1,P)}([a_{k+1}, a_m])$. If $a_k = a_m$, then set $\beta_{S(m+1,P)}([a_{m+1}, a_m]) = 0$. The probability of the alternative a_k selected at the preference profile P is defined as the difference between these two probabilities, i.e., $\phi_{a_k}(P) = \beta_{S(k,P)}([a_k, a_m]) - \beta_{S(k+1,P)}([a_{k+1}, a_m])$.[7] For an example, consider the PFBR ϕ w.r.t. the parameters presented in the predecessor paragraph. Consider a preference profile $P = (P_1, P_2)$ where $r_1(P_1) = a_2$ and $r_1(P_2) = a_4$. Then, we calculate

$$\begin{aligned}
&\phi_{a_1}(P) = \beta_{S(1,P)}([a_1, a_4]) - \beta_{S(2,P)}([a_2, a_4]) = \beta_N([a_1, a_4]) - \beta_N([a_2, a_4]) = 0,\\
&\phi_{a_2}(P) = \beta_{S(2,P)}([a_2, a_4]) - \beta_{S(3,P)}([a_3, a_4]) = \beta_N([a_2, a_4]) - \beta_{\{2\}}([a_3, a_4]) = 1 - 0.3 = 0.7,\\
&\phi_{a_3}(P) = \beta_{S(3,P)}([a_3, a_4]) - \beta_{S(4,P)}([a_4, a_4]) = \beta_{\{2\}}([a_3, a_4]) - \beta_{\{2\}}([a_4, a_4]) = 0.3 - 0.1 = 0.2, \text{ and}\\
&\phi_{a_4}(P) = \beta_{S(4,P)}([a_4, a_4]) - 0 = \beta_{\{2\}}([a_4, a_4]) = 0.1.
\end{aligned}$$

Clearly, the PFBR satisfies the tops-only property.

It is worth mentioning that the probabilistic ballot β_S for a coalition $S \subseteq N$ represents the outcome of ϕ at the "boundary profile" where agents in S have the preference $\overline{P}_i = (a_m \cdots a_k\, a_{k-1} \cdots a_1)$, while the others have the preference $\underline{P}_i = (a_1 \cdots a_{k-1}\, a_k \cdots a_m)$. For ease of presentation, we call such a preference profile a *S-boundary profile*.[8] If a PFBR ϕ is unanimous, then it follows that $\beta_{\emptyset}$ assigns probability 1 to a_1 and β_N assigns probability 1 to a_m, which in turn implies ballot unanimity. In what follows, we argue that if ϕ is strategy-proof, then $(\beta_S)_{S \subseteq N}$ must be monotonic. Consider a proper subset $S \subset N$ and $i \in N \setminus S$. Let P and P' be the S-boundary and $S \cup \{i\}$-boundary profiles, respectively. In other words, only agent i changes her preference $\overline{P}_i$ in the $S \cup \{i\}$-boundary profile to $\underline{P}_i$. Strategy-proofness of ϕ implies that the probability of each upper contour set of $\overline{P}_i$ is weakly increased from $\phi(P)$ to $\phi(P')$. Since the interval $[a_k, a_m]$ coincides with the upper contour set of a_k at $\overline{P}_i$, it follows that $\beta_S([a_k, a_m]) \leq \beta_{S \cup \{i\}}([a_k, a_m])$. Monotonicity of $(\beta_S)_{S \subseteq N}$ follows from the repeated application of this argument.

[7] Since $S(k+1, P) \subseteq S(k, P)$ and $[a_{k+1}, a_m] \subset [a_k, a_m]$, monotonicity ensures $\phi_{a_k}(P) = \beta_{S(k,P)}([a_k, a_m]) - \beta_{S(k+1,P)}([a_{k+1}, a_m]) \geq 0$. Moreover, note that $\sum_{k=1}^{m} \phi_{a_k}(P) = \sum_{k=1}^{m} \beta_{S(k,P)}([a_k, a_m]) - \beta_{S(k+1,P)}([a_{k+1}, a_m]) = \beta_{S(1,P)}([a_1, a_m]) = 1$. Therefore, $\phi(P) \in \Delta(A)$ and ϕ is a well defined RSCF.

[8] Note that for every $S \subseteq N$, there is a unique S-boundary profile.

Note that the outcome of a PFBR at any preference profile is uniquely determined by its outcomes at boundary profiles. It is shown in [46] that every PFBR is unanimous and strategy-proof on the single-peaked domain. Thus, unanimity and strategy-proofness of a PFBR at every preference profile can be ensured by imposing those only on the boundary profiles.

The deterministic versions of PFBRs can be obtained by additionally requiring the probabilistic ballots be degenerate, i.e., $\beta_S(a_k) \in \{0, 1\}$ for all $S \subseteq N$ and $a_k \in A$. These DSCFs were introduced by [72]; we refer to these as *Fixed Ballot Rules* (or FBRs).[9] [72] showed that a DSCF is unanimous, tops-only and strategy-proof on the single-peaked domain if and only if it is an FBR. It can be easily verified that an arbitrary mixture of FBRs is unanimous and strategy-proof on the single-peaked domain, and is a PFBR. Theorem 3 of [81] and Theorem 5 of [87] prove that the converse is also true.

Below, we present the formal definition of PFBRs.

Definition 6.4.1 *A RSCF $\phi : \mathbb{D}^n \to \Delta(A)$ is called a* ***Probabilistic Fixed Ballot Rule*** *(or* ***PFBR****) if there exists a collection of probabilistic ballots $(\beta_S)_{S \subseteq N}$ satisfying ballot unanimity and monotonicity such that for all $P \in \mathbb{D}^n$ and $a_k \in A$, we have*

$$\phi_{a_k}(P) = \beta_{S(k,P)}([a_k, a_m]) - \beta_{S(k+1,P)}([a_{k+1}, a_m]),$$

where $\beta_{S(m+1,P)}([a_{m+1}, a_m]) = 0$.

We are now ready to present the notion of RPFBRs. The structure of a $(\underline{k}, \overline{k})$-RPFBR depends on the values of $\underline{k}$ and $\overline{k}$. If $\overline{k} - \underline{k} = 1$, then the $(\underline{k}, \overline{k})$-RPFBR is the same as a PFBR. However, if $\overline{k} - \underline{k} > 1$, then the $(\underline{k}, \overline{k})$-RPFBR is a PFBR whose probabilistic ballots satisfy the following additional restriction: for each agent $i \in N$, there is a "conditional dictatorial coefficient" $\varepsilon_i \geq 0$ with $\sum_{i \in N} \varepsilon_i = 1$ such that for all $S \subseteq N$, $\beta_S([a_{\overline{k}}, a_m]) = \sum_{i \in S} \varepsilon_i$ and $\beta_S([a_1, a_{\underline{k}}]) = \sum_{i \in N \setminus S} \varepsilon_i$. Note that this, in particular, means that *no* β_S assigns positive probability to an alternative that lies (strictly) between $a_{\underline{k}}$ and $a_{\overline{k}}$, i.e., $\beta_S(a_k) = 0$ for all $S \subseteq N$ and $a_k \in [a_{\underline{k}+1}, a_{\overline{k}-1}]$. In what follows, we present an example of a RPFBR.

Example 6.4.2 Let $N = \{1, 2, 3\}$ and $A = \{a_1, a_2, a_3, a_4, a_5\}$. Take $\underline{k} = 2$ and $\overline{k} = 4$, and consider the $(2, 4)$-hybrid domain $\mathbb{D}_{\text{H}}(2, 4)$. Let $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{1}{3}$. Consider the 8 probabilistic ballots in Table 6.4.1, where both ballot unanimity and monotonicity can be easily verified. Note that they also satisfy the property that $\beta_S([a_4, a_5]) = \sum_{i \in S} \varepsilon_i$ and $\beta_S([a_1, a_2]) = \sum_{i \in N \setminus S} \varepsilon_i$ for all $S \subseteq N$. Therefore, the PFBR w.r.t. these probabilistic ballots is a $(2, 4)$-RPFBR. □

[9] [72] called these Augmented Median Voter Rules, while [12] called these Generalized Median Voter Schemes. For an FBR ϕ, the subtraction form in Definition 6.4.1 can be simplified to a max-min form [see Definition 10.3 in 76]. [72] originally defined an augmented median voter rule in the min-max form which can be equivalently translated to a max-min form.

	$\beta_\emptyset$	$\beta_{\{1\}}$	$\beta_{\{2\}}$	$\beta_{\{3\}}$	$\beta_{\{1,2\}}$	$\beta_{\{1,3\}}$	$\beta_{\{2,3\}}$	β_N
a_1	1	1/3	1/3	1/3	1/3	1/3	1/3	0
a_2	0	1/3	1/3	1/3	0	0	0	0
a_3	0	0	0	0	0	0	0	0
a_4	0	0	0	0	1/3	1/3	1/3	0
a_5	0	1/3	1/3	1/3	1/3	1/3	1/3	1

Table 6.4.1: The probabilistic ballots $(\beta_S)_{S\subseteq N}$

Below, we present a formal definition of RPFBRs.

Definition 6.4.3 *Let $1 \leq \underline{k} < \overline{k} \leq m$. A PFBR ϕ w.r.t. probabilistic ballots $(\beta_S)_{S\subseteq N}$ is called a $(\underline{k}, \overline{k})$-**Restricted Probabilistic Fixed Ballots Rule** (or $(\underline{k}, \overline{k})$-**RPFBR**) if $\overline{k} - \underline{k} > 1$ implies that for each $i \in N$, there exists $\varepsilon_i \geq 0$ with $\sum_{i\in N} \varepsilon_i = 1$ such that for all $S \subseteq N$, $\beta_S([a_{\overline{k}}, a_m]) = \sum_{i\in S} \varepsilon_i$ and $\beta_S([a_1, a_{\underline{k}}]) = \sum_{i\in N\setminus S} \varepsilon_i$.*

It is worth mentioning that when $\overline{k} - \underline{k} > 1$, at the preference profiles where all peaks are in the middle interval $M = [a_{\underline{k}}, a_{\overline{k}}]$, a $(\underline{k}, \overline{k})$-RPFBR behaves like a random dictatorship where each agent i's dictatorial coefficient is ε_i. More formally, if ϕ is a $(\underline{k}, \overline{k})$-RPFBR, then $\phi(P) = \sum_{i\in N} \varepsilon_i\, e_{r_1(P_i)}$ for all preference profile P such that $r_1(P_i) \in [a_{\underline{k}}, a_{\overline{k}}]$ for all $i \in N$. Therefore, in the extreme case where $\underline{k} = 1$ and $\overline{k} = m$, the $(1, m)$-RPFBR reduces to a random dictatorship. For ease of presentation, we call the condition satisfied by the probabilistic ballots $(\beta_S)_{S\subseteq N}$ in Definition 6.4.3 the **constrained random-dictatorship condition**.

6.5 A Characterization of Unanimous and Strategy-proof RSCFs on Hybrid Domains

In this section, we provide a characterization of unanimous and strategy-proof RSCFs on hybrid domains. Theorem 6.5.1 says that a RSCF ϕ is unanimous and strategy-proof on the $(\underline{k}, \overline{k})$-hybrid domain if and only if it is a $(\underline{k}, \overline{k})$-RPFBR. [46] consider the case of continuum of alternatives (for instance, the interval $[0, 1]$) and show that a RSCF is unanimous and strategy-proof on the single-peaked domain if and only if it is a PFBR. Since when $\overline{k} - \underline{k} = 1$, the $(\underline{k}, \overline{k})$-hybrid domain boils down to the single-peaked domain and the $(\underline{k}, \overline{k})$-RPFBR becomes a PFBR, Theorem 6.5.1 implies their result in the case of finite alternatives.

Theorem 6.5.1 *Let $1 \leq \underline{k} < \overline{k} \leq m$. A RSCF $\phi : \left[\mathbb{D}_H(\underline{k}, \overline{k})\right]^n \to \Delta(A)$ is unanimous and strategy-proof if and only if it is a $(\underline{k}, \overline{k})$-RPFBR.*

We present a formal proof of Theorem 6.5.1 in Appendix 6.8. Here, we provide an intuitive explanation. The "if part" of the theorem, i.e., the fact that every RPFBR on a hybrid domain is unanimous and strategy-proof, intuitively follows from the observations: (i) the $(\underline{k}, \overline{k})$-hybrid domain satisfies single-peakedness on the intervals $[a_1, a_{\underline{k}}]$ and $[a_{\overline{k}}, a_m]$, and (ii) the RPFBR behaves like a PFBR over these intervals. For the "only-if part", we first show how in a two-voter setting a PFBR fails to satisfy strategy-proofness on the $(\underline{k}, \overline{k})$-hybrid domain if any of its probabilistic ballots assigns a positive probability to some alternative in the interval $[a_{\underline{k}+1}, a_{\overline{k}-1}]$.

Consider the model with two agents. Suppose that some probabilistic ballot of ϕ, say $\beta_{\{2\}}$, assigns a strictly positive probability to some alternative $a_k \in [a_{\underline{k}+1}, a_{\overline{k}-1}]$. First, by the definition of the $(\underline{k}, \overline{k})$-hybrid domain, there is a preference where a_1 is the first-ranked alternative and $a_{\overline{k}}$ is preferred to a_k. Correspondingly, consider a preference profile where agent 1 has such a preference and the first-ranked alternative of agent 2 is $a_{\overline{k}}$. By the definition of PFBR, the probability of a_k at this profile equals $\beta_{\{2\}}(a_k)$, which is strictly positive by our assumption. However, using unanimity agent 1 can manipulate by misreporting a preference that has $a_{\overline{k}}$ as the first-ranked alternative.[10]

An important point to note is that the aforementioned argument only indicates that a PFBR which is strategy-proof on the $(\underline{k}, \overline{k})$-hybrid domain is a $(\underline{k}, \overline{k})$-RPFBR. In order to complete the verification of the "only-if part", a crucial step in the proof of Theorem 6.5.1 is to show that every unanimous and strategy-proof RSCF on the hybrid domain is some PFBR.

6.5.1 Decomposability of Anonymous RPFBRs

In this section, we investigate the decomposability property of RSCFs. We say that a unanimous and strategy-proof RSCF is *decomposable* if it can be expressed as a mixture (equivalently, a convex combination) of finitely many unanimous and strategy-proof DSCFs. Formally, a unanimous and strategy-proof RSCF $\phi : \mathbb{D}^n \to \Delta(A)$ is **decomposable** if there exist finitely many unanimous and strategy-proof DSCFs $f^k : \mathbb{D}^n \to A$, $k = 1, \ldots, q$ and weights $\alpha^1, \ldots, \alpha^q > 0$ with $\sum_{k=1}^{q} \alpha^k = 1$, such that $\phi(P) = \sum_{k=1}^{q} \alpha^k e_{f^k(P)}$ for all $P \in \mathbb{D}^n$.

Decomposability is an important property of RSCFs and has been widely investigated in a large class of domains [e.g., 54, 57, 81, 87]. As mentioned earlier, when $\overline{k} - \underline{k} = 1$, the $(\underline{k}, \overline{k})$-hybrid domain coincides with the single-peaked domain, and the $(\underline{k}, \overline{k})$-RPFBR becomes a PRBR. It is shown in [81] and [87] that every PFBR is a mixture of their deterministic counterparts. In the other extreme case where $\overline{k} - \underline{k} = m - 1$, every $(\underline{k}, \overline{k})$-RPFBR becomes a random dictatorship, which is, by definition, a mixture of dictatorships. Thus, a $(\underline{k}, \overline{k})$-RPFBR is decomposable when $\overline{k} - \underline{k} = 1$ or $\overline{k} - \underline{k} = m - 1$. However, for the

[10] Note that the strength of unanimity reduces considerably as the number of agents increases. So, the argument presented above does not extend straightforwardly to the case of arbitrary number of agents. We provide these details in our formal proof.

remaining cases $1 < \overline{k} - \underline{k} < m - 1$, we observe that decomposability fails in some RPFBRs (see Example 6.6.1 below). A complete characterization of decomposable RPFBRs in the general case, appears to be difficult.In this section, we investigate the decomposition of *anonymous* RPFBRs for the remaining cases $1 < \overline{k} - \underline{k} < m - 1$.[11]

Formally, a RSCF $\phi : \mathbb{D}^n \to \Delta(A)$ is **anonymous** if for all permutations $\sigma : N \to N$ and profile $(P_1, \dots, P_n) \in \mathbb{D}^n$, we have $\phi(P_1, \dots, P_n) = \phi(P_{\sigma(1)}, \dots, P_{\sigma(n)})$. More specifically, one can easily verify that a $(\underline{k}, \overline{k})$-RPFBR $\phi : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$ is anonymous if and only if all probabilistic ballots are invariant to the size of coalitions, i.e., for all nonempty $S, S' \subseteq N$ with $|S| = |S'|$, we have $\beta_S = \beta_{S'}$. For instance, recall the probabilistic ballots in Table 6.4.1. The corresponding RPFBR is anonymous.

We next provide a necessary and sufficient condition, *per-capita monotonicity*, for the decomposition of all anonymous RPFBRs. Consider a $(\underline{k}, \overline{k})$-RPFBR ϕ w.r.t. the probabilistic ballots $(\beta_S)_{S \subseteq N}$. Recall the left interval $L = [a_1, a_{\underline{k}}]$ and the right interval $R = [a_{\overline{k}}, a_m]$. This condition imposes two restrictions that strengthen the monotonicity requirement between the probabilistic ballots of two nonempty coalitions $S, S' \subset N$ with $S \subset S'$. The first restriction says that the average probability, $\frac{\beta_{S'}}{|S'|}$, of any interval $[a_t, a_m]$ in R for the coalition S' is at least as much as the counterpart for the coalition S, i.e., for all $a_t \in R$, $\frac{\beta_{S'}([a_t, a_m])}{|S'|} \geq \frac{\beta_S([a_t, a_m])}{|S|}$. The second restriction is the analogue of the first one. Here, we consider any interval $[a_1, a_s]$ in L and the respective complements of S' and S. Recall from the constrained random-dictatorship condition that the probabilities $\beta_{N \backslash S'}([a_1, a_s])$ and $\beta_{N \backslash S}([a_1, a_s])$ are related to the conditional dictatorial coefficients of voters in S' and S respectively. We require here that the average probability $\frac{\beta_{N \backslash S'}([a_1, a_s])}{|S'|}$ be weakly higher than $\frac{\beta_{N \backslash S}([a_1, a_s])}{|S|}$.

Definition 6.5.2 *A RPFBR $\phi : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$ satisfies* ***per-capita monotonicity*** *if, for all nonempty $S \subset S' \subset N$, $a_t \in R$ and $a_s \in L$, we have*

$$\frac{\beta_{S'}([a_t, a_m])}{|S'|} \geq \frac{\beta_J([a_t, a_m])}{|S|} \text{ and } \frac{\beta_{N \backslash S'}([a_1, a_s])}{|S'|} \geq \frac{\beta_{N \backslash S}([a_1, a_s])}{|S|}.$$

Our main theorem of this section says that per-capita monotonicity is both necessary and sufficient for the decomposability of anonymous RPFBRs. The proof of Theorem 6.5.3 is contained in Appendix 6.9.

Theorem 6.5.3 *Let $1 < \overline{k} - \underline{k} < m - 1$. Then, an anonymous $(\underline{k}, \overline{k})$-RPFBR $\phi : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$ is decomposable if and only if it satisfies per-capita monotonicity.*

To conclude this section, we observe using an example that a non-decomposable RPFBR may dominate a decomposable one in terms of admitting "social compromises". This indicates that

[11] It is important to mention that in the case $1 < \overline{k} - \underline{k} < m - 1$, Theorem 6.5.1 implies that there exists no anonymous, unanimous and strategy-proof DSCFs on the $(\underline{k}, \overline{k})$-hybrid domain. Therefore, the decomposition of an anonymous $(\underline{k}, \overline{k})$-RPFBR (if it exists) is a mixture of finitely many unanimous and strategy-proof DSCFs, all of which violate anonymity.

randomization enhances possibilities for economic design in a meaningful way, since the non-decomposable RPFBRs we characterize may allow for more flexibility in assigning probabilities to compromise alternatives.

Example 6.5.4 Let $N = \{1, 2, 3\}$ and $A = \{a_1, a_2, a_3, a_4, a_5\}$. Recall the $(2, 4)$-hybrid domain $\mathbb{D}_{\mathrm{H}}(2, 4)$ and the probabilistic ballots $(\beta_S)_{S \subseteq N}$ in Table 6.4.1. It is easy to verify that $(\beta_S)_{S \subseteq N}$ satisfy ballot unanimity, monotonicity and the constrained random-dictatorship condition when the conditional dictatorial coefficients are $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{1}{3}$, and are invariant to the size of coalitions. Therefore, the PFBR $\phi : [\mathbb{D}_{\mathrm{H}}(2, 4)]^3 \to \Delta(A)$ w.r.t. $(\beta_S)_{S \subseteq N}$ is an anonymous $(2, 4)$-RPFBR. Furthermore, it can be verified that ϕ is not decomposable as it fails to satisfy per-capita monotonicity, i.e., $\frac{\beta_{\{1,2\}}(a_5)}{|\{1,2\}|} = \frac{1}{6} < \frac{1}{3} = \frac{\beta_{\{1\}}(a_5)}{|\{1\}|}$. □

6.6 Other Results on Decomposability

Throughout this section, we restrict attention to the $(\underline{k}, \overline{k})$-hybrid domain $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ where $1 < \overline{k} - \underline{k} < m - 1$, and establish three main results related to the decomposition of $(\underline{k}, \overline{k})$-RPFBRs. First, we show that every two-voter $(\underline{k}, \overline{k})$-RPFBR is *unconditionally* decomposable (see Proposition 6.6.1). Second, we provide an example of a non-decomposable $(\underline{k}, \overline{k})$-RPFBR in the case of more than two voters, and furthermore identify a necessary condition for the decomposition of a general $(\underline{k}, \overline{k})$-RPFBR (see Proposition 6.6.2). Last, we develop a notion of dominance for comparing RPFBRs. A RPFBR is said to dominate another one in admitting compromises if the former assigns to a social compromise alternative, at least as much probability as the latter at *every* preference profile, and a strictly higher probability at *some* preference profile. Accordingly, we characterize all RPFBRs that are dominated in admitting compromises, and investigate the salience of non-decomposability by identifying a condition under which each anonymous decomposable $(\underline{k}, \overline{k})$-RPFBR is dominated by another anonymous *non-decomposable* $(\underline{k}, \overline{k})$-RPFBR in admitting compromises (see Proposition 6.6.3).

The proposition below shows by construction that every two-voter RPFBR is decomposable.

Proposition 6.6.1 *Every two-voter strategy-proof $(\underline{k}, \overline{k})$-RPFBR $\phi : [\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})]^2 \to \Delta(A)$ is decomposable.*

For the case of more than two voters, the unconditional decomposition result of Proposition 6.6.1 fails. We first provide an example to illustrate the existence of a non-decomposable $(\underline{k}, \overline{k})$-RPFBR.

Example 6.6.1 Let $n \geq 3$ and $A = \{a_1, \dots, a_m\}$. Consider the $(1, m-1)$-hybrid domain $\mathbb{D}_{\mathrm{H}}(1, m-1)$. We assign voters 1, 2 and 3 the conditional dictatorial coefficients $\varepsilon_1 = 0.3$, $\varepsilon_2 = 0.3$ and $\varepsilon_3 = 0.4$, make all other voters dummies, i.e., $\varepsilon_i = 0$ for all $i \notin \{1,2,3\}$, and specify the probabilistic ballots below:

$$\beta_S = \begin{cases} 0.4e_{a_m} + 0.3e_{a_{m-1}} + 0.3e_{a_1} & \text{if } \{1,2,3\} \cap S = \{1,3\} \text{ or } \{2,3\}, \text{ and} \\ \sum_{i \in S} \varepsilon_i \, e_{a_m} + \sum_{j \in N \setminus S} \varepsilon_j \, e_{a_1} & \text{otherwise.} \end{cases}$$

It is easy to verify that the probabilistic ballots $(\beta_S)_{S \subseteq N}$ satisfy ballot unanimity, monotonicity and the constrained random-dictatorship condition. Therefore, the corresponding PFBR $\phi : [\mathbb{D}_{\mathrm{H}}(1, m-1)]^n \to \Delta(A)$ is a $(1, m-1)$-RPFBR.

We show that ϕ is not decomposable by contradiction. Suppose not, i.e., there are $(1, m-1)$-RFBRs $f^k : [\mathbb{D}_{\mathrm{H}}(1, m-1)]^n \to A$, $k = 1, \ldots, q$, and weights $\alpha^1, \ldots, \alpha^q > 0$ with $\sum_{k=1}^q \alpha^k = 1$ such that $\phi(P) = \sum_{k=1}^q \alpha^k e_{f^k(P)}$ for all $P \in [\mathbb{D}_{\mathrm{H}}(1, m-1)]^n$. According to the coalitions $\{1\}$, $\{3\}$ and $\{1,3\}$, we induce the following contradiction:

$$\begin{aligned} 0.4 = \beta_{\{1,3\}}(a_m) &= \sum_{k=1}^q \alpha^k \mathbb{1}\big(b^k_{\{1,3\}} = a_m\big) \\ &= \sum_{k=1}^q \alpha^k \mathbb{1}\big(i^k = 1 \text{ and } b^k_{\{1,3\}} = a_m\big) + \sum_{k=1}^q \alpha^k \mathbb{1}\big(i^k = 3 \text{ and } b^k_{\{1,3\}} = a_m\big) \\ &\geq \sum_{k=1}^q \alpha^k \mathbb{1}\big(b^k_{\{1\}} = a_m\big) + \sum_{k=1}^q \alpha^k \mathbb{1}\big(b^k_{\{3\}} = a_m\big) \\ &= \beta_{\{1\}}(a_m) + \beta_{\{3\}}(a_m) \\ &= 0.7. \end{aligned}$$

Therefore, ϕ is not decomposable. □

In what follows, we generalize the inequality $\beta_{\{1,3\}}(a_m) \geq \beta_{\{1\}}(a_m) + \beta_{\{3\}}(a_m)$ in Example 6.6.1, and establish a necessary condition, the *scale-effect condition*, for the decomposition of RPFBRs. Consider a $(\underline{k}, \overline{k})$-RPFBR ϕ with the probabilistic ballots $(\beta_S)_{S \subseteq N}$. Recall that L and R denote the intervals $[a_1, a_{\underline{k}}]$ and $[a_{\overline{k}}, a_m]$, respectively. The scale-effect condition imposes two restrictions on the probabilistic ballots. Firstly, the probability of any right interval towards a_m in a probabilistic ballot, which is associated to the union of two disjoint nonempty coalitions $S, T \subseteq N$, is at least as much as the sum of these two coalitions' counterpart probabilities, i.e., for all $a_t \in R$, $\beta_{S \cup T}([a_t, a_m]) \geq \beta_S([a_t, a_m]) + \beta_T([a_t, a_m])$. The second restriction is, in some sense, the complement of the first one. Here we consider left intervals towards a_1, and take the sum of probabilities over the complement of S and the complement of T. Technically, it says that for all $a_s \in L$, we have $\beta_{N \setminus [S \cup T]}([a_1, a_s]) \geq \beta_{N \setminus S}([a_1, a_s]) + \beta_{N \setminus T}([a_1, a_s])$.

Definition 6.6.2 *A $(\underline{k}, \overline{k})$-RPFBR $\phi : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$, $n \geq 3$, satisfies the* ***scale-effect condition*** *if for all nonempty $S, T \subseteq N$ with $S \cap T = \emptyset$, $a_t \in R$ and $a_s \in L$, we have*

$$\beta_{S\cup T}([a_t, a_m]) \geq \beta_S([a_t, a_m]) + \beta_T([a_t, a_m]) \text{ and } \beta_{N\setminus[S\cup T]}([a_1, a_s]) \geq \beta_{N\setminus S}([a_1, a_s]) + \beta_{N\setminus T}([a_1, a_s]).$$

Proposition 6.6.2 below shows that the scale-effect condition is necessary for the decomposition of a $(\underline{k}, \overline{k})$-RPFBR.

Proposition 6.6.2 *A $(\underline{k}, \overline{k})$-RPFBR $\phi : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$ is decomposable only if it satisfies the scale-effect condition.*

Last, we analyze the entire class of RPFBRs from the perspective of admitting social compromises. Given a preference profile P, we recognize an alternative a_k as a *social compromise alternative* if some voters disagree on the peaks while all voters agree on a_k as the second best. Formally, given a preference domain $\mathbb{D}$, let $\mathcal{C}(\mathbb{D}^n) = \{P \in \mathbb{D}^n : r_1(P_i) \neq r_1(P_j) \text{ for some } i, j \in N, \text{ and } r_2(P_1) = \cdots = r_2(P_n)\}$ denote the set of preference profiles which have the social compromise alternatives. Moreover, given $P \in \mathcal{C}(\mathbb{D}^n)$, let the common second best alternative $c(P) \equiv r_2(P_1) = \cdots = r_2(P_n)$ denote the social compromise alternative. We compare RPFBRs according to the probabilities they assign to social compromise alternatives.

Definition 6.6.3 *A RSCF $\phi : \mathbb{D}^n \to A$* ***dominates*** *another RSCF $\varphi : \mathbb{D}^n \to A$* ***in admitting compromises*** *if we have $\phi_{c(P)}(P) \geq \varphi_{c(P)}(P)$ for all $P \in \mathcal{C}(\mathbb{D}^n)$ and $\phi_{c(P)}(P) > \varphi_{c(P)}(P)$ for some $P \in \mathcal{C}(\mathbb{D}^n)$.*

The proposition below characterizes all RPFBRs that are dominated in admitting compromises, and identify a condition under which an anonymous decomposable RPFBR is dominated by an anonymous *non-decomposable* one.

Proposition 6.6.3 *Let $1 < \underline{k} < \overline{k} < m$ and $\overline{k} - \underline{k} > 1$. Fixing a $(\underline{k}, \overline{k})$-RPFBR $\varphi : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$, $n \geq 3$, let $(\beta_S)_{S\subseteq N}$ be the corresponding probabilistic ballots. RPFBR φ is dominated in admitting compromises if and only if there exists $S \subseteq N$ with $|S| = n-1$ such that $\beta_S(a_m) > 0$ or $\beta_{N\setminus S}(a_1) > 0$. Furthermore, let φ be anonymous and decomposable. If there exists $S \subseteq N$ with $|S| = n-2$ such that $\beta_S(a_m) > 0$ or $\beta_{N\setminus S}(a_1) > 0$, then φ is dominated in admitting compromises by an anonymous non-decomposable $(\underline{k}, \overline{k})$-RPFBR.*

6.7 The Salience of Hybrid Domains and RPFBRs

Our purpose in this section is two-fold. We first propose an axiomatic justification of hybrid domains. Specifically, we show that any domain that satisfies certain "connectedness" and "richness" properties must be contained in a hybrid domain (say the $(\overline{k}, \underline{k})$-hybrid domain). Secondly, and more importantly, the set of unanimous and strategy-proof RSCFs on this domain is precisely the set of unanimous and strategy-proof RSCFs on the $(\overline{k}, \underline{k})$-hybrid domain, i.e., $(\overline{k}, \underline{k})$-RPFBRs. Thus, the set of unanimous and strategy-proof RSCFs on such a domain is the set of RPFBRs associated with the minimal hybrid domain in which it is embedded.

Recall the notions of adjacency and path introduced in the beginning of Section 6.2. A domain is said *connected* if every pair of two distinct preferences is connected by a path in the domain. We restrict attention to a class of connected domains which in addition satisfies the *weak no-restoration property* of [95].

Definition 6.7.1 *A domain $\mathbb{D}$ satisfies the* ***weak no-restoration property*** *if for all distinct $P_i, P_i' \in \mathbb{D}$ and $a_p, a_q \in A$, there exists a path $\{P_i^k\}_{k=1}^t \subseteq \mathbb{D}$ connecting P_i and P_i' such that we have*

$$[a_p P_i^{k^*} a_q \text{ and } a_q P_i^{k^*+1} a_p \text{ for some } 1 \leq k^* < t]$$
$$\Rightarrow [a_p P_i^k a_q \text{ for all } k = 1, \ldots, k^*, \text{ and } a_q P_i^l a_p \text{ for all } l = k^* + 1, \ldots, t].$$

Evidently, the weak no-restoration property implies connectedness, and suggests that according to each pair of alternatives a_p and a_q, one path can be constructed in the domain to reconcile the difference of P_i and P_i' shortly in the manner that the relative ranking of a_p and a_q is switched for *at most* once on the path. In particular, if a_p and a_q are identically ranked in P_i and P_i', then their relative ranking does not change along the path.

Proposition 3.2 of [95] shows that the weak no-restoration property is necessary for all DSCFs which only forbid misrepresentations of preferences that are adjacent to the sincere one, to retain strategy-proofness. The weak no-restoration property is satisfied by many important voting domains in the literature, e.g., the complete domain, the single-peaked domain and some multiple single-peaked domains, and also covers our hybrid domains (see the proof of Fact 6.8 in Appendix 6.14).

Our last result establishes two features of domains that satisfy the weak no-restoration property and include two completely reversed preferences. The first is that every such domain is a subset of some hybrid domain. The second is that every unanimous and strategy-proof RSCF on such a domain is a RPFBR. The proof Theorem 6.7.2 is available in Appendix 6.13.

Theorem 6.7.2 *Let domain $\mathbb{D}$ satisfy the weak no-restoration property and contain two completely reversed preferences. Then, there exist $1 \leq \underline{k} < \overline{k} \leq m$ such that $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ and $\mathbb{D} \not\subseteq \mathbb{D}_{\mathrm{H}}(\underline{k}', \overline{k}')$ where $\underline{k}' > \underline{k}$ or $\overline{k}' < \overline{k}$. Moreover, a RSCF $\phi : \mathbb{D}^n \to \Delta(A)$ is unanimous and strategy-proof if and only if it is a $(\underline{k}, \overline{k})$-RPFBR, where $\underline{k}$ and $\overline{k}$ are as described above.*

Appendix

6.8 Proof of Theorem 6.5.1

When $\overline{k} - \underline{k} = 1$, $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k}) = \mathbb{D}_{\prec}$, and then Theorem 6.5.1 follows from Theorem 4.1 and Proposition 5.2 of [46]. Henceforth, we assume $\overline{k} - \underline{k} > 1$.

(**Sufficiency part**) Let $\phi : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$ be a $(\underline{k}, \overline{k})$-RPFBR. First, ballot unanimity implies that ϕ is unanimous. We next show strategy-proofness of ϕ in two steps. In the first step, we introduce a notion weaker than strategy-proofness, *local strategy-proofness*, which only requires a RSCF be immune to the misrepresentation of preferences that are adjacent to the sincere one.[12] Fact 6.8 below shows that every locally strategy-proof RSCF on $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ is strategy-proof. In the second step, we show that ϕ is locally strategy-proof.

Every locally strategy-proof RSCF on $\mathbb{D}_{H}(\underline{k}, \overline{k})$ is strategy-proof.

By Theorem 1 of [38], to prove Fact 6.8, it suffices to show that $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ satisfies the *no-restoration property* of [95]. Therefore, the verification of Fact 6.8 is independent of RPFBR ϕ, and for ease of presentation, is delegated to Appendix 6.14.

Now, to complete the verification, we show that ϕ is locally strategy-proof. Fixing $i \in N$, $P_i, P_i' \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ with $P_i \sim P_i'$ and $P_{-i} \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^{n-1}$, we show that $\phi(P_i, P_{-i})$ stochastically dominates $\phi(P_i', P_{-i})$ according to P_i. Let $r_1(P_i) = a_s$ and $r_1(P_i') = a_t$. Evidently, if $a_s = a_t$, the tops-only property implies $\phi(P_i, P_{-i}) = \phi(P_i', P_{-i})$. Next, assume $a_s \neq a_t$. Then, $P_i \sim P_i'$ implies $r_1(P_i) = r_2(P_i') = a_s$, $r_1(P_i') = r_2(P_i) = a_t$ and $r_k(P_i) = r_k(P_i')$ for all $k \notin \{1, 2\}$. Thus, to show local strategy-proofness, it suffices to show the following condition:

$$\begin{aligned} &\phi_{a_s}(P_i, P_{-i}) \geq \phi_{a_s}(P_i', P_{-i}) \text{ or } \phi_{a_t}(P_i, P_{-i}) \leq \phi_{a_t}(P_i', P_{-i}), \text{ and} \\ &\phi_{a_k}(P_i, P_{-i}) = \phi_{a_k}(P_i', P_{-i}) \text{ for all } a_k \notin \{a_s, a_t\}. \end{aligned} \qquad (\#)$$

By the definition of $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$, $P_i \sim P_i'$ implies one of the following three cases: (i) $a_s, a_t \in L$ and $a_t \in \{a_{s-1}, a_{s+1}\}$, (ii) $a_s, a_t \in R$ and $a_t \in \{a_{s-1}, a_{s+1}\}$, and (iii) $a_s, a_t \in M$. Note that the first two cases

[12] Formally, a RSCF $\phi : \mathbb{D}^n \to \Delta(A)$ is **locally strategy-proof** if for all $i \in N$, $P_i, P_i' \in \mathbb{D}$ with $P_i \sim P_i'$ and $P_{-i} \in \mathbb{D}^{n-1}$, $\phi(P_i, P_{-i})$ stochastically dominates $\phi(P_i', P_{-i})$ according to P_i.

are symmetric. Therefore, we focus on cases (i) and (iii).

Claim 1: In case (i), condition (#) holds.

If $a_t = a_{s-1}$, then we know $S(s, (P_i, P_{-i})) \supset S(s, (P'_i, P_{-i}))$ and $S(k, (P_i, P_{-i})) = S(k, (P'_i, P_{-i}))$ for all $a_k \in A \backslash \{a_s\}$, and derive

$$\begin{aligned}\phi_{a_s}(P_i, P_{-i}) &= \beta_{S(s,(P_i,P_{-i}))}([a_s, a_m]) - \beta_{S(s+1,(P_i,P_{-i}))}([a_{s+1}, a_m]) \\ &\geq \beta_{S(s,(P'_i,P_{-i}))}([a_s, a_m]) - \beta_{S(s+1,(P'_i,P_{-i}))}([a_{s+1}, a_m]) \quad \text{by monotonicity} \\ &= \phi_{a_s}(P'_i, P_{-i}),\end{aligned}$$

and for all $a_k \notin \{a_{s-1}, a_s\}$,

$$\begin{aligned}\phi_{a_k}(P_i, P_{-i}) &= \beta_{S(k,(P_i,P_{-i}))}([a_k, a_m]) - \beta_{S(k+1,(P_i,P_{-i}))}([a_{k+1}, a_m]) \\ &= \beta_{S(k,(P'_i,P_{-i}))}([a_k, a_m]) - \beta_{S(k+1,(P'_i,P_{-i}))}([a_{k+1}, a_m]) = \phi_{a_k}(P'_i, P_{-i}).\end{aligned}$$

If $a_t = a_{s+1}$, then we know $S(s+1, (P_i, P_{-i})) \subset S(s+1, (P'_i, P_{-i}))$ and $S(k, (P_i, P_{-i})) = S(k, (P'_i, P_{-i}))$ for all $a_k \in A \backslash \{a_{s+1}\}$, and derive

$$\begin{aligned}\phi_{a_{s+1}}(P_i, P_{-i}) &= \beta_{S(s+1,(P_i,P_{-i}))9}([a_{s+1}, a_m]) - \beta_{S(s+2,(P_i,P_{-i}))}([a_{s+2}, a_m]) \\ &\leq \beta_{S(s+1,(P'_i,P_{-i}))}([a_{s+1}, a_m]) - \beta_{S(s+2,(P'_i,P_{-i}))}([a_{s+2}, a_m]) \quad \text{by monotonicity} \\ &= \phi_{a_{s+1}}(P'_i, P_{-i}).\end{aligned}$$

and for all $a_k \notin \{a_s, a_{s+1}\}$,

$$\begin{aligned}\phi_{a_k}(P_i, P_{-i}) &= \beta_{S(k,(P_i,P_{-i}))}([a_k, a_m]) - \beta_{S(k+1,(P_i,P_{-i}))}([a_{k+1}, a_m]) \\ &= \beta_{S(k,(P'_i,P_{-i}))}([a_k, a_m]) - \beta_{S(k+1,(P'_i,P_{-i}))}([a_{k+1}, a_m]) = \phi_{a_k}(P'_i, P_{-i}).\end{aligned}$$

This completes the verification of the claim.

Claim 2: In case (iii), condition (#) holds.

We assume $a_t \prec a_s$. The verification related to the situation $a_s \prec a_t$ is symmetric, and we hence omit it. First, note that $S(a_k, (P_i, P_{-i})) = S(a_k, (P'_i, P_{-i}))$ for all $a_k \in A$ with $a_k \preceq a_t$ or $a_s \prec a_k$. Then, for each

$a_k \in A$ with $a_k \prec a_t$ or $a_s \prec a_k$, we have

$$\begin{aligned}\phi_{a_k}(P_i, P_{-i}) &= \beta_{S(k,(P_i,P_{-i}))}([a_k, a_m]) - \beta_{S(k+1,(P_i,P_{-i}))}([a_{k+1}, a_m]) \\ &= \beta_{S(k,(P'_i,P_{-i}))}([a_k, a_m]) - \beta_{S(k+1,(P'_i,P_{-i}))}([a_{k+1}, a_m]) = \phi_{a_k}(P'_i, P_{-i}).\end{aligned}$$

Next, given $a_t \prec a_k \prec a_s$, we know $a_{\underline{k}} \prec a_k \prec a_{\overline{k}}$ and $a_{\underline{k}} \prec a_{k+1} \preceq a_{\overline{k}}$. Then, Definition 6.4.3 implies that for all $S \subseteq N$, $\beta_S([a_k, a_m]) = \sum_{j\in S} \varepsilon_j = \beta_S([a_{k+1}, a_m])$. Moreover, note that $S(k, (P_i, P_{-i}))\backslash S(k+1, (P_i, P_{-i})) = \{j \in N\backslash\{i\} : r_1(P_j) = a_k\} = S(k, (P'_i, P_{-i}))\backslash S(k+1, (P'_i, P_{-i}))$. Therefore, we have

$$\begin{aligned}\phi_{a_k}(P_i, P_{-i}) &= \beta_{S(k,(P_i,P_{-i}))}([a_k, a_m]) - \beta_{S(k+1,(P_i,P_{-i}))}([a_{k+1}, a_m]) \\ &= \sum\nolimits_{j\in S(k,(P_i,P_{-i}))\backslash S(k+1,(P_i,P_{-i}))} \varepsilon_j \\ &= \sum\nolimits_{j\in S(k,(P'_i,P_{-i}))\backslash S(k+1,(P'_i,P_{-i}))} \varepsilon_j \\ &= \beta_{S(k,(P'_i,P_{-i}))}([a_k, a_m]) - \beta_{S(k+1,(P'_i,P_{-i}))}([a_{k+1}, a_m]) = \phi_{a_k}(P'_i, P_{-i}).\end{aligned}$$

Overall, we have $\phi_{a_k}(P_i, P_{-i}) = \phi_{a_k}(P'_i, P_{-i})$ for all $a_k \notin \{a_s, a_t\}$. Last, since $a_t \prec a_s$ implies $S(s, (P_i, P_{-i})) \supset S(s, (P'_i, P_{-i}))$ and $S(a_{s+1}, (P_i, P_{-i})) = S(a_{s+1}, (P'_i, P_{-i}))$, we have

$$\begin{aligned}\phi_{a_s}(P_i, P_{-i}) &= \beta_{S(s,(P_i,P_{-i}))}([a_s, a_m]) - \beta_{S(s+1,(P_i,P_{-i}))}([a_{s+1}, a_m]) \\ &\geq \beta_{S(s,(P'_i,P_{-i}))}([a_s, a_m]) - \beta_{S(s+1,(P'_i,P_{-i}))}([a_{s+1}, a_m]) \quad \text{by monotonicity} \\ &= \phi_{a_s}(P'_i, P_{-i}).\end{aligned}$$

This completes the verification of the claim.

Therefore, ϕ is locally strategy-proof, as required. This hence completes the verification of the sufficiency part of Theorem 6.5.1.

(**Necessity part**) Let $\phi : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$ be a unanimous and strategy-proof RSCF. Since $\mathbb{D}_{\prec} \subseteq \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$, we can elicit a unanimous and strategy-proof RSCF $\varphi : [\mathbb{D}_{\prec}]^n \to \Delta(A)$ such that $\varphi(P) = \phi(P)$ for all $P \in [\mathbb{D}_{\prec}]^n$. First, Theorem 3 of [81] or Theorem 5 of [87] and Proposition 3 of [72] together imply that φ is a mixture of finitely many FBRs. Then, it follows immediately that φ is a PFBR. Let $(\beta_S)_{S\subseteq N}$ be the probabilistic ballots of φ. Evidently, $(\beta_S)_{S\subseteq N}$ satisfies ballot unanimity and monotonicity. Next, by the proof of Fact 6.8 and Proposition 1 of [31], we know that ϕ satisfies the tops-only property. Last, since both $\mathbb{D}_{\prec}$ and $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ are minimally rich, the tops-only property of ϕ implies that ϕ is also a PFBR and inherits φ's probabilistic ballots $(\beta_S)_{S\subseteq N}$. Therefore, for all

$P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n$ and $a_k \in A$, we have $\phi_{a_k}(P) = \beta_{S(k,P)}([a_k, a_m]) - \beta_{S(k+1,P)}([a_{k+1}, a_m])$, where $\beta_{S(m+1,P)}([a_{m+1}, a_m]) = 0$. To complete the proof, we show that ϕ is a $(\underline{k}, \overline{k})$-RPFBR.

Let $\overline{\mathbb{D}} = \left\{P_i \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k}) : r_1(P_i) \in M\right\}$ denote the subdomain of hybrid preferences whose peaks are in M. Since $|M| \geq 3$ and $\overline{\mathbb{D}}$ has no restriction on the ranking of alternatives in M, according to the random dictatorship characterization theorem of [57], we easily infer that there exists a "conditional dictatorial coefficient" $\varepsilon_i \geq 0$ for each $i \in N$ with $\sum_{i\in N} \varepsilon_i = 1$ such that $\phi(P) = \sum_{i\in N} \varepsilon_i \, e_{r_1(P_i)}$ for all $P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n$ with $r_1(P_i) \in M$ for all $i \in N$.

Fix an arbitrary coalition $S \subseteq N$. We first show $\beta_S([a_{\overline{k}}, a_m]) = \sum_{j\in S} \varepsilon_j$. We construct a profile $P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n$ where every voter of S has the preference with the peak $a_{\overline{k}}$ and all other voters have the preference with the peak $a_{\underline{k}}$. Thus, $S = S(\overline{k}, P)$ and $\phi(P) = \sum_{j\in S} \varepsilon_j \, e_{a_{\overline{k}}} + \sum_{j\in N\setminus S} \varepsilon_j \, e_{a_{\underline{k}}}$. We then have $\beta_S([a_{\overline{k}}, a_m]) = \beta_{S(\overline{k},P)}([a_{\overline{k}}, a_m]) = \sum_{k=\overline{k}}^{m} \left[\beta_{S(k,P)}([a_k, a_m] - \beta_{S(k+1,P)}([a_{k+1}, a_m])\right] = \sum_{k=\overline{k}}^{m} \phi_{a_k}(P) = \phi_{a_{\overline{k}}}(P^*) = \sum_{j\in S} \varepsilon_j$.

Last, we show $\beta_S([a_1, a_{\underline{k}}]) = \sum_{j\in N\setminus S} \varepsilon_j$. Since $\beta_S([a_1, a_{\underline{k}}]) = 1 - \beta_S([a_{\overline{k}}, a_m]) - \beta_S([a_{\underline{k}+1}, a_{\overline{k}-1}]) = \sum_{j\in N\setminus S} \varepsilon_j - \beta_S([a_{\underline{k}+1}, a_{\overline{k}-1}])$, it suffices to show $\beta_S(a_k) = 0$ for all $a_k \in [a_{\underline{k}+1}, a_{\overline{k}-1}]$. Given $a_{\underline{k}} \prec a_k \prec a_{\overline{k}}$, since $S(k, P) = S = S(k+1, P)$, we have $\beta_S(a_k) = \beta_S([a_k, a_m]) - \beta_S([a_{k+1}, a_m]) = \beta_{S(k,P)}([a_k, a_m]) - \beta_{S(k+1,P)}([a_{k+1}, a_m]) = \phi_{a_k}(P) = 0$, as required. This completes the verification of the necessity part of Theorem 6.5.1.

6.9 Proof of Theorem 6.5.3

We first show the sufficiency part of Theorem 6.5.3, and then turn to proving the necessity part. Before proceeding the proof, we formally introduce the deterministic version of a $(\underline{k}, \overline{k})$-RPFBR, which we call a $(\underline{k}, \overline{k})$-*Restricted Fixed Ballot Rule* (or $(\underline{k}, \overline{k})$-*RFBR*).

Definition 6.9.1 *A DSCF $f : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$ is called a $(\underline{k}, \overline{k})$-**Restricted Fixed Ballot Rule** (or $(\underline{k}, \overline{k})$-**RFBR**) if it is an FBR, i.e., there exists a collection of deterministic ballots $(b_S)_{S\subseteq N}$ satisfying ballot unanimity, i.e., $b_N = a_m$ and $b_\emptyset = a_1$, and monotonicity, i.e., $[S \subset T \subseteq N] \Rightarrow [b_S \preceq b_T]$, such that for all $P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n$, we have $f(P) = \max^{\prec}_{S\subseteq N} \left(\min^{\prec}_{j\in S} \left(r_1(P_j), b_S\right)\right)$, and in addition, $(b_S)_{S\subseteq N}$ satisfy the **constrained dictatorship condition**, i.e., $\overline{k} - \underline{k} > 1$ implies that there exists $i \in N$ such that $[i \in S] \Rightarrow [b_S \in R]$ and $[i \notin S] \Rightarrow [b_S \in L]$.*

(**Sufficiency part**) Fixing an anonymous $(\underline{k}, \overline{k})$-RPFBR $\phi : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$, assume that ϕ satisfy per-capita monotonicity. Let $(\beta_S)_{S\subseteq N}$ be the corresponding probabilistic ballots. By anonymity and the constrained random-dictatorship condition, $\beta_S = \beta_{S'}$ for all nonempty $S, S' \subseteq N$ with $|S| = |S'|$, and each voter has the conditional dictatorial coefficient $\frac{1}{n}$. We are going to decompose ϕ as a mixture of finitely many $(\underline{k}, \overline{k})$-RFBRs.

We provide some new notation which will be repeatedly used in the proof. Given $S \subseteq N$, let $supp(\beta_S) = \{a_k \in A : \beta_S(a_k) > 0\}$ denote the support of β_S. Given $S \subseteq N$ with $S \neq \emptyset$ and $N\backslash S \neq \emptyset$, the constrained random-dictatorship condition implies $supp(\beta_S) \cap R \neq \emptyset$ and $supp(\beta_S) \cap L \neq \emptyset$. Hence, we define

$$\hat{b}_S^R = \min\nolimits^{\prec}\big(supp(\beta_S) \cap R\big) \text{ and } \hat{b}_S^L = \max\nolimits^{\prec}\big(supp(\beta_S) \cap L\big).$$

Evidently, $\hat{b}_S^L \prec \hat{b}_S^R$. Moreover, let $\hat{b}_N^R = a_m$ and let $\hat{b}_\emptyset^L = a_1$. It is evident that (i) $\beta_N(\hat{b}_N^R) = 1$ and $\beta_\emptyset(\hat{b}_\emptyset^L) = 1$, and (ii) for all nonempty $S \subset N$, $\beta_S(\hat{b}_S^R) > 0$, $\beta_S(\hat{b}_S^L) > 0$ and $\beta_S(a_k) = 0$ for all $a_k \in A$ with $\hat{b}_S^L \prec a_k \prec \hat{b}_S^R$. Note that anonymity of ϕ implies $\hat{b}_S^R = \hat{b}_{S'}^R$ and $\hat{b}_S^L = \hat{b}_{S'}^L$ for all nonempty $S, S' \subseteq N$ with $|S| = |S'|$.

Lemma 6.9.1 *For all nonempty $S \subset S' \subseteq N$, we have $\hat{b}_S^R \preceq \hat{b}_{S'}^R$.*

Proof: If $S' = N$, it is evident that $\hat{b}_S^R \preceq a_m = \hat{b}_{S'}^R$. Next, let $S' \subset N$. Suppose $\hat{b}_S^R \succ \hat{b}_{S'}^R$. We then have $\frac{\beta_{S'}([\hat{b}_S^R, a_m])}{|S'|} \leq \frac{\beta_{S'}([a_{\overline{k}}, a_m]) - \beta_{S'}(\hat{b}_{S'}^R)}{|S'|} < \frac{|S'|/n}{|S'|} = \frac{1}{n} = \frac{\beta_S([a_{\overline{k}}, a_m])}{|S|} = \frac{\beta_S([\hat{b}_S^R, a_m])}{|S|}$, which contradicts per-capita monotonicity. ■

Lemma 6.9.2 *For all $S \subset S' \subset N$, we have $\hat{b}_S^L \preceq \hat{b}_{S'}^L$.*

Proof: If $S = \emptyset$, it is evident that $\hat{b}_S^L = a_1 \preceq \hat{b}_{S'}^L$. Next, let $S \neq \emptyset$. Suppose $\hat{b}_S^L \succ \hat{b}_{S'}^L$. For notational convenience, let $\hat{S} = N\backslash S$ and $\hat{S}' = N\backslash S'$. Thus, $\hat{S} \neq \emptyset$, $\hat{S}' \neq \emptyset$, $\hat{S} \supset \hat{S}'$ and $\hat{b}_{N\backslash\hat{S}}^L = \hat{b}_S^L \succ \hat{b}_{S'}^L = \hat{b}_{N\backslash\hat{S}'}^L$. We then have $\frac{\beta_{N\backslash\hat{S}}([a_1, \hat{b}_{N\backslash\hat{S}'}^L])}{|\hat{S}|} \leq \frac{\beta_{N\backslash\hat{S}}([a_1, a_{\underline{k}}]) - \beta_{N\backslash\hat{S}}(\hat{b}_{N\backslash\hat{S}}^L)}{|\hat{S}|} < \frac{|\hat{S}|/n}{|\hat{S}|} = \frac{1}{n} = \frac{\beta_{N\backslash\hat{S}'}([a_1, a_{\underline{k}}])}{|\hat{S}'|} = \frac{\beta_{N\backslash\hat{S}'}([a_1, \hat{b}_{N\backslash\hat{S}'}^L])}{|\hat{S}'|}$, which contradicts per-capita monotonicity. ■

Given an arbitrary $i \in N$, we construct deterministic ballots $(b_S^i)_{S \subseteq N}$:

$$b_S^i = \hat{b}_S^R \text{ and } b_{N\backslash S}^i = \hat{b}_{N\backslash S}^L \text{ for all } S \subseteq N \text{ with } i \in S.$$

Since $b_N^i = \hat{b}_N^R = a_m$ and $b_\emptyset^i = \hat{b}_{N\backslash N}^L = \hat{b}_\emptyset^L = a_1$, ballot unanimity is satisfied. Next, we show monotonicity is satisfied. Fix $S \subset S' \subset N$. If $i \in S$, then $i \in S'$, and Lemma 6.9.1 implies $b_S^i = \hat{b}_S^R \preceq \hat{b}_{S'}^R = b_{S'}^i$. If $i \notin S'$, then $i \notin S$, and Lemma 6.9.2 implies $b_S^i = b_{N\backslash[N\backslash S]}^i = \hat{b}_{N\backslash[N\backslash S]}^L = \hat{b}_S^L \preceq \hat{b}_{S'}^L = \hat{b}_{N\backslash[N\backslash S']}^L = b_{N\backslash[N\backslash S']}^i = b_{S'}^i$. If $i \in S'\backslash S$, then $b_S^i \in L$ and $b_{S'}^i \in R$, and hence $b_S^i \prec b_{S'}^i$. Overall, $b_S^i \preceq b_{S'}^i$, as required. Correspondingly, let f^i be the FBR w.r.t. the deterministic ballots $(b_S^i)_{S \subseteq N}$. Moreover, given $S \subseteq N$, we have $[i \in S] \Rightarrow [b_S^i = \hat{b}_S^R \in R]$, and $[i \in N\backslash S] \Rightarrow \big[b_S^i = b_{N\backslash[N\backslash S]}^i = \hat{b}_{N\backslash[N\backslash S]}^L \in L\big]$ which meet the constrained dictatorship condition. Therefore, f^i is a $(\underline{k}, \overline{k})$-RFBR which is strategy-proof on $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ by Theorem 6.5.1.

Next, we mix all $(\underline{k}, \overline{k})$-RFBRs $(f^i)_{i\in N}$ with the equal weight $\frac{1}{n}$, and construct the $(\underline{k}, \overline{k})$-RPFBR:

$$\varphi(P) = \sum_{i\in N} \frac{1}{n} e_{f^i(P)} \text{ for all } P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n.$$

Let $(\gamma_S)_{S\subseteq N}$ denote the corresponding probabilistic ballots, which obviously satisfies ballot unanimity, monotonicity and the constrained random-dictatorship condition. We make two observations on $(\gamma_S)_{S\subseteq N}$: (1) $\gamma_S = \sum_{i\in N} \frac{1}{n} e_{b_S^i} = \frac{1}{n} \sum_{i\in S} e_{\hat{b}_S^R} + \frac{1}{n} \sum_{i\in N\setminus S} e_{\hat{b}_S^L} = \frac{|S|}{n} e_{\hat{b}_S^R} + \frac{n-|S|}{n} e_{\hat{b}_S^L}$ for all $S \subseteq N$, and (2) φ is anonymous. Given distinct $S, S' \subseteq N$ with $|S| = |S'|$, anonymity of ϕ implies $e_{\hat{b}_S^R} = e_{\hat{b}_{S'}^R}$ and $e_{\hat{b}_S^L} = e_{\hat{b}_{S'}^L}$. We then have $\gamma_S = \frac{1}{n} e_{\hat{b}_S^R} + \frac{n-|S|}{n} e_{\hat{b}_S^L} = \frac{1}{n} e_{\hat{b}_{S'}^R} + \frac{n-|S'|}{n} e_{\hat{b}_{S'}^L} = \gamma_{S'}$, as required.

Furthermore, we identify the real number:

$$a = \min_{S\subset N: S\neq\emptyset} \Big(\min\Big(\frac{\beta_S(\hat{b}_S^R)}{|S|}, \frac{\beta_S(\hat{b}_S^L)}{n-|S|} \Big) \Big).$$

Evidently, $0 < a \leq \frac{\beta_S(\hat{b}_S^R)}{|S|}$ for all nonempty $S \subset N$. Moreover, given a nonempty $S \subset N$, the constrained random-dictatorship condition implies $a \leq \frac{\beta_S(\hat{b}_S^R)}{|S|} \leq \frac{\sum_{j\in S} \frac{1}{n}}{|S|} = \frac{1}{n}$.

Lemma 6.9.3 *We have $a = \frac{1}{n}$ if and only if $|supp(\beta_S)| = 2$ for all nonempty $S \subset N$. Moreover, if $a = \frac{1}{n}$, then $\phi(P) = \varphi(P)$ for all $P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n$, and hence ϕ is decomposable.*

Proof: First, assume $|supp(\beta_S)| = 2$ for all nonempty $S \subset N$. Thus, for all nonempty $S \subset N$, we know $supp(\beta_S) = \{\hat{b}_S^R, \hat{b}_S^L\}$, $\beta_S(\hat{b}_S^R) = \frac{|S|}{n}$ and $\beta_S(\hat{b}_S^L) = \frac{n-|S|}{n}$ by the constrained random-dictatorship condition. Consequently, $a = \frac{1}{n}$ by definition.

Next, assume $a = \frac{1}{n}$. Fix an arbitrary nonempty $S \subset N$. By definition, $\frac{\beta_S(\hat{b}_S^R)}{|S|} \geq a = \frac{1}{n}$ and $\frac{\beta_S(\hat{b}_S^L)}{n-|S|} \geq a = \frac{1}{n}$. Meanwhile, the constrained random-dictatorship condition implies $\beta_S(\hat{b}_S^R) \leq \frac{|S|}{n}$ and $\beta_S(\hat{b}_S^L) \leq \frac{n-|S|}{n}$. Therefore, $\beta_S(\hat{b}_S^R) = \frac{|S|}{n}$ and $\beta_S(\hat{b}_S^L) = \frac{n-|S|}{n}$, and hence $|supp(\beta_S)| = 2$.

Furthermore, note that (i) $\beta_N = e_{a_m} = \gamma_N$ and $\beta_\emptyset = e_{a_m} = \gamma_\emptyset$, and (ii) for all nonempty $S \subset N$, $\beta_S = \frac{|S|}{n} e_{\hat{b}_S^R} + \frac{n-|S|}{n} e_{\hat{b}_S^L} = \sum_{i\in N} \frac{1}{n} e_{b_S^i} = \gamma_S$. Therefore, $\phi(P) = \varphi(P)$ for all $P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n$, and hence, ϕ is decomposable. ■

Henceforth, we assume $0 < a < \frac{1}{n}$, and define the following

$$\hat{\beta}_S = \frac{\beta_S - an\gamma_S}{1 - an} = \frac{\beta_S - a|S|e_{\hat{b}_S^R} - a(n-|S|)e_{\hat{b}_S^L}}{1 - an} \text{ for all } S \subseteq N, \text{ and}$$

$$\psi(P) = \frac{\phi(P) - an\varphi(P)}{1 - an} \text{ for all } P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n.$$

It is easy to show that $\hat{\beta}_S \in \Delta(A)$ for each $S \subseteq N$. Hence, $(\hat{\beta}_S)_{S\subseteq N}$ are probabilistic ballots. It is evident that $(\hat{\beta}_S)_{S\subseteq N}$ satisfy ballot unanimity. Since both ϕ and φ are anonymous, ψ is also anonymous by construction. Next, let each voter have the conditional dictatorial coefficient $\frac{1}{n}$. We show that $(\hat{\beta}_S)_{S\subseteq N}$ satisfy the constrained random-dictatorship condition. Given nonempty $S \subset N$, we have $\hat{\beta}_S([a_{\overline{k}}, a_m]) = \frac{\beta_S([a_{\overline{k}},a_m])-\alpha|S|}{1-\alpha n} = \frac{\frac{|S|}{n}-\alpha|S|}{1-\alpha n} = \frac{|S|}{n}$ and $\hat{\beta}_S([a_1, a_{\underline{k}}]) = \frac{\beta_S([a_1,a_{\underline{k}}])-\alpha(n-|S|)}{1-\alpha n} = \frac{\frac{n-|S|}{n}-\alpha(n-|S|)}{1-\alpha n} = \frac{n-|S|}{n}$, as required. Next, we show that ψ is a PFBR w.r.t. $(\hat{\beta}_S)_{S\subseteq N}$. Given $P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n$ and $a_k \in A$, we have $\psi_{a_k}(P) = \frac{\phi_{a_k}(P)-\alpha n\varphi_{a_k}(P)}{1-\alpha n} = \frac{\left(\beta_{S(k,P)}([a_k,a_m])-\beta_{S(k+1,P)}([a_{k+1},a_m])\right)-\alpha n\left(\gamma_{S(k,P)}([a_k,a_m])-\gamma_{S(k+1,P)}([a_{k+1},a_m])\right)}{1-\alpha n} = \frac{\beta_{S(k,P)}([a_k,a_m])-\alpha n\gamma_{S(k,P)}([a_k,a_m])}{1-\alpha n} - \frac{\beta_{S(k+1,P)}([a_{k+1},a_m])-\alpha n\gamma_{S(k+1,P)}([a_{k+1},a_m])}{1-\alpha n} = \hat{\beta}_{S(k,P)}([a_k, a_m]) - \hat{\beta}_{S(k+1,P)}([a_{k+1}, a_m])$, as required.

The next two lemmas show that $(\hat{\beta}_S)_{S\subseteq N}$ satisfy monotonicity and ψ satisfies per-capita monotonicity respectively. Hence, we conclude that ψ is an anonymous $(\underline{k}, \overline{k})$-RPFBR and satisfies per-capita monotonicity.

Lemma 6.9.4 *Probabilistic ballots $(\hat{\beta}_S)_{S\subseteq N}$ satisfy monotonicity.*

Proof: Given $S \subset S' \subseteq N$, if $S = \emptyset$ or $S' = N$, monotonicity holds evidently. We hence assume $S \neq \emptyset$ and $S' \neq N$. We first identify $\hat{b}^L_S \preceq \hat{b}^L_{S'} \preceq a_{\underline{k}} \prec a_{\overline{k}} \preceq \hat{b}^R_S \preceq \hat{b}^R_{S'}$ by Lemmas 6.9.1 and 6.9.2. We assume w.l.o.g. that $|S'| = |S| + 1$. Given $a_t \in A$, we have five cases: (1) $\hat{b}^R_{S'} \prec a_t$, (2) $\hat{b}^R_S \prec a_t \preceq \hat{b}^R_{S'}$, (3) $\hat{b}^L_{S'} \prec a_t \preceq \hat{b}^R_S$, (4) $\hat{b}^L_S \prec a_t \preceq \hat{b}^L_{S'}$, and (5) $a_t \preceq \hat{b}^L_S$. We show $\hat{\beta}_{S'}([a_t, a_m]) \geq \hat{\beta}_S([a_t, a_m])$ in each case.

First, in either case (1) or case (5), $\hat{\beta}_{S'}([a_t, a_m]) - \hat{\beta}_S([a_t, a_m]) = \frac{\beta_{S'}([a_t,a_m])-\beta_S([a_t,a_m])}{1-\alpha n} \geq 0$.

In case (2), $\hat{\beta}_{S'}([a_t, a_m]) - \hat{\beta}_S([a_t, a_m]) = \frac{\beta_{S'}([a_t,a_m])-\alpha|S'|-\beta_S([a_t,a_m])}{1-\alpha n} \geq \frac{\frac{|S'|}{n}-\alpha(|S'|)-\left[\beta_S([\hat{b}^R_S,a_m])-\beta_S(\hat{b}^R_S)\right]}{1-\alpha n} = \frac{\frac{|S|+1}{n}-\alpha(|S|+1)-\frac{|S|}{N}+\beta_S(\hat{b}^R_S)}{1-\alpha n} = \frac{(\frac{1}{n}-\alpha)+|S|\left(\frac{\beta_S(\hat{b}^R_S)}{|S|}-\alpha\right)}{1-\alpha n} > 0$, where the first inequality follows from $\hat{b}^L_S \prec a_t \preceq \hat{b}^L_{S'}$ and the constrained random dictatorship condition of ϕ, and the last inequality follows from the hypothesis $\alpha < \frac{1}{n}$ and the definition of α.

In case (3),
$\hat{\beta}_{S'}([a_t, a_m]) - \hat{\beta}_S([a_t, a_m]) = \frac{\beta_{S'}([a_t,a_m])-\alpha|S'|-[\beta_S([a_t,a_m])-\alpha|S|]}{1-\alpha n} = \frac{\frac{|S'|}{n}-\alpha|S'|-(\frac{|S|}{n}-\alpha|S|)}{1-\alpha n} = \frac{\frac{1}{n}-\alpha}{1-\alpha n} > 0$.

Last, in case (4), we have $\hat{\beta}_{S'}([a_t, a_m]) - \hat{\beta}_S([a_t, a_m]) = \frac{\beta_{S'}([a_t,a_m])-\alpha|S'|-\alpha(n-|S'|)-\left[\beta_S([a_t,a_m])-\alpha|S|\right]}{1-\alpha n} = \frac{\frac{|S'|}{n}+\beta_{S'}([a_t,a_{\underline{k}}])-\alpha(n-|S|)-\left[\frac{|S|}{n}+\beta_S([a_t,a_{\underline{k}}])\right]}{1-\alpha n} \geq \frac{\frac{1}{n}+\beta_{S'}(\hat{b}^L_{S'})-\alpha(n-|S'|+1)}{1-\alpha n} = \frac{(\frac{1}{n}-\alpha)+(n-|S'|)\left(\frac{\beta_{S'}(\hat{b}^L_{S'})}{n-|S'|}-\alpha\right)}{1-\alpha n} > 0$, where the first inequality follows from $\hat{b}^R_S \prec a_t \preceq \hat{b}^R_{S'}$ and the constrained random dictatorship condition of ϕ, and the last inequality follows from the hypothesis $\alpha < \frac{1}{n}$ and the definition of α.

In conclusion, $\hat{\beta}_{S'}([a_t, a_m]) \geq \hat{\beta}_S([a_t, a_m])$ for all $a_t \in A$. ■

Lemma 6.9.5 *RPFBR ψ satisfies per-capita monotonicity.*

Proof: Fixing $S \subset S' \subseteq N$, we have $\hat{b}_S^R \preceq \hat{b}_{S'}^R$ and $\hat{b}_{N\setminus S'}^L \preceq \hat{b}_{N\setminus S}^L$ by Lemmas 6.9.1 and 6.9.2. If $S = \emptyset$ or $S' = N$, per-capita monotonicity holds evidently. We hence assume $S \neq \emptyset$ and $S' \neq N$.

Given $a_t \in R$, either one of the three cases occurs: (1) $\hat{b}_{S'}^R \prec a_t$, (2) $\hat{b}_S^R \prec a_t \preceq \hat{b}_{S'}^R$, and (3) $a_t \preceq \hat{b}_S^R$.

In case (1), $\frac{\hat{\beta}_{S'}([a_t,a_m])}{|S'|} = \frac{1}{1-\alpha n}\frac{\beta_{S'}([a_t,a_m])}{|S'|} \geq \frac{1}{1-\alpha n}\frac{\beta_S([a_t,a_m])}{|S|} = \frac{\hat{\beta}_S([a_t,a_m])}{|S|}$, where the inequality follows from per-capita monotonicity of ϕ.

In case (2), $\frac{\hat{\beta}_{S'}([a_t,a_m])}{|S'|} = \frac{1}{1-\alpha n}\frac{\beta_{S'}([a_t,a_m])-\alpha|S'|}{|S'|} = \frac{1}{1-\alpha n}\frac{\frac{|S'|}{n}-\alpha|S'|}{|S'|} = \frac{1}{1-\alpha n}\left(\frac{1}{n}-\alpha\right) \geq \frac{1}{1-\alpha n}\left(\frac{1}{n}-\frac{\beta_S(\hat{b}_S^R)}{|S|}\right) = \frac{1}{1-\alpha n}\frac{\frac{|S|}{n}-\beta_S(\hat{b}_S^R)}{|S|} = \frac{1}{1-\alpha n}\frac{\beta_S([a_{\overline{k}},a_m])-\beta_S(\hat{b}_S^R)}{|S|} \geq \frac{1}{1-\alpha n}\frac{\beta_S([a_t,a_m])}{|S|} = \frac{\hat{\beta}_S([a_t,a_m])}{|S|}$, where the first inequality follows from the definition of α and the second inequality follows from $\hat{b}_S^R \prec a_t$.

Last, in case (3), $\frac{\hat{\beta}_{S'}([a_t,a_m])}{|S'|} = \frac{1}{1-\alpha n}\frac{\beta_{S'}([a_t,a_m])-\alpha|S'|}{|S'|} = \frac{1}{(1-\alpha n)}\left[\frac{\beta_{S'}([a_t,a_m])}{|S'|}-\alpha\right] \geq \frac{1}{(1-\alpha n)}\left[\frac{\beta_S([a_t,a_m])}{|S|}-\alpha\right]$ $= \frac{1}{1-\alpha n}\frac{\beta_S([a_{\overline{k}},a_t])-\alpha|S|}{|S|} = \frac{\hat{\beta}_S([a_t,a_m])}{|S|}$, where the inequality follows from per-capita monotonicity of ϕ.

Symmetrically, given $a_s \in L$, either one of the three cases occurs: (i) $a_s \prec \hat{b}_{N\setminus S'}^L$, (ii) $\hat{b}_{N\setminus S'}^L \preceq a_s \prec \hat{b}_{N\setminus S}^L$, and (iii) $\hat{b}_{N\setminus S}^L \preceq a_s$.

In case (i), $\frac{\hat{\beta}_{N\setminus S'}([a_1,a_s])}{|S'|} = \frac{1}{1-\alpha n}\frac{\beta_{N\setminus S'}([a_1,a_s])}{|S'|} \geq \frac{1}{1-\alpha n}\frac{\beta_{N\setminus S}([a_1,a_s])}{|S|} = \frac{\hat{\beta}_{N\setminus S}([a_1,a_s])}{|S|}$, where the inequality follows from per-capita monotonicity of ϕ.

In case (ii), $\frac{\hat{\beta}_{N\setminus S'}([a_1,a_s])}{|S'|} = \frac{1}{1-\alpha n}\frac{\beta_{N\setminus S'}([a_1,a_s])-\alpha[n-(n-|S'|)]}{|S'|} = \frac{1}{1-\alpha n}\left(\frac{1}{n}-\alpha\right) \geq \frac{1}{1-\alpha n}\left(\frac{1}{n}-\frac{\beta_{N\setminus S}(\hat{b}_{N\setminus S}^L)}{n-(n-|S|)}\right) = \frac{1}{1-\alpha n}\frac{\frac{|S|}{n}-\beta_{N\setminus S}(\hat{b}_{N\setminus S}^L)}{|S|} = \frac{1}{1-\alpha n}\frac{\beta_{N\setminus S}([a_1,a_{\underline{k}}])-\beta_{N\setminus S}(\hat{b}_{N\setminus S}^L)}{|S|} \geq \frac{1}{1-\alpha n}\frac{\beta_{N\setminus S}([a_1,a_s])}{|S|} = \frac{\hat{\beta}_{N\setminus S}([a_1,a_s])}{|S|}$, where the first inequality follows from the definition of α and the second inequality follows from $a_s \prec \hat{b}_{N\setminus S}^L$.

Last, in case (iii), $\frac{\hat{\beta}_{N\setminus S'}([a_1,a_s])}{|S'|} = \frac{1}{1-\alpha n}\frac{\beta_{N\setminus S'}([a_1,a_s])-\alpha[n-(n-|S'|)]}{|S'|} = \frac{1}{1-\alpha n}\left[\frac{\beta_{N\setminus S'}([a_1,a_s])}{|S'|}-\alpha\right] \geq$ $\frac{1}{1-\alpha n}\left[\frac{\beta_{N\setminus S}([a_1,a_s])}{|S|}-\alpha\right] = \frac{1}{1-\alpha n}\frac{\beta_{N\setminus S}([a_1,a_s])-\alpha[n-(n-|S|)]}{|S|} = \frac{\hat{\beta}_{N\setminus S}([a_1,a_s])}{|S|}$, where the inequality follows from per-capita monotonicity of ϕ.

In conclusion, ψ satisfies per-capita monotonicity. ■

The next lemma shows that the support of every ϕ's probabilistic ballot is refined by that of ψ, and the support of some ϕ's probabilistic ballot is strictly refined.

Lemma 6.9.6 *For all nonempty $S \subset N$, $supp(\hat{\beta}_S) \subseteq supp(\beta_S)$, and for some nonempty $S^* \subset N$, $supp(\hat{\beta}_{S^*}) \subset supp(\beta_{S^*})$.*

Proof: Given nonempty $S \subset N$, since $\hat{\beta}_S = \frac{\beta_S - \alpha|S|e_{\hat{b}_S^R} - \alpha(n-|S|)e_{\hat{b}_S^L}}{1-\alpha n}$, it is true that $supp(\hat{\beta}_S) \subseteq supp(\beta_S)$. Next, by the definition of α, there exists a nonempty $S^* \subset N$ such that $\alpha = \frac{\beta_{S^*}(\hat{b}_{S^*}^R)}{|S^*|}$ or $\alpha = \frac{\beta_{S^*}(\hat{b}_{S^*}^L)}{n-|S^*|}$. Hence, either $\hat{\beta}_{S^*}(\hat{b}_{S^*}^R) = 0$ or $\hat{\beta}_{S^*}(\hat{b}_{S^*}^L) = 0$ holds. Therefore, $supp(\hat{\beta}_{S^*}) \subset supp(\beta_{S^*})$. ■

By spirit of Lemma 6.9.6, we call ψ the *refined* $(\underline{k}, \overline{k})$-RPFBR of ϕ. Now, we have $(\underline{k}, \overline{k})$-RFBRs $(f^i)_{i\in N}$ and an anonymous $(\underline{k}, \overline{k})$-RPFBR ψ which satisfies per-capita monotonicity. More importantly, the

original $(\underline{k}, \overline{k})$-RPFBR ϕ can be specified as a mixture of $(f^i)_{i\in N}$ and ψ, i.e.,
$\phi(P) = an\varphi(P) + (1 - an)\psi(P) = a\sum_{i\in N} e_{f^i(P)} + (1 - an)\psi(P)$ for all $P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n$.

Note that if we repeat the procedure above on the anonymous $(\underline{k}, \overline{k})$-RPFBR ψ, we can further decompose ϕ. Therefore, by repeatedly applying the procedure, we eventually can decompose ϕ as a mixture of finitely many $(\underline{k}, \overline{k})$-RFBRs, *provided that the procedure can terminate in finite steps*. In each step of the procedure, Lemma 6.9.6 implies that the support of the refined $(\underline{k}, \overline{k})$-RPFBR's probabilistic ballots strictly shrinks. Since the alternative set A is finite, it must be the case that after finite steps, the support of the refined $(\underline{k}, \overline{k})$-RPFBR's every probabilistic ballot becomes a binary set. Furthermore, by Lemma 6.9.3, the refined $(\underline{k}, \overline{k})$-RPFBR becomes a mixture of n $(\underline{k}, \overline{k})$-RFBRs. Hence, the procedure terminates, and we finish the decomposition of ϕ. This completes the verification of the sufficiency part of Theorem 6.5.3.

(**Necessity part**) Fix an anonymous decomposable $(\underline{k}, \overline{k})$-RPFBR $\phi : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$. Let $(\beta_S)_{S\subseteq N}$ be the corresponding probabilistic ballots. By Theorem 6.5.1, we know that $(\beta_S)_{S\subseteq N}$ satisfy ballot unanimity, monotonicity and the constrained random-dictatorship condition. Moreover, anonymity of ϕ implies that every voter has the conditional dictatorial coefficient $\frac{1}{n}$, and $\beta_S = \beta_{S'}$ for all $S, S' \subseteq N$ with $|S| = |S'|$. By decomposability and Theorem 6.5.1, we have finitely many $(\underline{k}, \overline{k})$-RFBRs $f^k : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$, $k = 1 \dots, q$, and weights $\alpha^1, \dots, \alpha^q > 0$ with $\sum_{k=1}^q \alpha^k = 1$ such that $\phi(P) = \sum_{k=1}^q \alpha^k e_{f^k(P)}$ for all $P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n$. For each $1 \leq k \leq q$, let $(b_S^k)_{S\subseteq N}$ denote the deterministic ballots of f^k. Evidently, for each $1 \leq k \leq q$, $(b_S^k)_{S\subseteq N}$ satisfy ballot unanimity, monotonicity and the constrained-dictatorship condition. For ease of presentation, we call the voter specified in the constrained dictatorship condition of f^k the *constrained dictator*, denoted by i^k. Moreover, let $I_i = \left\{k \in \{1, \dots, q\} : i^k = i\right\}$ collect the indexes of RFBRs where i is the constrained dictator. Last, by monotonicity of both $(\beta_S)_{S\subseteq N}$ and $(b_S^k)_{S\subseteq N}$, $k = 1, \dots, q$, it is true that $\beta_S = \sum_{k=1}^q \alpha^k e_{b_S^k}$ for all $S \subseteq N$.

Lemma 6.9.7 *For all $i \in N$, $\sum_{k\in I_i} \alpha^k = \frac{1}{n}$.*

Proof: Suppose that it is not true. Then, there exist $i, j \in N$ such that $\sum_{k\in I_i} \alpha^k \neq \sum_{k\in I_j} \alpha^k$. Then, by the constrained random dictatorship condition, we have
$\beta_{\{i\}}([a_{\overline{k}}, a_m]) = \sum_{k=1}^q \alpha^k \mathbb{1}\big(b_{\{i\}}^k \in R\big) = \sum_{k\in I_i} \alpha^k \neq \sum_{k\in I_j} \alpha^k = \sum_{k=1}^q \alpha^k \mathbb{1}\big(b_{\{j\}}^k \in R\big) = \beta_{\{j\}}([a_{\overline{k}}, a_m])$,
which contradicts the fact $\beta_{\{i\}} = \beta_{\{j\}}$.[13] ■

For each $i \in N$, let $\phi^i(P) = \sum_{k\in I_i} \alpha^k n e_{f^k(P)}$ for all $P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n$. By Lemma 6.9.7, ϕ^i is a mixture of RFBRs $(f^k)_{k\in I_i}$ according to the weights $(\alpha^k n)_{k\in I_i}$, and hence is a $(\underline{k}, \overline{k})$-RPFBR. Let $(\beta_S^i)_{S\subseteq N}$ denote the corresponding probabilistic ballots. Evidently, $(\beta_S^i)_{S\subseteq N}$ satisfy ballot unanimity and monotonicity, and ϕ^i satisfies the constrained random-dictatorship condition. Note that voter i has the conditional dictatorial coefficient 1 in ϕ^i.

[13] The notation $\mathbb{1}(\cdot)$ denotes an indicator function.

Lemma 6.9.8 *For all $S \subseteq N$, $\beta_S = \sum_{i\in N} \frac{1}{n}\beta^i_S$.*

Proof: By the definition RPFBRs $(\phi^i)_{i\in N}$, we can rewrite ϕ as follows:
$\phi(P) = \sum_{k=1}^{q} \alpha^k e_{f^k(P)} = \sum_{i\in N}\sum_{k\in I_i} \alpha^k e_{f^k(P)} = \sum_{i\in N}\frac{1}{n}\Big(\sum_{k\in I_i}\alpha^k n e_{f^k(P)}\Big) = \sum_{i\in N}\frac{1}{n}\phi^i(P)$ for all $P \in \big[\mathbb{D}_{\mathrm{H}}(\underline{k},\overline{k})\big]^n$. Therefore, $\beta_S = \sum_{i\in N}\frac{1}{n}\beta^i_S$ for all $S \subseteq N$. ■

Now, for each $i \in N$, we construct another collection of probabilistic ballots $(\bar{\beta}^i_S)_{S\subseteq N}$ by equally mixing probabilistic ballots $\{(\beta^j_S)_{S\subseteq N} : j \in N\}$ in a particular way. Specifically, given $S \subseteq N$, say $|S| = k$, we construct $\bar{\beta}^i_S$ in two steps. In the first step, we refer to each coalition $S' \subseteq N$ that has the same size as S, the k corresponding probabilistic ballots $(\beta^j_{S'})_{j\in S'}$ and the $n-k$ corresponding probabilistic ballots $(\beta^j_{S'})_{j\in N\backslash S'}$. We then make two equal mixtures $\sum_{j\in S'}\frac{1}{k}\beta^j_{S'}$ and $\sum_{j\in N\backslash S'}\frac{1}{n-k}\beta^j_{S'}$. In the second step, we check whether i is included in S or not. If $i \in S$, we refer to $\sum_{j\in S'}\frac{1}{k}\beta^j_{S'}$ for all $C^k_n = \frac{n!}{k!(n-k)!}$ subsets S' of N that have the same size as S, and make their equal mixture as $\bar{\beta}^i_{S'}$, i.e.,

$$\bar{\beta}^i_S = \sum_{S'\subseteq N:|S'|=k}\frac{1}{C^k_n}\Big(\sum_{j\in S'}\frac{1}{k}\beta^j_{S'}\Big) = \frac{1}{C^k_n}\frac{1}{k}\sum_{S'\subseteq N:|S'|=k}\sum_{j\in S'}\beta^j_{S'};$$

otherwise we refer to $\sum_{j\in N\backslash S'}\frac{1}{n-k}\beta^j_{S'}$ for all $C^k_n = \frac{n!}{k!(n-k)!}$ subsets S' of N that have the same size as S, and make their equal mixture as $\bar{\beta}^i_{S'}$, i.e.,

$$\bar{\beta}^i_S = \sum_{S'\subseteq N:|S'|=k}\frac{1}{C^k_n}\Big(\sum_{j\in N\backslash S'}\frac{1}{n-k}\beta^j_{S'}\Big) = \frac{1}{C^k_n}\frac{1}{n-k}\sum_{S'\subseteq N:|S'|=k}\sum_{j\in N\backslash S'}\beta^j_{S'}.$$

We are going to show that $(\bar{\beta}^i_S)_{S\subseteq N}$ satisfy ballot unanimity, monotonicity and the constrained random-dictatorship condition. First, it is easy to verify the following four statements:

(i) $\bar{\beta}^i_S \in \Delta(A)$ for all $S \subseteq N$ and $i \in N$.

(ii) $(\bar{\beta}^i_S)_{S\subseteq N}$ satisfy ballot unanimity, i.e., $\bar{\beta}^i_\emptyset = \frac{1}{n}\sum_{S'\subseteq N:|S'|=0}\sum_{j\notin S'}\beta^j_{S'} = \frac{1}{n}\sum_{j\in N}\beta^j_\emptyset = e_{a_1}$ and $\bar{\beta}^i_N = \frac{1}{n}\sum_{S'\subseteq N:|S'|=n}\sum_{j\in S'}\beta^j_{S'} = \frac{1}{n}\sum_{j\in N}\beta^j_N = e_{a_m}$.

(iii) $(\bar{\beta}^i_S)_{S\subseteq N}$ satisfy the constrained random dictatorship condition, i.e., given $S \subset N$, say $|S| = k$, if $i \in S$, we have $\bar{\beta}^i_S([a_{\overline{k}}, a_m]) = \sum_{S'\subseteq N:|S'|=k}\frac{1}{C^k_n}\big(\sum_{j\in S'}\frac{1}{k}\beta^j_{S'}([a_{\overline{k}}, a_m])\big) = 1$; otherwise, we have $\bar{\beta}^i_S([a_1, a_{\underline{k}}]) = \sum_{S'\subseteq N:|S'|=k}\frac{1}{C^k_n}\big(\sum_{j\in N\backslash S'}\frac{1}{n-k}\beta^j_{S'}([a_1, a_{\underline{k}}])\big) = 1$.

(iv) For all nonempty $S \subset N$ and distinct $i, j \in S$ or $i, j \notin S$, we have $\bar{\beta}^i_S = \bar{\beta}^j_S$.

Next, we focus on showing monotonicity of $(\bar{\beta}^i_S)_{S\subseteq N}$.

Lemma 6.9.9 *Given nonempty $S \subset N$, $\beta_S = \sum_{i\in N}\frac{1}{n}\bar{\beta}^i_S$.*

Proof: Let $|S| = k$. Thus, $0 < k < n$. We then have

$$\begin{aligned}
\beta_S =& \frac{1}{C_n^k} \sum_{S' \subseteq N: |S'|=k} \beta_{S'} \quad \text{(by anonymity)} \\
=& \frac{1}{C_n^k} \sum_{S' \subseteq N: |S'|=k} \sum_{i \in N} \frac{1}{n} \beta_{S'}^i \quad \text{(by Lemma 6.9.8)} \\
=& \frac{1}{C_n^k} \frac{1}{n} \sum_{S' \subseteq N: |S'|=k} \Big(\sum_{i \in S'} \beta_{S'}^i + \sum_{i \in N \setminus S'} \beta_{S'}^i \Big) \\
=& \frac{k}{n} \Big(\frac{1}{C_n^k} \frac{1}{k} \sum_{S' \subseteq N: |S'|=k} \sum_{i \in S'} \beta_{S'}^i \Big) + \frac{n-k}{n} \Big(\frac{1}{C_n^k} \frac{1}{n-k} \sum_{S' \subseteq N: |S'|=k} \sum_{i \in N \setminus S'} \beta_{S'}^i \Big) \\
=& \frac{k}{n} \bar{\beta}_S^i + \frac{n-k}{n} \bar{\beta}_S^j \quad \text{for some } i \in S \text{ and some } j \in N \setminus S \quad \text{(by the definition of } \bar{\beta}_S^i \text{ and } \bar{\beta}_S^j\text{)} \\
=& \sum_{i \in S} \frac{1}{n} \bar{\beta}_S^i + \sum_{j \in N \setminus S} \frac{1}{n} \bar{\beta}_S^j \quad \text{(by statement (iv) above)} \\
=& \sum_{i \in N} \frac{1}{n} \bar{\beta}_S^i .
\end{aligned}$$

This completes the verification of the lemma. ■

Lemma 6.9.10 *Probabilistic ballots* $\left(\bar{\beta}_S^i\right)_{S \subseteq N}$ *satisfy monotonicity.*

Proof: Fix $S \subset S' \subseteq N$. If $S = \emptyset$ or $S' = N$, the condition of monotonicity holds evidently. Henceforth, let $S \neq \emptyset$ and $S' \neq N$. We assume w.l.o.g. that $|S| = k$ and $|S'| = k + 1$. If $S' \setminus S = \{i\}$, we have $\bar{\beta}_{S'}^i([a_{\bar{k}}, a_m]) = 1$ and $\bar{\beta}_S^i[a_1, a_{\underline{k}}] = 1$ by the constrained random-dictatorship condition, which immediately imply the condition of monotonicity.

Next, assume $i \in S$. Then, $i \in S'$. Now, given $a_t \in A$, we have

$$\begin{aligned}
\bar{\beta}_{S'}^i([a_t, a_m]) - \bar{\beta}_S^i([a_t, a_m]) =& \frac{1}{C_n^{k+1}} \frac{1}{k+1} \sum_{\bar{S} \subseteq N: |\bar{S}|=k+1} \sum_{j \in \bar{S}} \beta_{\bar{S}}^j([a_t, a_m]) - \frac{1}{C_n^k} \frac{1}{k} \sum_{\bar{S} \subseteq N: |\bar{S}|=k} \sum_{j \in \bar{S}} \beta_{\bar{S}}^j([a_t, a_m]) \\
=& \frac{1}{C_n^{k+1}} \frac{1}{k+1} \frac{1}{k} \left[\sum_{\bar{S} \subseteq N: |\bar{S}|=k+1} \left(k \sum_{j \in \bar{S}} \beta_{\bar{S}}^j([a_t, a_m]) \right) - \sum_{\bar{S} \subseteq N: |\bar{S}|=k} \left((n-k) \sum_{j \in \bar{S}} \beta_{\bar{S}}^j([a_t, a_m]) \right) \right] \\
=& \frac{1}{C_n^{k+1}} \frac{1}{k+1} \frac{1}{k} \left[\sum_{\bar{S} \subseteq N: |\bar{S}|=k} \left(\sum_{v \in N \setminus \bar{S}} \sum_{j \in \bar{S}} \beta_{\bar{S} \cup \{v\}}^j([a_t, a_m]) \right) - \sum_{\bar{S} \subseteq N: |\bar{S}|=k} \left((n-k) \sum_{j \in \bar{S}} \beta_{\bar{S}}^j([a_t, a_m]) \right) \right] \\
=& \frac{1}{C_n^{k+1}} \frac{1}{k+1} \frac{1}{k} \sum_{\bar{S} \subseteq N: |\bar{S}|=k} \sum_{v \in N \setminus \bar{S}} \sum_{j \in \bar{S}} \left(\beta_{\bar{S} \cup \{v\}}^j([a_t, a_m]) - \beta_{\bar{S}}^j([a_t, a_m]) \right) \\
\geq& 0. \quad \text{(by monotonicity of } (\beta_J^j)_{J \subseteq N}, j \in \bar{S}\text{)}
\end{aligned}$$

Last, assume $i \notin S'$. Then, $i \notin S$. Now, given $a_t \in A$, we have

$$
\begin{aligned}
&\overline{\beta}^i_{S'}([a_t, a_m]) - \overline{\beta}^i_S([a_t, a_m]) \\
=&\frac{1}{C_n^{k+1}} \frac{1}{n-(k+1)} \sum_{\overline{S}\subseteq N:|\overline{S}|=k+1} \sum_{j\in N\setminus\overline{S}} \beta^j_{\overline{S}}([a_t, a_m]) - \frac{1}{C_n^k}\frac{1}{n-k} \sum_{\overline{S}\subseteq N:|\overline{S}|=k} \sum_{j\in N\setminus\overline{S}} \beta^j_{\overline{S}}([a_t, a_m]) \\
=&\frac{1}{C_n^k}\frac{1}{n-k}\frac{1}{n-(k+1)} \left[\sum_{\overline{S}\subseteq N:|\overline{S}|=k+1} \left((k+1)\sum_{j\in N\setminus\overline{S}} \beta^j_{\overline{S}}([a_t, a_m])\right) - \sum_{\overline{S}\subseteq N:|\overline{S}|=k} \left([n-(k+1)]\sum_{j\in N\setminus\overline{S}} \beta^j_{\overline{S}}([a_t, a_m])\right)\right] \\
=&\frac{1}{C_n^k}\frac{1}{n-k}\frac{1}{n-(k+1)} \left[\sum_{\overline{S}\subseteq N:|\overline{S}|=k+1} \left((k+1)\sum_{j\in N\setminus\overline{S}} \beta^j_{\overline{S}}([a_t, a_m])\right) - \sum_{\overline{S}\subseteq N:|\overline{S}|=k+1} \left(\sum_{v\in\overline{S}}\sum_{j\in N\setminus\overline{S}} \beta^j_{\overline{S}\setminus\{v\}}([a_t, a_m])\right)\right] \\
=&\frac{1}{C_n^k}\frac{1}{n-k}\frac{1}{n-(k+1)} \sum_{\overline{S}\subseteq N:|\overline{S}|=k+1} \sum_{v\in\overline{S}}\sum_{j\in N\setminus\overline{S}} \left[\beta^j_{\overline{S}}([a_t, a_m]) - \beta^j_{\overline{S}\setminus\{v\}}([a_t, a_m])\right] \\
\geq& 0. \quad \text{(by monotonicity of } (\beta^j_J)_{J\subseteq N}, j\in N\setminus\overline{S})
\end{aligned}
$$

This completes the verification of the lemma. ■

Now, we are ready to show per-capita monotonicity of ϕ. Given nonempty $S \subset S' \subset N$, $a_t \in R$ and $a_s \in L$, we have

$$
\begin{aligned}
\frac{\beta_{S'}([a_t, a_m])}{|S'|} - \frac{\beta_S([a_t, a_m])}{|S|} &= \frac{\sum_{i\in N}\frac{1}{n}\overline{\beta}^i_{S'}([a_t, a_m])}{|S'|} - \frac{\sum_{i\in N}\frac{1}{n}\overline{\beta}^i_S([a_t, a_m])}{|S|} \quad \text{(by Lemma 6.9.9)} \\
&= \frac{\sum_{i\in S'}\frac{1}{n}\overline{\beta}^i_{S'}([a_t, a_m])}{|S'|} - \frac{\sum_{i\in S}\frac{1}{n}\overline{\beta}^i_S([a_t, a_m])}{|S|} \quad \text{(by statement (iii))} \\
&= \frac{\overline{\beta}^i_{S'}([a_t, a_m]) - \overline{\beta}^i_S([a_t, a_m])}{n} \quad \text{(select } i\in S \text{ and apply statement (iv))} \\
&\geq 0 \quad \text{(by Lemma 6.9.10), and}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\beta_{N\setminus S'}([a_1, a_s])}{|S'|} - \frac{\beta_{N\setminus S}([a_1, a_s])}{|S|} &= \frac{\sum_{i\in N}\frac{1}{n}\overline{\beta}^i_{N\setminus S'}([a_1, a_s])}{|S'|} - \frac{\sum_{i\in N}\frac{1}{n}\overline{\beta}^i_{N\setminus S}([a_1, a_s])}{|S|} \quad \text{(by Lemma 6.9.9)} \\
&= \frac{\sum_{i\in S'}\frac{1}{n}\overline{\beta}^i_{N\setminus S'}([a_1, a_s])}{|S'|} - \frac{\sum_{i\in S}\frac{1}{n}\overline{\beta}^i_{N\setminus S}([a_1, a_s])}{|S|} \quad \text{(by statement (iii))} \\
&= \frac{\overline{\beta}^i_{N\setminus S'}([a_1, a_s]) - \overline{\beta}^i_{N\setminus S}([a_1, a_s])}{n} \quad \text{(select } i\in J \text{ and apply statement (iv))} \\
&= \frac{\overline{\beta}^i_{N\setminus S}([a_{s+1}, a_m]) - \overline{\beta}^i_{N\setminus S'}([a_{s+1}, a_m])}{n} \\
&\geq 0. \quad \text{(by Lemma 6.9.10)}
\end{aligned}
$$

This completes the verification of the necessity part of Theorem 6.5.3.

6.10 Proof of Proposition 6.6.1

Proof: We first recall the deterministic version of a $(\underline{k}, \overline{k})$-RPFBR, which we call a $(\underline{k}, \overline{k})$-*Restricted Fixed Ballot Rule* (or $(\underline{k}, \overline{k})$-*RFBR*). Formally, a DSCF $f : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$ is called a $(\underline{k}, \overline{k})$-**Restricted Fixed Ballot Rule** (or $(\underline{k}, \overline{k})$-**RFBR**) if it is an Fixed Ballot Rule (or FBR), i.e., there exists a collection of deterministic ballots $(b_S)_{S\subseteq N}$ satisfying ballot unanimity, i.e., $b_N = a_m$ and $b_\emptyset = a_1$, and monotonicity, i.e., $[S \subset T \subseteq N] \Rightarrow [b_S \preceq b_T]$, such that for all $P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n$, we have $f(P) = \max^{\prec}_{S\subseteq N} \left(\min^{\prec}_{j\in S} \left(r_1(P_j), b_S\right)\right)$, and in addition, $(b_S)_{S\subseteq N}$ satisfy the **constrained dictatorship condition**, i.e., $\overline{k} - \underline{k} > 1$ implies that there exists $i \in N$ such that $[i \in S] \Rightarrow [b_S \in R]$ and $[i \notin S] \Rightarrow [b_S \in L]$.

Now, let $N = \{i, j\}$ and fix a two-voter $(\underline{k}, \overline{k})$-RPFBR $\phi : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^2 \to \Delta(A)$. Let $(\beta_S)_{S\subseteq N} = \left(\beta_\emptyset = e_{a_1}, \beta_{\{i\}}, \beta_{\{j\}}, \beta_N = e_{a_m}\right)$ be the corresponding probabilistic ballots. We are going to decompose ϕ as a mixture of finitely many $(\underline{k}, \overline{k})$-RFBRs.

Since $(\beta_S)_{S\subseteq N}$ satisfies the constrained random-dictatorship condition, let ε be the dictatorial coefficient of voter i, and $1 - \varepsilon$ be the dictatorial coefficient of voter j. Thus, ϕ behaves like a random dictatorship at all preference profiles where both voters' peaks are in M, i.e., $\phi(P_i, P_j) = \varepsilon\, e_{r_1(P_i)} + (1 - \varepsilon)\, e_{r_1(P_j)}$ for all $P_i, P_j \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ with $r_1(P_i), r_1(P_j) \in M$.

By the proof of the necessity part of Theorem 1 of our paper, we know that ϕ can be written as a mixture of several FBRs, i.e., there exist FBRs $f^k : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^2 \to A$, $k = 1, \ldots, q$, and weights $\alpha^1, \ldots, \alpha^q > 0$ with $\sum_{k=1}^q \alpha^k = 1$ such that $\phi(P_i, P_j) = \sum_{k=1}^q \alpha^k e_{f^k(P_i,P_j)}$ for all $P_i, P_j \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$. However, we only know that all FBRs $f^1, \ldots, f^q$ are strategy-proof on the single-peaked domain $\mathbb{D}_{\prec}$, and cannot ensure their strategy-proofness on the $(\underline{k}, \overline{k})$-hybrid domain $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$. For each $k = 1, \ldots, q$, let $(b^k_S)_{S\subseteq N}$ denote the deterministic ballots of f^k. For notational convenience, we slightly simplify the max-min form of each FBR f^k as follows: for all $P_i, P_j \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$,

$$\begin{aligned} f^k(P_i, P_j) &= \max^{\prec}\left(b^k_\emptyset = a_1, \min^{\prec}\left(r_1(P_i), b^k_{\{i\}}\right), \min^{\prec}\left(r_1(P_j), b^k_{\{j\}}\right), \min^{\prec}\left(r_1(P_i), r_1(P_j), b^k_N = a_m\right)\right) \\ &= \max^{\prec}\left(\min^{\prec}\left(r_1(P_i), b^k_{\{i\}}\right), \min^{\prec}\left(r_1(P_j), b^k_{\{j\}}\right), \min^{\prec}\left(r_1(P_i), r_1(P_j)\right)\right). \end{aligned}$$

Note that by Theorem 1 of our paper, f^k is strategy-proof if and only if $(b^k_S)_{S\subseteq N}$ satisfies the *constrained dictatorship condition*, i.e., either $b^k_{\{i\}} \in R$ and $b^k_{\{j\}} \in L$, or $b^k_{\{j\}} \in R$ and $b^k_{\{i\}} \in L$ hold.

Claim 1: For each $k = 1, \ldots, q$, we have $b^k_{\{i\}}, b^k_{\{j\}} \in L \cup R$.

Given $P_i, P_j \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ with $r_1(P_i) = a_{\underline{k}}$ and $r_1(P_j) = a_{\overline{k}}$, we have $\sum_{k=1}^q \alpha^k\, e_{f^k(P_i,P_j)} = \phi(P_i, P_j) = \varepsilon e_{a_{\underline{k}}} + (1 - \varepsilon) e_{a_{\overline{k}}}$. This implies that for each $k = 1, \ldots, q$,

$\max^{\prec}\Big(\min^{\prec}\big(a_{\underline{k}}, b^k_{\{i\}}\big), \min^{\prec}\big(a_{\overline{k}}, b^k_{\{j\}}\big), a_{\underline{k}}\Big) = f^k(P_i, P_j) \in \{a_{\underline{k}}, a_{\overline{k}}\}$. Consequently, it must be the case that $b^k_{\{i\}}, b^k_{\{j\}} \in L \cup R$ for all $k = 1, \ldots, q$. This completes the verification of the claim.

By Claim 1, we know that an FBR f^k is manipulable on $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ if and only if $b^k_{\{i\}}, b^k_{\{j\}} \in L$ or $b^k_{\{i\}}, b^k_{\{j\}} \in R$. Accordingly, we separate all FBRs $f^1, \ldots, f^q$ into three groups:

$$\begin{aligned}\Lambda\ &= \Big\{f^k : \text{either } b^k_{\{i\}} \in R \text{ and } b^k_{\{j\}} \in L, \text{ or } b^k_{\{j\}} \in R \text{ and } b^k_{\{i\}} \in L\Big\},\\ \Lambda^L &= \Big\{f^k : b^k_{\{i\}}, b^k_{\{j\}} \in L\Big\} \text{ and } \Lambda^R = \Big\{f^k : b^k_{\{i\}}, b^k_{\{j\}} \in R\Big\}.\end{aligned}$$

If $\Lambda^L = \emptyset$ and $\Lambda^R = \emptyset$, then ϕ is decomposable. Henceforth, assume either $\Lambda^L \neq \emptyset$ or $\Lambda^R \neq \emptyset$. We are going to reshuffle the deterministic ballots of all FBRs in $\Lambda^L \cup \Lambda^R$ to "cure" all FBRs of $\Lambda^L \cup \Lambda^R$. The next claim shows that the total weights of FBRs in Λ^L equals that in Λ^R.

Claim 2: $\sum_{k: f^k \in \Lambda^L} \alpha^k = \sum_{k: f^k \in \Lambda^R} \alpha^k$.

Fix $P_i, P_j \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ with $r_1(P_i) = a_{\underline{k}}$ and $r_1(P_j) = a_{\overline{k}}$, and $P'_i, P'_j \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ with $r_1(P'_i) = a_{\overline{k}}$ and $r_1(P'_j) = a_{\underline{k}}$. We first know that

(i) ϕ behaves like a random dictatorship at both (P_i, P_j) and (P'_i, P'_j),

(ii) each $f^k \in \Lambda$ behaves like a dictatorship at both (P_i, P_j) and (P'_i, P'_j), and let i^k denote the corresponding constrained dictator,

(iii) for each $f^k \in \Lambda^L$,

$$\begin{aligned} f^k(P_i, P_j) &= \max^{\prec}\Big(\min^{\prec}\big(a_{\underline{k}}, b^k_{\{i\}}\big), \min^{\prec}\big(a_{\overline{k}}, b^k_{\{j\}}\big), \min^{\prec}\big(a_{\underline{k}}, a_{\overline{k}}\big)\Big) = a_{\underline{k}}, \text{ and}\\ f^k(P'_i, P'_j) &= \max^{\prec}\Big(\min^{\prec}\big(a_{\overline{k}}, b^k_{\{i\}}\big), \min^{\prec}\big(a_{\underline{k}}, b^k_{\{j\}}\big), \min^{\prec}\big(a_{\overline{k}}, a_{\underline{k}}\big)\Big) = a_{\underline{k}}, \end{aligned}$$

(iv) for each $f^k \in \Lambda^R$,

$$\begin{aligned} f^k(P_i, P_j) &= \max^{\prec}\Big(\min^{\prec}\big(a_{\underline{k}}, b^k_{\{i\}}\big), \min^{\prec}\big(a_{\overline{k}}, b^k_{\{j\}}\big), \min^{\prec}\big(a_{\underline{k}}, a_{\overline{k}}\big)\Big) = a_{\overline{k}}, \text{ and}\\ f^k(P'_i, P'_j) &= \max^{\prec}\Big(\min^{\prec}\big(a_{\overline{k}}, b^k_{\{i\}}\big), \min^{\prec}\big(a_{\underline{k}}, b^k_{\{j\}}\big), \min^{\prec}\big(a_{\overline{k}}, a_{\underline{k}}\big)\Big) = a_{\overline{k}}. \end{aligned}$$

First, item (i) implies $\phi_{a_{\underline{k}}}(P_i, P_j) = \varepsilon = \phi_{a_{\overline{k}}}(P'_i, P'_j)$. Next, by items (ii), (iii) and (iv), we have

$$\phi_{a_{\underline{k}}}(P_i,P_j)=\sum_{k=1}^{q}\alpha^k \mathbb{1}\big(f^k(P_i,P_j)=a_{\underline{k}}\big)=\sum_{k:f^k\in\Lambda}\alpha^k\,\mathbb{1}(i^k=i)+\sum_{k:f^k\in\Lambda^L\cup\Lambda^R}\alpha^k \mathbb{1}(f^k(P_i,P_j)=a_{\underline{k}})$$
$$=\sum_{k:f^k\in\Lambda}\alpha^k\,\mathbb{1}(i^k=i)+\sum_{k:f^k\in\Lambda^L}\alpha^k,\ \text{and}$$

$$\phi_{a_{\overline{k}}}(P_i',P_j')=\sum_{k=1}^{q}\alpha^k \mathbb{1}\big(f^k(P_i',P_j')=a_{\overline{k}}\big)=\sum_{k:f^k\in\Lambda}\alpha^k\,\mathbb{1}(i^k=i)+\sum_{k:f^k\in\Lambda^L\cup\Lambda^R}\alpha^k \mathbb{1}(f^k(P_i,P_j)=a_{\overline{k}})$$
$$=\sum_{k:f^k\in\Lambda}\alpha^k\,\mathbb{1}(i^k=i)+\sum_{k:f^k\in\Lambda^R}\alpha^k.$$

Therefore, $\sum_{k:f^k\in\Lambda^L}\alpha^k=\sum_{k:f^k\in\Lambda^R}\alpha^k$. This completes the verification of the claim.

By Claim 2, the hypothesis that either $\Lambda^L\neq\emptyset$ or $\Lambda^R\neq\emptyset$ implies $\Lambda^L\neq\emptyset$ and $\Lambda^R\neq\emptyset$. Fixing $f^s\in\Lambda^L$ and $f^t\in\Lambda^R$, according to their deterministic ballots $\big(b^s_\emptyset=a_1, b^s_{\{i\}}\in L, b^s_{\{j\}}\in L, b^s_I=a_m\big)$ and $\big(b^t_\emptyset=a_1, b^t_{\{i\}}\in R, b^t_{\{j\}}\in R, b^t_I=a_m\big)$, we swap $b^s_{\{j\}}$ and $b^t_{\{j\}}$, and create two new sets of deterministic ballots

$$(\bar{b}^s_S)_{S\subseteq N}=\big(\bar{b}^s_\emptyset=a_1, \bar{b}^s_{\{i\}}=b^s_{\{i\}}\in L, \bar{b}^s_{\{j\}}=b^t_{\{j\}}\in R, \bar{b}^s_N=a_m\big)\ \text{and}$$
$$(\bar{b}^t_S)_{S\subseteq N}=\big(\bar{b}^t_\emptyset=a_1, \bar{b}^t_{\{i\}}=b^t_{\{i\}}\in R, \bar{b}^t_{\{j\}}=b^s_{\{j\}}\in L, \bar{b}^t_N=a_m\big).$$

Note that both $(\bar{b}^s_S)_{S\subseteq N}$ and $(\bar{b}^t_S)_{S\subseteq N}$ satisfy ballot unanimity, monotonicity and the constrained dictatorship condition. Correspondingly, we generate two $(\underline{k},\overline{k})$-RFBRs $\bar{f}^s:\big[\mathbb{D}_{\mathrm{H}}(\underline{k},\overline{k})\big]^2\to\Delta(A)$ and $\bar{f}^t:\big[\mathbb{D}_{\mathrm{H}}(\underline{k},\overline{k})\big]^2\to\Delta(A)$ which are strategy-proof by Theorem 1 of our paper. More importantly, since $e_{\bar{b}^s_{\{i\}}}+e_{\bar{b}^t_{\{i\}}}=e_{b^s_{\{i\}}}+e_{b^t_{\{i\}}}$ and $e_{\bar{b}^s_{\{j\}}}+e_{\bar{b}^t_{\{j\}}}=e_{b^t_{\{j\}}}+e_{b^s_{\{j\}}}$, it is true that $e_{\bar{f}^s(P_i,P_j)}+e_{\bar{f}^t(P_i,P_j)}=e_{f^s(P_i,P_j)}+e_{f^t(P_i,P_j)}$ for all $P_i,P_j\in\mathbb{D}_{\mathrm{H}}(\underline{k},\overline{k})$. Assume w.l.o.g. that $\alpha^s\geq\alpha^t$. We then reformulate ϕ by using $\bar{f}^s,\bar{f}^t$ and $(f^k)_{k\neq t}$: for all $P_i,P_j\in\mathbb{D}_{\mathrm{H}}(\underline{k},\overline{k})$, we have

$$\phi(P_i,P_j)=\sum_{k:f^k\in\Lambda}\alpha^k e_{f^k(P_i,P_j)}+\left[\Big[\alpha^s e_{f^s(P_i,P_j)}+\alpha^t e_{f^t(P_i,P_j)}\Big]+\sum_{k\notin\{s,t\}:f^k\in\Lambda^L\cup\Lambda^R}\alpha^k e_{f^k(P_i,P_j)}\right]$$
$$=\left[\sum_{k:f^k\in\Lambda}\alpha^k e_{f^k(P_i,P_j)}+\alpha^t\Big[e_{\bar{f}^s(P_i,P_j)}+e_{\bar{f}^t(P_i,P_j)}\Big]\right]+\left[\sum_{k\notin\{s,t\}:f^k\in\Lambda^L\cup\Lambda^R}\alpha^k e_{f^k(P_i,P_j)}+(\alpha^s-\alpha^t)e_{f^s(P_i,P_j)}\right].$$

In the reformulation, two new $(\underline{k},\overline{k})$-RFBRs are added, the manipulable FBR f^t is eliminated, and the weight of the manipulable FBR f^s reduces to $\alpha_s-\alpha_t$. Since Λ^L and Λ^R are finite and $\sum_{k:f^k\in\Lambda^L}\alpha^k=\sum_{k:f^k\in\Lambda^R}\alpha^k$ by Claim 2, by repeatedly reshuffling deterministic ballots and reformulating ϕ, we eventually are able to write ϕ as a mixture of finitely many $(\underline{k}, \overline{k})$-RFBRs. Therefore, we assert that ϕ is decomposable. ■

6.11 Proof of Proposition 6.6.2

Proof: Fix a $(\underline{k}, \overline{k})$-RPFBR $\phi : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$. Let $(\beta_S)_{S\subseteq N}$ denote the probabilistic ballots of ϕ. Thus, $(\beta_S)_{S\subseteq N}$ satisfies ballot unanimity, monotonicity and the constrained random-dictatorship condition. Let $\varepsilon_i \geq 0$ be the dictatorial coefficient of voter i and $\sum_{i\in N} \varepsilon_i = 1$. Thus, for all $S \subseteq N$, $\beta_S([a_{\overline{k}}, a_m]) = \sum_{i\in S} \varepsilon_i$ and $\beta_S([a_1, a_{\underline{k}}]) = \sum_{i\in N\setminus S} \varepsilon_i$. Next, since ϕ is decomposable, there are $(\underline{k}, \overline{k})$-PFBR $f^k : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to A$, $k = 1, \ldots, q$, and weights $\alpha^1, \ldots, \alpha^q > 0$ with $\sum_{k=1}^q \alpha^k = 1$ such that $\phi(P) = \sum_{k=1}^q \alpha^k e_{f^k(P)}$ for all $P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n$. For each $k = 1, \ldots, q$, let $(b_S^k)_{S\subseteq N}$ denote the deterministic ballots of f^k. Thus, $(b_S^k)_{S\subseteq N}$ satisfies ballot unanimity, monotonicity and the constrained dictatorship condition. Correspondingly, let i^k denote the constrained dictator in f^k.

Fixing a nonempty $S, T \subseteq N$ with $S \cap T = \emptyset$, $a_t \in R$ and $a_s \in L$, we have

$$\begin{aligned}
\beta_S([a_t, a_m]) + \beta_T([a_t, a_m]) &= \sum_{k=1}^q \alpha^k \mathbb{1}\big(b_S^k \in [a_t, a_m]\big) + \sum_{k=1}^q \alpha^k \mathbb{1}\big(b_T^k \in [a_t, a_m]\big) \\
&= \sum_{k=1}^q \alpha^k \mathbb{1}\big(i^k \in S \text{ and } b_S^k \in [a_t, a_m]\big) + \sum_{k=1}^q \alpha^k \mathbb{1}\big(i^k \in T \text{ and } b_T^k \in [a_t, a_m]\big) \\
&\leq \sum_{k=1}^q \alpha^k \mathbb{1}\big(i^k \in S \text{ and } b_{S\cup T}^k \in [a_t, a_m]\big) + \sum_{k=1}^q \alpha^k \mathbb{1}\big(i^k \in T \text{ and } b_{S\cup T}^k \in [a_t, a_m]\big) \\
&= \sum_{k=1}^q \alpha^k \mathbb{1}\big(i^k \in S \cup T \text{ and } b_{S\cup T}^k \in [a_t, a_m]\big) \\
&= \beta_{S\cup T}([a_t, a_m]), \text{ and} \\
\beta_{N\setminus S}([a_1, a_s]) + \beta_{N\setminus T}([a_1, a_s]) &= \sum_{k=1}^q \alpha^k \mathbb{1}\big(b_{N\setminus S}^k \in [a_1, a_s]\big) + \sum_{k=1}^q \alpha^k \mathbb{1}\big(b_{N\setminus T}^k \in [a_1, a_s]\big) \\
&= \sum_{k=1}^q \alpha^k \mathbb{1}\big(i^k \in S \text{ and } b_{N\setminus S}^k \in [a_1, a_s]\big) + \sum_{k=1}^q \alpha^k \mathbb{1}\big(i^k \in T \text{ and } b_{N\setminus T}^k \in [a_1, a_s]\big) \\
&\leq \sum_{k=1}^q \alpha^k \mathbb{1}\big(i^k \in S \text{ and } b_{N\setminus[S\cup T]}^k \in [a_1, a_s]\big) + \sum_{k=1}^q \alpha^k \mathbb{1}\big(i^k \in T \text{ and } b_{N\setminus[S\cup T]}^k \in [a_1, a_s]\big) \\
&= \sum_{k=1}^q \alpha^k \mathbb{1}\big(i^k \in S \cup T \text{ and } b_{N\setminus[S\cup T]}^k \in [a_1, a_s]\big) \\
&= \beta_{N\setminus[S\cup T]}([a_1, a_s]).
\end{aligned}$$

Therefore, ϕ satisfies the scale-effect condition. ■

6.12 Proof of Proposition 6.6.3

Proof: We first provide a lemma which will be repeated adopted.

Lemma 6.12.1 *Fixing a $(\underline{k}, \overline{k})$-RPFBR $\phi : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$, let $(\hat{\beta}_S)_{S \subseteq N}$ be the corresponding probabilistic ballots. RPFBR ϕ dominates φ in admitting compromises if and only if for all $S \subseteq N$ with $1 \leq |S| \leq n-1$ and $a_k \in [a_2, \ldots, a_{\underline{k}}] \cup [a_{\overline{k}}, a_{m-1}]$, $\hat{\beta}_S(a_k) \geq \beta_S(a_k)$, and there exist $S \subseteq N$ with $1 \leq |S| \leq n-1$ and $a_k \in [a_2, \ldots, a_{\underline{k}}] \cup [a_{\overline{k}}, a_{m-1}]$ such that $\hat{\beta}_S(a_k) > \beta_S(a_k)$.*

Proof: We first show the necessity part of Lemma 6.12.1. Given $S \subseteq N$ with $1 \leq |S| \leq n-1$ and $a_k \in [a_2, \ldots, a_{\underline{k}}] \cup [a_{\overline{k}}, a_{m-1}]$, we consider the preference profile P where every voter of S has the preference peak a_{k+1}, every voter of $N \backslash S$ has the preference peak a_{k-1}, and all voters share the common second best alternative a_k. Such a preference profile is admissible in $\left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n$. Thus, $P \in \mathcal{C}\left(\left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n\right)$ and $c(P) = a_k$. Note that $S(k, P) = S(k+1, P) = S$. Then, we have

$$
\begin{aligned}
\hat{\beta}_S(a_k) - \beta_S(a_k) &= \left[\hat{\beta}_S([a_k, a_m]) - \hat{\beta}_S([a_{k+1}, a_m])\right] - \left[\beta_S([a_k, a_m]) - \beta_S([a_{k+1}, a_m])\right] \\
&= \left[\hat{\beta}_{S(k,P)}([a_k, a_m]) - \hat{\beta}_{S(k+1,P)}([a_{k+1}, a_m])\right] - \left[\beta_{S(k,P)}([a_k, a_m]) - \beta_{S(k+1,P)}([a_{k+1}, a_m])\right] \\
&= \phi_{a_k}(P) - \varphi_{a_k}(P) \geq 0.
\end{aligned}
$$

Next, by definition, there exists a profile $P \in \mathcal{C}\left(\left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n\right)$ such that $\phi_{c(P)}(P) > \varphi_{c(P)}(P)$. Evidently, $\phi_{c(P)}(P) > 0$. Let $c(P) = a_k$. We first show that $a_k \in [a_2, \ldots, a_{\underline{k}}] \cup [a_{\overline{k}}, a_{m-1}]$. Suppose not, i.e., either $a_k \in \{a_1, a_m\}$ or $a_k \in [a_{\underline{k}+1}, a_{\overline{k}-1}]$. If $a_k = a_1$, by the definition of $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$, $c(P) = a_1$ implies $r_1(P_i) = a_2$ for all $i \in N$ which contradicts the hypothesis that $P \in \mathcal{C}\left(\left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n\right)$. The similar contradiction arises if $a_k = a_m$. Next, if $a_k \in [a_{\underline{k}+1}, a_{\overline{k}-1}]$, by the definition of $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$, $c(P) = a_k$ implies $r_1(P_i) \in M$ for all $i \in N$. Consequently, the constrained random-dictatorship condition implies $\phi_{a_k}(P) = 0$. Contradiction! Therefore, $a_k \in [a_2, \ldots, a_{\underline{k}}] \cup [a_{\overline{k}}, a_{m-1}]$.

Now, we consider three cases: (1) $a_k \in [a_2, \ldots, a_{\underline{k}-1}] \cup [a_{\overline{k}+1}, a_{m-1}]$, (2) $a_k = a_{\underline{k}}$ and (3) $a_k = a_{\overline{k}}$. In case (1), by the definition of $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$, $P \in \mathcal{C}\left(\left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n\right)$ implies that there exists $S' \subseteq N$ with $1 \leq |S| \leq n-1$ such that $r_1(P_i) = a_{k+1}$ for all $i \in S$ and $r_1(P_j) = a_{k-1}$ for all $j \in N \backslash S$. We then have $\hat{\beta}_S(a_k) = \phi_{a_k}(P) > \varphi_{a_k}(P) = \beta_S(a_k)$. In case (2), by the definition of $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$, $P \in \mathcal{C}\left(\left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n\right)$ implies that there exists $S \subseteq N$ with $1 \leq |S| \leq n-1$ such that $r_1(P_i) \in M \backslash \{a_{\underline{k}}\}$ for all $i \in S$ and $r_1(P_j) = a_{\underline{k}-1}$ for all $j \in N \backslash S$. Then, similar to case (1), we have $\hat{\beta}_S(a_k) > \beta_S(a_k)$. Last, in case (3), by the definition of $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$, $P \in \mathcal{C}\left(\left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n\right)$ implies that there exists $S \subseteq N$ with $1 \leq |S| \leq n-1$ such that $r_1(P_i) = a_{\overline{k}+1}$ for all $i \in S$ and $r_1(P_j) \in M \backslash \{a_{\overline{k}}\}$ for all $j \in N \backslash S$. Then, similar to case (1), we have $\hat{\beta}_S(a_k) > \beta_S(a_k)$. This completes the verification of the necessity part.

Next, we turn to showing the sufficiency part of Lemma 6.12.1. Given a profile $P \in \mathcal{C}\big([\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})]^n \big)$, let $c(P) = a_k$. We first show $\phi_{a_k}(P) \geq \varphi_{a_k}(P)$. One of the following four cases must occur:

(i) $a_k \in [a_{\underline{k}+1}, a_{\overline{k}-1}]$ and $r_1(P_i) \in M$ for all $i \in M$,

(ii) $a_k \in [a_2, \ldots, a_{\underline{k}-1}] \cup [a_{\overline{k}+1}, a_{m-1}]$, and there exists $S \subseteq N$ with $1 \leq |S| \leq n-1$ such that $r_1(P_i) = a_{k+1}$ for all $i \in S$ and $r_1(P_j) = a_{k-1}$ for all $j \in N\backslash S$,

(iii) $a_k = a_{\underline{k}}$, and there exists $S \subseteq N$ with $1 \leq |S| \leq n-1$ such that $r_1(P_i) \in M\backslash\{a_{\underline{k}}\}$ for all $i \in S$ and $r_1(P_j) = a_{\underline{k}-1}$ for all $j \in N\backslash S$, and

(iv) $a_k = a_{\overline{k}}$, and there exists $S \subseteq N$ with $1 \leq |S'| \leq n-1$ such that $r_1(P_i) = a_{\overline{k}+1}$ for all $i \in S'$ and $r_1(P_j) \in M\backslash\{a_{\overline{k}}\}$ for all $j \in N\backslash S$.

In case (i), the constrained random-dictatorship condition implies $\phi_{a_k}(P) = \varphi_{a_k}(P) = 0$. In all cases (ii) - (iv), first note that $S(k, P) = S(k+1, P) = S$. Then, we have

$$
\begin{aligned}
\phi_{a_k}(P) - \varphi_{a_k}(P) = &\Big[\hat{\beta}_{S(k,P)}([a_k, a_m]) - \hat{\beta}_{S(k+1,P)}([a_{k+1}, a_m])\Big] - \big[\beta_{S(k,P)}([a_k, a_m]) - \beta_{S(k+1,P)}([a_{k+1}, a_m])\big] \\
= &\Big[\hat{\beta}_S([a_k, a_m]) - \hat{\beta}_S([a_{k+1}, a_m])\Big] - \big[\beta_S([a_k, a_m]) - \beta_S([a_{k+1}, a_m])\big] \\
= &\hat{\beta}_S(a_k) - \beta_S(a_k) \geq 0.
\end{aligned}
$$

Last, note that there exist $S \subseteq N$ with $1 \leq |S| \leq n-1$ and $a_k \in [a_2, \ldots, a_{\underline{k}}] \cup [a_{\overline{k}}, a_{m-1}]$ such that $\hat{\beta}_S(a_k) > \beta_S(a_k)$. According to the coalition S, we construct a preference profile $P \in [\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})]^n$ where every voter of S has the preference peak a_{k+1}, every voter of $N\backslash S$ has the preference peak a_{k-1}, and all voters share the common second best alternative a_k. Thus, $P \in \mathcal{C}\big([\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})]^n\big)$ and $c(P) = a_k$. Since $S(k, P) = S(k+1, P) = S$, we have

$$
\begin{aligned}
\phi_{a_k}(P) - \varphi_{a_k}(P) = &\Big[\hat{\beta}_{S(k,P)}([a_k, a_m]) - \hat{\beta}_{S(k+1,P)}([a_{k+1}, a_m])\Big] - \big[\beta_{S(k,P)}([a_k, a_m]) - \beta_{S(k+1,P)}([a_{k+1}, a_m])\big] \\
= &\Big[\hat{\beta}_S([a_k, a_m]) - \hat{\beta}_S([a_{k+1}, a_m])\Big] - \big[\beta_S([a_k, a_m]) - \beta_S([a_{k+1}, a_m])\big] \\
= &\hat{\beta}_S(a_k) - \beta_S(a_k) > 0.
\end{aligned}
$$

Therefore, ϕ dominates φ in admitting compromises. This completes the verification of the sufficiency part, and hence proves Lemma 6.12.1. ■

Now, we start to prove Proposition 6.6.3. Let $(\underline{k}, \overline{k})$-RPFBR $\phi : [\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})]^n \to \Delta(A)$ dominate φ in admitting compromises. Let $(\hat{\beta}_S)_{S \subseteq N}$ denote the probabilistic ballots of ϕ. We show that there exists $S \subseteq N$ with $|S| = n-1$ such that $\beta_S(a_m) > 0$ or $\beta_{N\backslash S}(a_1) > 0$. Suppose not, i.e., for all $S \subseteq N$ with

$|S| = n-1$, $\beta_S(a_m) = 0$ and $\beta_{N\backslash S}(a_1) = 0$. First, monotonicity implies $\beta_{S'}(a_m) \leq \beta_S(a_m) = 0$ and $\beta_{N\backslash S'}(a_1) \leq \beta_{N\backslash S}(a_1) = 0$ for all $S' \subseteq N$ with $1 \leq |S'| < n-1$. Hence, for all $S \subseteq N$ with $1 \leq |S| \leq n-1$, we have $\beta_S(a_m) = 0$ and $\beta_{N\backslash S}(a_1) = 0$. By Lemma 6.12.1, there exists a coalition $S \subseteq N$ with $1 \leq |S| \leq n-1$ such that $\hat{\beta}_S(a_k) \geq \beta_S(a_k)$ for all $a_k \in [a_2, \ldots, a_{\underline{k}}] \cup [a_{\overline{k}}, a_{m-1}]$ and $\hat{\beta}_S(a_v) > \beta_S(a_v)$ for some $a_v \in [a_2, \ldots, a_{\underline{k}}] \cup [a_{\overline{k}}, a_{m-1}]$. Note that (i) $\beta_S(a_m) = 0$ and $\beta_S(a_1) = \beta_{N\backslash[N\backslash S]}(a_1) = 0$, and (ii) $\beta_S(a_k) = 0$ for all $a_k \in [a_{\underline{k}+1}, a_{\overline{k}-1}]$ by the constrained random-dictatorship condition. Hence, $\beta_S([a_2, a_{\underline{k}}]) + \beta_S([a_{\overline{k}}, a_{m-1}]) = 1$. Consequently, we induce the following contradiction:

$$\begin{aligned}\sum_{a_k \in A} \hat{\beta}_S(a_k) =& \hat{\beta}_S(a_1) + \hat{\beta}_S(a_m) + \hat{\beta}_S([a_{\underline{k}+1}, a_{\overline{k}-1}]) + \Big[\hat{\beta}_S([a_2, a_{\underline{k}}]) + \hat{\beta}_S([a_{\overline{k}}, a_{m-1}])\Big] \\ >& \hat{\beta}_S(a_1) + \hat{\beta}_S(a_m) + \hat{\beta}_S([a_{\underline{k}+1}, a_{\overline{k}-1}]) + \big[\beta_S([a_2, a_{\underline{k}}]) + \beta_S([a_{\overline{k}}, a_{m-1}])\big] \geq 1.\end{aligned}$$

Next, let $\beta_S(a_m) > 0$ or $\beta_S(a_1) > 0$ for some $S \subseteq N$ with $|S| = n-1$. We construct a $(\underline{k}, \overline{k})$-RPFBR $\phi : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$, and show that ϕ dominates φ in admitting compromises. For notational convenience, let $S = \{1, \ldots, n-2, n-1\}$. We construct the following probabilistic ballots: for all $S' \subseteq N$ with $1 \leq |S'| \leq n-1$,

$$\hat{\beta}_{S'}(a_k) = \begin{cases} 0 & \text{if } a_k \in \{a_1, a_m\}, \\ \beta_{S'}(a_m) + \beta_S(a_{m-1}) & \text{if } a_k = a_{m-1}, \\ \beta_{S'}(a_1) + \beta_S(a_2) & \text{if } a_k = a_2, \text{ and} \\ \beta_{S'}(a_k) & \text{otherwise.} \end{cases}$$

In other words, we construct $\hat{\beta}_{S'}$ by transferring the probability of a_m in $\beta_{S'}$ to a_{m-1}, transferring the probability of a_1 in $\beta_{S'}$ to a_2, and keeping the probability of every other alternative in $\beta_{S'}$ unchanged. Meanwhile, let $\hat{\beta}_N = e_{a_m}$ and $\hat{\beta}_\emptyset = e_{a_1}$. It is easy to verify that $(\hat{\beta}_{S'})_{S' \subseteq N}$ satisfy ballot unanimity, monotonicity and the constrained random-dictatorship condition. Therefore, the corresponding PFBR $\phi : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$ is a $(\underline{k}, \overline{k})$-RPFBR. Furthermore, by construction, we know that $\hat{\beta}_{S'}(a_k) \geq \beta_{S'}(a_k)$ for all $S' \subseteq N$ with $1 \leq |S'| \leq n-1$ and $a_k \in [a_2, \ldots, a_{\underline{k}}] \cup [a_{\overline{k}}, a_{m-1}]$, and $\hat{\beta}_S(a_{m-1}) = \beta_S(a_{m-1}) + \beta_S(a_m) > \beta_S(a_{m-1})$ or $\hat{\beta}_S(a_2) = \beta_S(a_2) + \beta_S(a_1) > \beta_S(a_2)$. Then, Lemma 6.12.1 implies that ϕ dominates φ in admitting compromises. This completes the verification of the first part of Proposition 6.6.3.

Last, let φ be anonymous and decomposable, and $S \subseteq N$ be such that $|S| = n-2$, and $\beta_S(a_m) > 0$ or $\beta_{N\backslash S}(a_1) > 0$. We assume w.l.o.g. that $\beta_S(a_m) > 0$. We construct an anonymous non-decomposable $(\underline{k}, \overline{k})$-RPFBR $\phi : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$, and show that ϕ dominates φ in admitting compromises. For notational convenience, let $S = \{1, \ldots, n-2\}$ and $\overline{S} = \{1, \ldots, n-2, n-1\}$. Given an arbitrary $\hat{S} \subseteq N$

with $|\hat{S}| = n-1$, by anonymity and monotonicity, we know $\beta_{\hat{S}}(a_m) = \beta_{\overline{S}}(a_m) \geq \beta_S(a_m) > 0$. Moreover, since φ is decomposable, by anonymity and Theorem 2 of our paper, we have $\frac{\beta_{\hat{S}}(a_m)}{n-1} = \frac{\beta_{\overline{S}}(a_m)}{n-1} \geq \frac{\beta_S(a_m)}{n-2}$. Thus, $\beta_{\hat{S}}(a_m) > \beta_S(a_m)$. Now, we construct new probabilistic ballots: for all $\hat{S} \subseteq N$ with $1 \leq |\hat{S}| \leq n-1$,

$$\hat{\beta}_{\hat{S}} = \begin{cases} \beta_{\hat{S}} & \text{if } |\hat{S}| < n-1, \\ \beta_{\hat{S}} - [\beta_{\hat{S}}(a_m) - \beta_S(a_m)]e_{a_m} + [\beta_{\hat{S}}(a_m) - \beta_S(a_m)]e_{a_{m-1}} & \text{otherwise.} \end{cases}$$

In other word, when coalition $\hat{S}$ has less than $n-1$ voters, we fix $\hat{\beta}_{\hat{S}}$ to $\beta_{\hat{S}}$, and when coalition $\hat{S}$ has $n-1$ voters, we lower the probability of a_m to that in β_S, and transfer the remaining probability of a_m to a_{m-1}. Moreover, let $\hat{\beta}_N = e_{a_m}$ and $\hat{\beta}_\emptyset = e_{a_1}$. It is easy to verify that $(\hat{\beta}_{\hat{S}})_{\hat{S}\subseteq N}$ satisfy ballot unanimity, monotonicity and the constrained random-dictatorship condition. Therefore, the corresponding PFBR $\phi : [\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})]^n \to \Delta(A)$ is a $(\underline{k}, \overline{k})$-RPFBR. Moreover, it is easy to show that $(\hat{\beta}_{\hat{S}})_{\hat{S}\subseteq N}$ is invariant to the size of coalitions. Therefore, ϕ is anonymous. However, $(\hat{\beta}_{\hat{S}})_{\hat{S}\subseteq N}$ violate per-capita monotonicity, i.e., $\frac{\hat{\beta}_{\overline{S}}(a_m)}{n-1} = \frac{\hat{\beta}_S(a_m)}{n-1} < \frac{\hat{\beta}_S(a_m)}{n-2}$. Therefore, ϕ is not decomposable by Theorem 2 of our paper. Last, by construction, we know that $\hat{\beta}_{\hat{S}}(a_k) \geq \beta_S(a_k)$ for all $\hat{S} \subseteq N$ with $1 \leq |\hat{S}| \leq n-1$ and $a_k \in [a_2, \dots, a_{\underline{k}}] \cup [a_{\overline{k}}, a_{m-1}]$, and $\hat{\beta}_{\overline{S}}(a_{m-1}) = \beta_{\overline{S}}(a_{m-1}) + \beta_{\overline{S}}(a_m) - \beta_S(a_m) > \beta_{\overline{S}}(a_{m-1})$. Then, Lemma 6.12.1 implies that ϕ dominates φ in admitting compromises. This completes the verification of the second part of Proposition 6.6.3. ■

6.13 Proof of Theorem 6.7.2

Let domain $\mathbb{D}$ satisfy the weak no-restoration property and contain two completely reversed preferences. Thus, $\mathbb{D}$ is connected. Note that $\mathbb{D}$ is minimally richness. We first show that $\mathbb{D}$ is $(\underline{k}, \overline{k})$-hybrid for some unique $\underline{k}$ and $\overline{k}$. The proof consists of Lemmas 6.13.1 - 6.13.7.

We first introduce an important new notion. A pair of distinct alternatives $a_s, a_t \in A$ is said **adjacent** in $\mathbb{D}$, denoted $a_s \sim a_t$, if there exist $P_i, P_i' \in \mathbb{D}$ with $r_1(P_i) = a_s$ and $r_1(P_i') = a_t$ such that $P_i \sim P_i'$. Then, we induce a graph, denoted by $G_{\mathbb{D}}$, such that the set of vertex is A, and in the set of edges, every pair of alternatives forms an edge if and only if they are adjacent in $\mathbb{D}$. An **alternative-path**, denoted by $\mathcal{P}$, connecting a_s and a_t is a sequence of (non-repeated) vertices $\{x_k\}_{k=1}^l \subseteq A$ such that $x_1 = a_s$, $x_l = a_t$ and $x_k \sim x_{k+1}$ for all $k = 1, \dots, l-1$. For notational convenience, let $\Pi(a_s, a_t)$ denote the set of *all* alternative-paths connecting a_s and a_t,[14] and $\langle a_s, a_t \rangle$ denote *one* alternative-path connecting a_s and a_t.

Lemma 6.13.1 *Every pair of distinct alternatives $a_s, a_t \in A$ is connected via an alternative-path, i.e., $\Pi(a_s, a_t) \neq \emptyset$.*

[14] In particular, if $a_s = a_t$, then $\Pi(a_s, a_t) = \{\{a_s\}\}$ is a singleton set of a null alternative-path.

Proof: Given $P_i \in \mathbb{D}$ with $r_1(P_i) = a_s$ and $P_i' \in \mathbb{D}$ with $r_1(P_i') = a_t$ by minimal richness, since $\mathbb{D}$ is connected, we have a path $\{P_i^k\}_{k=1}^t \subseteq \mathbb{D}$ connecting P_i and P_i'. We partition $\{P_i^k\}_{k=1}^t$ according to the peaks of preferences (without rearranging preferences in the path), and elicit all preference peaks:

$$\left\{ \frac{P_i^1, \dots, P_i^{k_1}}{\text{the same peak } x_1}, \frac{P_i^{k_1+1}, \dots, P_i^{k_2}}{\text{the same peak } x_2}, \dots, \frac{P_i^{k_{q-1}+1}, \dots, P_i^t}{\text{the same peak } x_q} \right\} \longrightarrow \text{Elicit peaks}\{x_1, x_2, \dots, x_q\},$$

where $x_k \neq x_{k+1}$ and $x_k \sim x_{k+1}$ for all $k = 1, \dots, q-1$. Note that $\{x_1, x_2, \dots, x_q\}$ may contain repetitions. Whenever a repetition appears, we remove all alternatives strictly between the repetition and one alternative of the repetition. For instance, if $x_k = x_l$ where $1 \leq k < l \leq q$, we remove $x_k, x_{k+1}, \dots, x_{l-1}$, and refine the sequence to $\{x_1, \dots, x_{k-1}, x_l, \dots, x_q\}$. By repeatedly eliminating repetitions, we finally elicit an alternative-path $\{x_k\}_{k=1}^p$ connecting a_s and a_t. ■

Let $\underline{P}_i$ and $\overline{P}_i$ be the pair of completely reversed preferences contained in $\mathbb{D}$. Assume w.l.o.g. that $\underline{P}_i = (a_1 \cdots a_{k-1} a_k \cdots a_m)$ and $\overline{P}_i = (a_m \cdots a_k a_{k-1} \cdots a_1)$. Note that the way we specify $\underline{P}_i$ and $\overline{P}_i$ determines the labeling of all alternatives.

Lemma 6.13.2 *Given distinct $a_p, a_s, a_t \in A$, let a_t be included in every alternative-path of $\Pi(a_p, a_s)$. Given $P_i \in \mathbb{D}$, we have $[r_1(P_i) = a_p] \Rightarrow [a_t P_i a_s]$ and $[r_1(P_i) = a_s] \Rightarrow [a_t P_i a_p]$.*

Proof: Suppose that $r_1(P_i) = a_p$ and $a_s P_i a_t$. Pick an arbitrary preference $P_i' \in \mathbb{D}$ with $r_1(P_i') = a_s$ by minimal richness. By the weak no-restoration property, there exists a path $\{P_i^k\}_{k=1}^l \subseteq \mathbb{D}$ connecting P_i and P_i' such that $a_s P_i^k a_t$ for all $k = 1, \dots, l$. Thus, $r_1(P_i^k) \neq a_t$ for all $k = 1, \dots, l$. According to path $\{P_i^k\}_{k=1}^l$, we elicit an alternative-path $\langle a_p, a_s \rangle$ which excludes a_t. This contradicts the hypothesis of the lemma. Therefore, $a_t P_i a_s$. Symmetrically, if $r_1(P_i) = a_s$, then $a_t P_i a_p$. ■

Lemma 6.13.3 *Given $a_s, a_t \in A \backslash \{a_1, a_m\}$ with $a_s \sim a_t$, If one alternative-path of $\Pi(a_1, a_m)$ includes a_t, there exists an alternative-path of $\Pi(a_1, a_m)$ including a_s.*

Proof: Let $\{x_k\}_{k=1}^p \in A$ and $a_t = x_\eta$ for some $1 < \eta < p$. If $a_s \in \{x_k\}_{k=1}^p$, the lemma holds evidently. Henceforth, assume $a_s \notin \{x_k\}_{k=1}^p$. Note the alternative-path $\{a_1 = x_1, x_2, \dots, x_\eta = a_t, a_s\} \in \Pi(a_1, a_s)$, and the alternative-path $\{a_s, a_t = x_\eta, \dots, x_{p-1}, x_p = a_m\} \in \Pi(a_s, a_m)$.

Since $\underline{P}$ and $\overline{P}_i$ are completely reversed, either $a_s \underline{P}_i a_t$ or $a_s \overline{P}_i a_t$ holds. Assume w.l.o.g. that $a_s \underline{P}_i a_t$. The verification related to $a_s \overline{P}_i a_t$ is symmetric and we hence omit it. Pick an arbitrary preference $P_i \in \mathbb{D}$ with $r_1(P_i) = a_s$ by minimal richness. By the weak no-restoration property, we have a path $\{P_i^k\}_{k=1}^v \subseteq \mathbb{D}$ connecting $\underline{P}_i$ and P_i such that $a_s P_i^k a_t$ for all $k = 1, \dots, v$. Thus, $r_1(P_i^k) \neq a_t$ for all $k = 1, \dots, v$. According to $\{P_i^k\}_{k=1}^v$, we elicit an alternative-path $\{y_k\}_{k=1}^q \in \Pi(a_1, a_s)$ such that $a_t \notin \{y_k\}_{k=1}^q$.

Evidently, $\{y_k\}_{k=1}^q \cap \{x_k\}_{k=1}^p \supseteq \{a_1\}$. If $\{y_k\}_{k=1}^q \cap \{x_k\}_{k=1}^p = \{a_1\}$, then the concatenated alternative-path $\{a_1 = y_1, \ldots, y_q = a_s; a_t = x_\eta, \ldots, x_p = a_m\} \in \Pi(a_1, a_m)$ includes a_s. Next, we assume $\{y_k\}_{k=1}^q \cap \{x_k\}_{k=1}^p \supset \{a_1\}$. We identify the alternative in $\{y_k\}_{k=1}^q$ that has the maximum index and is also included in $\{x_k\}_{k=1}^p$, i.e., $y_{\hat{k}} = x_{k^*}$ for some $1 < \hat{k} < q$ and $1 < k^* \leq p$ and $\{y_{\hat{k}+1}, \ldots, y_q\} \cap \{x_k\}_{k=1}^p = \emptyset$. Note that $a_t = x_\eta$, $1 < \eta < p$ and $a_t \neq y_{\hat{k}}$. Therefore, either $1 < k^* < \eta$ or $\eta < k^* \leq p$ must hold. If $1 < k^* < \eta$, the concatenated alternative-path $\{a_1 = x_1, \ldots, x_{k^*} = y_{\hat{k}}; y_{\hat{k}+1}, \ldots, y_q = a_s; a_t = x_\eta, \ldots, x_p = a_m\} \in \Pi(a_1, a_m)$ includes a_s. If $\eta < k^* \leq p$, the concatenated alternative-path $\{a_1 = x_1, \ldots, x_\eta = a_t; a_s = y_q, \ldots, y_{\hat{k}+1}; y_{\hat{k}} = x_{k^*}, \ldots, x_p = a_m\} \in \Pi(a_1, a_m)$ includes a_s. ■

Lemma 6.13.4 *Given $a_s \in A \backslash \{a_1, a_m\}$, there exists an alternative-path of $\Pi(a_1, a_m)$ including a_s.*

Proof: Pick an arbitrary preference $P_i \in \mathbb{D}$ with $r_1(P_i) = a_s$ by minimal richness. Note that $a_s \underline{P}_i a_m$ and $a_s P_i a_m$. By the weak no-restoration property, we have a path $\{P_i^k\}_{k=1}^l \subseteq \mathbb{D}$ connecting $\underline{P}_i$ and P_i such that $a_s P_i^k a_m$ for all $k = 1, \ldots, l$. Thus, $r_1(P_i^k) \neq a_m$ for all $k = 1, \ldots, l$. According to $\{P_i^k\}_{k=1}^l$, we elicit an alternative-path $\{x_k\}_{k=1}^p \in \Pi(a_1, a_s)$ that excludes a_m. Symmetrically, we have an alternative-path $\{y_k\}_{k=1}^q \in \Pi(a_s, a_m)$ that excludes a_1. Thus, $\{x_k\}_{k=1}^p \cap \{y_k\}_{k=1}^q \supseteq \{a_s\}$. If $\{x_k\}_{k=1}^p \cap \{y_k\}_{k=1}^q = \{a_s\}$, then the concatenated alternative-path $\{a_1 = x_1, \ldots, x_p = a_s = y_1, \ldots, y_q = a_m\} \in \Pi(a_1, a_m)$ includes a_s. If $\{x_k\}_{k=1}^p \cap \{y_k\}_{k=1}^q \supset \{a_s\}$, we identify the alternative a_t included in both $\{x_k\}_{k=1}^p$ and $\{y_k\}_{k=1}^q$ with the maximum index in $\{x_k\}_{k=1}^p$ and the minimum index in $\{y_k\}_{k=1}^q$, i.e., $a_t = x_{\hat{k}} = y_{k^*}$ for some $1 < \hat{k} < p$ and $1 < k^* < q$ such that $\{x_1, \ldots, x_{\hat{k}-1}\} \cap \{y_{k^*+1}, \ldots, y_q\} = \emptyset$. Thus, the concatenated alternative-path $\{x_1, \ldots, x_{\hat{k}-1}, x_{\hat{k}} = a_t = y_{k^*}, y_{k^*+1}, \ldots, y_q\} \in \Pi(a_1, a_m)$ includes a_t, and excludes a_s. Furthermore, we refer to the sub-alternative-path $\{a_t = x_{\hat{k}}, \ldots, x_p = a_s\}$, by repeatedly applying Lemma 6.13.3 step by step from a_t to a_s along the sub-alternative-path, we eventually find an alternative-path of $\Pi(a_1, a_m)$ that includes a_s. ■

Note that $\Pi(a_1, a_m)$ is a finite nonempty set. Hence, we label $\Pi(a_1, a_m) = \{\mathcal{P}_1, \ldots, \mathcal{P}_n\}$, and make sure that each alternative-path of $\Pi(a_1, a_m)$ starts from a_1 and ends at a_m. Given $\mathcal{P}_l \in \Pi(a_1, a_m)$ and $a_s, a_t \in \mathcal{P}_l$, let $\langle a_s, a_t \rangle^{\mathcal{P}_l}$ denote the interval between a_s and a_t on $\mathcal{P}_l$.

Lemma 6.13.5 *If $\Pi(a_1, a_m)$ is a singleton set, $\mathbb{D}$ is $(\underline{k}, \overline{k})$-hybrid for all $1 \leq \underline{k} < \overline{k} \leq m$ with $\overline{k} - \underline{k} = 1$.*

Proof: Since $\Pi(a_1, a_m)$ is a singleton set, Lemma 6.13.4 implies that all alternatives must be included in a unique alternative-path. Thus, $G_{\mathbb{D}}$ must be a line and include all alternatives. More importantly, Lemma 6.13.2 implies that all preferences of $\mathbb{D}$ must be single-peaked w.r.t. $G_{\mathbb{D}}$. Since $\underline{P}_i$ and $\overline{P}_i$ are single-peaked w.r.t. $G_{\mathbb{D}}$, it must be the case that $G_{\mathbb{D}}$ is a line of $\{a_1, a_2, \ldots, a_k, a_{k+1}, \ldots, a_m\}$ which coincides to the natural order $\prec$. Hence, $\mathbb{D} \subseteq \mathbb{D}_{\prec} = \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ for all $1 \leq \underline{k} < \overline{k} \leq m$ with $\overline{k} - \underline{k} = 1$. Evidently, as $\mathbb{D}_{\mathrm{H}}(\underline{k}', \overline{k}')$, where $\underline{k}' > \underline{k}$ or $\overline{k}' < \overline{k}$, is not well defined, $\mathbb{D} \nsubseteq \mathbb{D}_{\mathrm{H}}(\underline{k}', \overline{k}')$. ■

Henceforth, we assume that $\Pi(a_1, a_m)$ is not a singleton set. Since all alternative-paths of $\Pi(a_1, a_m)$ start from a_1 and end at a_m, we can identify the left maximum common part and the right maximum common part of all alternative-paths of $\Pi(a_1, a_m)$, i.e., there exist two alternatives $a_{\underline{k}}, a_{\overline{k}} \in A$ (either $\underline{k} \leq \overline{k}$ or $\underline{k} \geq \overline{k}$ so far) such that the following three conditions are satisfied:

(i) $a_{\underline{k}}, a_{\overline{k}} \in \mathcal{P}_l$ for all $\mathcal{P}_l \in \Pi(a_1, a_m)$,

(ii) $\langle a_1, a_{\underline{k}} \rangle^{\mathcal{P}_l} = \langle a_1, a_{\underline{k}} \rangle^{\mathcal{P}_v}$, and $\langle a_{\overline{k}}, a_m \rangle^{\mathcal{P}_l} = \langle a_{\overline{k}}, a_m \rangle^{\mathcal{P}_v}$ for all $\mathcal{P}_l, \mathcal{P}_v \in \Pi(a_1, a_m)$, and

(iii) there exist *no* $a_{\underline{k}'}, a_{\overline{k}'} \in A$ such that $a_{\underline{k}'}, a_{\overline{k}'} \in \mathcal{P}_l$ for all $\mathcal{P}_l \in \Pi(a_1, a_m)$, and $\langle a_1, a_{\underline{k}} \rangle^{\mathcal{P}_l} \subset \langle a_1, a_{\underline{k}'} \rangle^{\mathcal{P}_l}$ or $\langle a_{\overline{k}}, a_m \rangle^{\mathcal{P}_l} \subset \langle a_{\overline{k}'}, a_m \rangle^{\mathcal{P}_l}$ for all $\mathcal{P}_l \in \Pi(a_1, a_m)$.

We claim that $a_{\underline{k}} \neq a_{\overline{k}}$. Otherwise, $\Pi(a_1, a_m)$ degenerates to a singleton set. Note that condition (iii) implies that $a_{\underline{k}}$ and $a_{\overline{k}}$ are unique. Fix an arbitrary $\mathcal{P}_l \in \Pi(a_1, a_m)$. We first claim $\langle a_1, a_{\underline{k}} \rangle^{\mathcal{P}_l} \cap \langle a_{\overline{k}}, a_m \rangle^{\mathcal{P}_l} = \emptyset$. Suppose not, i.e., there exists $a_s \in \langle a_1, a_{\underline{k}} \rangle^{\mathcal{P}_l} \cap \langle a_{\overline{k}}, a_m \rangle^{\mathcal{P}_l}$ such that $\langle a_1, a_s \rangle^{\mathcal{P}_l} \cap \langle a_s, a_m \rangle^{\mathcal{P}_l} = \{a_s\}$. Since $a_{\underline{k}} \neq a_{\overline{k}}$, we know either $a_s \neq a_{\underline{k}}$ or $a_s \neq a_{\overline{k}}$. Consequently, the concatenated alternative-path $\{\langle a_1, a_s \rangle^{\mathcal{P}_l}, \langle a_s, a_m \rangle^{\mathcal{P}_l}\} \in \Pi(a_1, a_m)$ excludes either $a_{\underline{k}}$ or $a_{\overline{k}}$, which contradicts condition (i). Therefore, $\langle a_1, a_{\underline{k}} \rangle^{\mathcal{P}_l} \cap \langle a_{\overline{k}}, a_m \rangle^{\mathcal{P}_l} = \emptyset$. Next, we claim that $\langle a_1, a_{\underline{k}} \rangle^{\mathcal{P}_l} \cup \langle a_{\overline{k}}, a_m \rangle^{\mathcal{P}_l} \neq A$. Otherwise, condition (ii) implies $\langle a_1, a_{\underline{k}} \rangle^{\mathcal{P}_v} \cup \langle a_{\overline{k}}, a_m \rangle^{\mathcal{P}_v} = A$ for all $\mathcal{P}_v \in \Pi(a_1, a_m)$, and consequently, $\Pi(a_1, a_m)$ degenerates to a singleton set.

Lemma 6.13.6 *The following two statements hold:*

(i) $\Pi(a_1, a_{\underline{k}})$ *is a singleton set of the unique alternative-path* $\{a_1, \ldots, a_k, a_{k+1}, \ldots, a_{\underline{k}}\}$.

(ii) $\Pi(a_{\overline{k}}, a_m)$ *is a singleton set of the unique alternative-path* $\{a_{\overline{k}}, \ldots, a_k, a_{k+1}, \ldots, a_m\}$.

Proof: By symmetry, we show the first statement, and omit the verification of the second statement.

First, let $\Pi(a_1, a_{\underline{k}})$ be a singleton set. We show that $\Pi(a_1, a_{\underline{k}}) = \big\{\{a_1, \ldots, a_k, a_{k+1}, \ldots, a_{\underline{k}}\}\big\}$, which coincides to the nature order $\prec$ from a_1 to $a_{\underline{k}}$. Since $\Pi(a_1, a_{\underline{k}})$ is a singleton set, Lemma 6.13.2 implies that all preferences of $\mathbb{D}$ must be single-peaked w.r.t. the unique alternative-path of $\Pi(a_1, a_{\underline{k}})$. Moreover, since the completely reversed preferences $\underline{P}_i = (a_1 \cdots a_k a_{k+1} \cdots a_{\underline{k}} \cdots a_{\overline{k}} \cdots a_m)$ and $\overline{P}_i = (a_m \cdots a_{\overline{k}} \cdots a_{\underline{k}} \cdots a_{k+1} a_k \cdots a_1)$ are contained in $\mathbb{D}$, this implies that the unique alternative-path of $\Pi(a_1, a_{\underline{k}})$ must be $\{a_1, \ldots, a_k, a_{k+1}, \ldots, a_{\underline{k}}\}$.

Next, we show that $\Pi(a_1, a_{\underline{k}})$ is a singleton set. If $a_1 = a_{\underline{k}}$, statement (i) holds by the definition of $\Pi(a_1, a_{\underline{k}})$. We next assume $a_1 \neq a_{\underline{k}}$. Pick an arbitrary alternative-path

$\mathcal{P}_l = \{a_1 = x_1, \ldots, x_v = a_{\underline{k}}, \ldots, x_t = a_m\} \in \Pi(a_1, a_m)$. Given an arbitrary alternative-path $\langle a_1, a_{\underline{k}}\rangle = \{a_1 = y_1, \ldots, y_u = a_{\underline{k}}\}$, we show $\langle a_1, a_{\underline{k}}\rangle = \langle a_1, a_{\underline{k}}\rangle^{\mathcal{P}_l}$. Since $a_{\underline{k}} = x_v = y_u$, we can identify the alternative $y_{\hat{k}} = x_{k^*}$ for some $1 < \hat{k} \leq u$ and $v \leq k^* \leq t$ such that $\{y_1, \ldots, y_{\hat{k}-1}\} \cap \{x_{k^*+1}, \ldots, x_t\} = \emptyset$. Then, we have a concatenated alternative-path $\mathcal{P}_v = \{y_1, \ldots, y_{\hat{k}-1}, y_{\hat{k}} = x_{k^*}, x_{k^*+1}, \ldots, x_t\} \in \Pi(a_1, a_m)$. By condition (i) above, we know $a_{\underline{k}} \in \mathcal{P}_v$. Since $a_{\underline{k}} \notin \{y_1, \ldots, y_{\hat{k}-1}\}$ and $a_{\underline{k}} \notin \{x_{k^*+1}, \ldots, x_t\}$, it must be the case $y_{\hat{k}} = a_{\underline{k}}$ and $x_{k^*} = a_{\underline{k}}$. Hence, $\langle a_1, a_{\underline{k}}\rangle = \langle a_1, a_{\underline{k}}\rangle^{\mathcal{P}_v}$. Last, by condition (ii) above, we have $\langle a_1, a_{\underline{k}}\rangle = \langle a_1, a_{\underline{k}}\rangle^{\mathcal{P}_v} = \langle a_1, a_{\underline{k}}\rangle^{\mathcal{P}_l}$. Since both $\mathcal{P}_l$ and $\langle a_1, a_{\underline{k}}\rangle$ are arbitrarily selected, $\langle a_1, a_{\underline{k}}\rangle = \langle a_1, a_{\underline{k}}\rangle^{\mathcal{P}_l}$ implies that $\Pi(a_1, a_{\underline{k}})$ is a singleton set. ■

Henceforth, let $L = \{a_1, \ldots, a_k, a_{k+1}, \ldots, a_{\underline{k}}\}$, $R = \{a_{\overline{k}}, \ldots, a_k, a_{k+1}, \ldots, a_m\}$ and $M = \{a_{\underline{k}}, \ldots, a_k, a_{k+1}, \ldots, a_{\overline{k}}\}$. As mentioned before, we know $\overline{k} - \underline{k} > 1$.

Lemma 6.13.7 *Domain $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$, and $\mathbb{D} \not\subseteq \mathbb{D}_{\mathrm{H}}(\underline{k}', \overline{k}')$ where $\underline{k}' > \underline{k}$ or $\overline{k}' < \overline{k}$.*

Proof: By Lemma 6.13.2, we know that all preferences of $\mathbb{D}$ are single-peaked w.r.t. the natural order $\prec$ on both L and R. Therefore, the first restriction of Definition 6.3.1 is satisfied. We focus on showing the second restriction of Definition 6.3.1.

Fix $P_i \in \mathbb{D}$ with $r_1(P_i) = a_p \in L$ and $a_r \in M\backslash\{a_{\underline{k}}\}$. If $a_p = a_{\underline{k}}$, $a_{\underline{k}} P_i a_r$ holds evidently. We next assume $a_p \neq a_{\underline{k}}$. By Lemma 6.13.2, to prove $a_{\underline{k}} P_i a_r$, it suffices to show that $a_{\underline{k}}$ is included in every alternative-path of $\Pi(a_p, a_r)$. Suppose not, i.e., there exists an alternative-path $\langle a_p, a_r\rangle$ such that $a_{\underline{k}} \notin \langle a_p, a_r\rangle$. Since $a_p \neq a_{\underline{k}}$, we have the alternative-path $\langle a_1, a_p\rangle = \{a_1, \ldots, a_k, a_{k+1}, \ldots, a_p\}$ which excludes $a_{\underline{k}}$. Next, if $a_r = a_{\overline{k}}$, we have the alternative-path $\langle a_r, a_m\rangle = \{a_{\overline{k}}, \ldots, a_m\}$ which excludes $a_{\underline{k}}$. If $a_r \in M\backslash\{a_{\underline{k}}, a_{\overline{k}}\}$, by Lemma 6.13.4, we have an alternative-path $\mathcal{P}_l \in \Pi(a_1, a_m)$ that includes a_r. Moreover, by condition (i) above and Lemma 6.13.6, we write $\mathcal{P}_l = \{a_1, \ldots, a_{\underline{k}}, x_1, \ldots, x_t, a_{\overline{k}}, \ldots, a_m\}$ where $a_r = x_v \in \{x_1, \ldots, x_t\} \subseteq M\backslash\{a_{\underline{k}}, a_{\overline{k}}\}$ for some $1 \leq v \leq t$. Then, we have an alternative-path $\{a_r = x_v, \ldots, x_t, a_{\overline{k}}, \ldots, a_m\}$ which excludes $a_{\underline{k}}$. Overall, we have an alternative-path $\langle a_r, a_m\rangle$ that excludes $a_{\underline{k}}$. Now, we have three alternative-paths $\langle a_1, a_p\rangle$, $\langle a_p, a_r\rangle$ and $\langle a_r, a_m\rangle$ which all exclude $a_{\underline{k}}$. By combining them and removing repeated alternatives, we can construct an alternative-path of $\Pi(a_1, a_m)$ that excludes $a_{\underline{k}}$. This contradicts condition (i) above. Therefore, $a_{\underline{k}}$ is included in every alternative-path of $\Pi(a_p, a_r)$, as required. Symmetrically, given $P_i \in \mathbb{D}$ with $r_1(P_i) \in R$ and $a_s \in M\backslash\{a_{\overline{k}}\}$, we have $a_{\overline{k}} P_i a_s$.

Last, recall condition (iii) above. Since $a_{\underline{k}}$ and $a_{\overline{k}}$ are uniquely identified, $\mathbb{D} \not\subseteq \mathbb{D}_{\mathrm{H}}(\underline{k}', \overline{k}')$ where $\underline{k}' > \underline{k}$ or $\overline{k}' < \overline{k}$. This completes the verification of the lemma, and hence proves the first part of Theorem 6.7.2. ■

Now, we turn to the second part of Theorem 6.7.2. By the first part of Theorem 6.7.2, we know that $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ for some $1 \leq \underline{k} < \overline{k} \leq m$ and $\mathbb{D} \not\subseteq \mathbb{D}_{\mathrm{H}}(\underline{k}', \overline{k}')$ where $\underline{k}' > \underline{k}$ and $\overline{k}' < \overline{k}$. By the sufficiency part of Theorem 6.5.1, it is evident that every $(\underline{k}, \overline{k})$-RPFBR is unanimous and strategy-proof on $\mathbb{D}$.

Therefore, we focus on showing that every unanimous and strategy-proof on $\mathbb{D}$ is a $(\underline{k}, \overline{k})$-RPFBR. We provides four independent lemmas which show some important properties on all unanimous and strategy-proof RSCFs defined on $\mathbb{D}$. Then, these four lemmas together enable us to complete the characterization of $(\underline{k}, \overline{k})$-RPFBRs.

Lemma 6.13.8 *Every unanimous and strategy-proof RSCF $\phi : \mathbb{D}^n \to \Delta(A)$ satisfies the tops-only property.*

Proof: Fix a unanimous and strategy-proof RSCF $\phi : \mathbb{D}^n \to \Delta(A)$. To prove the tops-only property, it suffices to show that for all $i \in N$, $P_i, P'_i \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{n-1}$, $[r_1(P_i) = r_1(P'_i)] \Rightarrow [\phi(P_i, P_{-i}) = \phi(P'_i, P_{-i})]$.

We prove this in two steps. In the first step, by the proof of Theorem 1 of [31], we know that ϕ satisfies the following property: for all $i \in N$, $P_i, P'_i \in \mathbb{D}$ with $P_i \sim P'_i$ and $P_{-i} \in \mathbb{D}^{n-1}$, $[r_1(P_i) = r_1(P'_i)] \Rightarrow [\phi(P_i, P_{-i}) = \phi(P'_i, P_{-i})]$.[15] In the second step, we consider $P_i, P'_i \in \mathbb{D}$ such that $r_1(P_i) = r_1(P'_i) \equiv a_s$, but P_i is not adjacent to P'_i.

First, strategy-proofness implies $\phi_{a_s}(P_i, P_{-i}) = \phi_{a_s}(P'_i, P_{-i})$. Next, pick an arbitrary $a_t \in A \backslash \{a_s\}$, we show $\phi_{a_t}(P_i, P_{-i}) = \phi_{a_t}(P'_i, P_{-i})$. By the weak no-restoration property, there exists a path $\{P_i^k\}_{k=1}^q \subseteq \mathbb{D}$ connecting P_i and P'_i such that $a_s P_i^k a_t$ for all $k = 1, \ldots, q$. Start from P_i^2. If $r_1(P_i^2) = r_1(P_i^1)$, the result in the first step implies $\phi_{a_t}(P_i^1, P_{-i}) = \phi_{a_t}(P_i^2, P_{-i})$. If $r_1(P_i^2) = a_r \neq a_s = r_1(P_i^1)$, then $P_i^1 \sim P_i^2$ implies $r_1(P_i^1) = r_2(P_i^2) = a_s$, $r_1(P_i^2) = r_2(P_i^1) = a_r$ and $r_l(P_i^1) = r_l(P_i^2)$ for all $l = 3, \ldots, m$. Hence, it must be the case that $a_t = r_l(P_i^1) = r_l(P_i^2)$ for some $3 \leq l \leq m$, and then strategy-proofness implies $\phi_{a_t}(P_i^1, P_{-i}) = \phi_{a_t}(P_i^2, P_{-i})$. Overall, we have $\phi_{a_t}(P_i^1, P_{-i}) = \phi_{a_t}(P_i^2, P_{-i})$. By repeatedly applying this argument along the path from P_i^2 to P_i^q, we eventually have $\phi_{a_t}(P_i^k, P_{-i}) = \phi_{a_t}(P_i^{k+1}, P_{-i})$ for all $k = 1, \ldots, q-1$. Hence, $\phi_{a_t}(P_i, P_{-i}) = \phi_{a_t}(P'_i, P_{-i})$. Therefore, $\phi(P_i, P_{-i}) = \phi(P'_i, P_{-i})$, as required. ■

Since $\mathbb{D}$ is minimally rich, the tops-only property implies that every unanimous and strategy-proof $\varphi : \mathbb{D}^n \to \Delta(A)$ degenerates to a *random voting scheme* $\varphi : A^n \to \Delta(A)$. Given an arbitrary random voting scheme $\varphi : A^n \to \Delta(A)$, we say that (i) φ is *unanimous* on $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ if for all $(P_1, \ldots, P_N) \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n$, $[r_1(P_1) = \cdots = r_1(P_n) = a_k] \Rightarrow [\varphi(a_k, \ldots, a_k) = e_{a_k}]$, and (ii) φ is *strategy-proof* (respectively, *locally strategy-proof*) on $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ if for all $i \in N$, $P_i, P'_i \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ (respectively, $P_i \sim P'_i$) and $P_{-i} \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^{n-1}$, $\varphi\big(r_1(P_i), r_1(P_{-i})\big)$ stochastically dominates $\varphi\big(r_1(P'_i), r_1(P_{-i})\big)$ according to P_i, where $r_1(P_{-i}) = \big(r_1(P_1), \ldots, r_1(P_{i-1}), r_1(P_{i+1}), \ldots, r_1(P_n)\big)$.

To show a unanimous and strategy-proof $\phi : \mathbb{D}^n \to \Delta(A)$ is a $(\underline{k}, \overline{k})$-RPFBR, by Lemma 6.13.8, Fact 6.8 and the necessity part of Theorem 6.5.1, it suffices to show that the corresponding random voting

[15] [31] introduce the interior and exterior properties on a domain and show that they together are sufficient for endogenizing the tops-only property on all unanimous and strategy-proof RSCFs. The weak no-restoration property implies the exterior property, but may not be compatible with the interior property. However, the proof of their Theorem 1 can be directly applied to show the first-step result here.

scheme $\phi : A^n \to \Delta(A)$ is unanimous and locally strategy-proof on $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$. Note that both $\mathbb{D}$ and $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ are minimally rich. Consequently, since RSCF ϕ is unanimous and satisfies the tops-only property, it follows immediately that the random voting scheme $\phi : A^n \to \Delta(A)$ is unanimous on $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$. In the rest of the proof, we show that every random voting scheme, which is induced from a unanimous and strategy-proof RSCF $\phi : \mathbb{D}^n \to \Delta(A)$, is locally strategy-proof on $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$.

For notational convenience, with a little notational abuse, we write (a_s, a_t) as a two-voter preference profile where the first voter presents a preference with peak a_s while the second reports a preference with peak a_t. We also write (a_s, P_{-i}) as an n-voter preference profile where voter i presents a preference with peak a_s and $P_{-i} = (P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n)$.

Lemma 6.13.9 (The uncompromising property) *Let $\phi : \mathbb{D}^n \to \Delta(A)$ be a unanimous and strategy-proof RSCF. Given an alternative-path $\{x_k\}_{k=1}^t$, $i \in I$ and $P_{-i} \in \mathbb{D}^{n-1}$, we have $\phi_{a_s}(x_1, P_{-i}) = \phi_{a_s}(x_t, P_{-i})$ for all $a_s \notin \{x_k\}_{k=1}^t$, and hence $\sum_{k=1}^t \phi_{x_k}(x_1, P_{-i}) = \sum_{k=1}^t \phi_{x_k}(x_t, P_{-i})$.*

Proof: We start with $\phi(x_1, P_{-i})$ and $\phi(x_2, P_{-i})$. Since $x_1 \sim x_2$, we have $P_i \in \mathbb{D}^{x_1}$ and $P_i' \in \mathbb{D}^{x_2}$ such that $P_i \sim P_i'$. Then, the tops-only property and strategy-proofness imply
$\phi_{a_s}(x_1, P_{-i}) = \phi_{a_s}(P_i, P_{-i}) = \phi_{a_s}(P_i', P_{-i}) = \phi_{a_s}(x_2, P_{-i})$ for all $a_s \notin \{x_1, x_2\}$.

We next introduce an induction hypothesis: Given $2 < k \leq t$, for all $2 \leq k' < k$, $\phi_{a_s}(x_1, P_{-i}) = \phi_{a_s}(x_{k'}, P_{-i})$ for all $a_s \notin \{x_l\}_{l=1}^{k'}$. We show $\phi_{a_s}(x_1, P_{-i}) = \phi_{a_s}(x_k, P_{-i})$ for all $a_s \notin \{x_l\}_{l=1}^{k}$. Since $x_k \sim x_{k-1}$, we have $P_i \in \mathbb{D}^{x_k}$ and $P_i' \in \mathbb{D}^{x_{k-1}}$ such that $P_i \sim P_i'$. Then, the tops-only property and strategy-proofness imply $\phi_{a_s}(x_k, P_{-i}) = \phi_{a_s}(P_i, P_{-i}) = \phi_{a_s}(P_i', P_{-i}) = \phi_{a_s}(x_{k-1}, P_{-i})$ for all $a_s \notin \{x_{k-1}, x_k\}$. Moreover, since $\phi_{a_s}(x_1, P_{-i}) = \phi_{a_s}(x_{k-1}, P_{-i})$ for all $a_s \notin \{x_l\}_{l=1}^{k-1}$ by the induction hypothesis, it is true that $\phi_{a_s}(x_1, P_{-i}) = \phi_{a_s}(x_k, P_{-i})$ for all $a_s \notin \{x_l\}_{l=1}^{k}$. This completes the verification of the induction hypothesis. Therefore, $\phi_{a_s}(x_1, P_{-i}) = \phi_{a_s}(x_t, P_{-i})$ for all $a_s \notin \{x_k\}_{k=1}^t$. Then, we have $\sum_{k=1}^t \phi_{x_k}(x_1, P_{-i}) = 1 - \sum_{a_s \notin \{x_k\}_{k=1}^t} \phi_{a_s}(x_1, P_{-i}) = 1 - \sum_{a_s \notin \{x_k\}_{k=1}^t} \phi_{a_s}(x_t, P_{-i}) = \sum_{k=1}^t \phi_{x_k}(x_t, P_{-i})$. ■

Now, we can show that if $\overline{k} - \underline{k} = 1$, every unanimous and strategy-proof $\phi : \mathbb{D}^n \to \Delta(A)$ is a PFBR. Recall that $\overline{k} - \underline{k} = 1$ implies $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k}) = \mathbb{D}_{\prec}$. Correspondingly, Lemma 6.13.9 degenerates to the uncompromising property of [46], and the random voting scheme $\phi : A^n \to \Delta(A)$ satisfies the uncompromising property on $\mathbb{D}_{\prec}$. Furthermore, Lemma 3.2 of [46] implies that the random voting scheme ϕ is strategy-proof on $\mathbb{D}_{\prec}$, as required. This completes the verification of the second part of Theorem 6.7.2 in the case $\overline{k} - \underline{k} = 1$. Henceforth, we assume $\overline{k} - \underline{k} > 1$. We first make two observations on graph $G_{\mathbb{D}}$, which will be repeatedly used in the following-up proof. Given $a_s \in M \backslash \{a_{\underline{k}}, a_{\overline{k}}\}$, there exists an alternative-path $\langle a_{\underline{k}}, a_{\overline{k}} \rangle \subseteq M$ that includes a_s. □ There exists a cycle $\mathcal{C}_1 = \{x_k\}_{k=1}^p \subseteq M$, $p \geq 3$, i.e., $x_k \sim x_{k+1}$ for all $k = 1, \ldots, p$ where $x_{p+1} = x_1$, such that $a_{\underline{k}} \in \mathcal{C}_1$.[16] There exists a cycle

[16]By the identification of $a_{\underline{k}}$, we know that there exist at least two distinct alternatives of M that are adjacent to $a_{\underline{k}}$ in $\mathbb{D}$. Then,

$\mathcal{C}_2 = \{y_k\}_{k=1}^{q} \subseteq M$, $q \geq 3$, i.e., $y_k \sim y_{k+1}$ for all $k = 1, \ldots, p-1$ where $y_{q+1} = y_1$, such that $a_{\overline{k}} \in \mathcal{C}_2$. □

Lemma 6.13.10 *Every unanimous and strategy-proof RSCF $\phi : \mathbb{D}^n \to \Delta(A)$ behaves like a random dictatorship on the subdomain $\overline{\mathbb{D}} = \{P_i \in \mathbb{D} : r_1(P_i) \in M\}$, i.e., there exists a conditional dictatorial coefficient $\varepsilon_i \geq 0$ for each $i \in N$ with $\sum_{i\in N} \varepsilon_i = 1$ such that $\phi(P) = \sum_{i\in N} \varepsilon_i \, e_{r_1(P_i)}$ for all $P \in \overline{\mathbb{D}}^n$.*

Proof: We verify this lemma in two steps. In the first step, we restrict attention to the case $n = 2$, i.e., $N = \{1, 2\}$, and show by Claims 1 - 4 below that every two-voter unanimous and strategy-proof RSCF on $\mathbb{D}$ behaves like a random dictatorship on subdomain $\overline{\mathbb{D}}$. In the second step, we extend the result to the case $n > 2$ by adopting the Ramification Theorem of [35].

Fix a unanimous and strategy-proof RSCF $\phi : \mathbb{D}^2 \to \Delta(A)$. By Lemma 6.13.8, ϕ satisfies the tops-only property.

Claim 1: The following two statements hold:

(i) Given an alternative-path $\{z_k\}_{k=1}^{l}$, we have $\sum_{k=1}^{l} \phi_{z_k}(z_1, z_l) = 1$.

(ii) Given a circle $\{z_k\}_{k=1}^{l}$, we have $\phi_{z_s}(z_s, z_t) + \phi_{z_t}(z_s, z_t) = 1$ for all $s \neq t$.

The first statement follows immediately from unanimity and the uncompromising property. Next, consider the circle $\{z_k\}_{k=1}^{l}$. Fixing z_s and z_t, assume w.l.o.g. that $s < t$. There are two alternative-paths connecting z_s and z_t: the clockwise alternative-path $\mathcal{P} = \{z_s, z_{s+1}, \ldots, z_t\}$ and the counter clockwise alternative-path $\mathcal{P}' = \{z_s, z_{s-1}, \ldots, z_1, z_l, z_{l-1}, \ldots, z_t\}$. It follows immediately from statement (i) that $\sum_{z\in\mathcal{P}} \phi_z(z_s, z_t) = 1$ and $\sum_{z\in\mathcal{P}'} \phi_z(z_s, z_t) = 1$. Last, since $\mathcal{P} \cap \mathcal{P}' = \{z_s, z_t\}$, it is true that $\phi_{z_s}(z_s, z_t) + \phi_{z_t}(z_s, z_t) = 1$. This completes the verification of the claim.

Claim 2: According to the cycle $\mathcal{C}_1 = \{x_k\}_{k=1}^{p}$ of Observation 6.13, ϕ behaves like a random dictatorship on the subdomain $\mathbb{D}^{\mathcal{C}_1} = \{P_i \in \mathbb{D} : r_1(P_i) \in \mathcal{C}_1\}$, i.e., there exists $0 \leq \varepsilon \leq 1$ such that $\phi(x_k, x_{k'}) = \varepsilon e_{x_k} + (1-\varepsilon) e_{x_{k'}}$ for all $x_k, x_{k'} \in \mathcal{C}_1$.

Claim 1(ii) first implies $\phi_{x_1}(x_1, x_2) + \phi_{x_2}(x_1, x_2) = 1$. Let $\varepsilon = \phi_{x_1}(x_1, x_2)$ and $1 - \varepsilon = \phi_{x_2}(x_1, x_2)$. Fix another profile $(x_k, x_{k'})$. If $x_k = x_{k'}$, unanimity implies $\phi(x_k, x_{k'}) = \varepsilon e_{x_k} + (1-\varepsilon) e_{x_{k'}}$. We next assume $x_k \neq x_{k'}$. There are four possible cases: (i) $x_1 \neq x_k$ and $x_2 = x_{k'}$, (ii) $x_1 = x_k$ and $x_2 \neq x_{k'}$, (iii) $x_1 \neq x_k$, $x_2 \neq x_{k'}$ and $(x_k, x_{k'}) \neq (x_2, x_1)$, and (iv) $(x_k, x_{k'}) = (x_2, x_1)$.

Since cases (i) and (ii) are symmetric, we focus on the verification of case (i), and omit the consideration of case (ii). We first have $\phi_{x_k}(x_k, x_2) + \phi_{x_2}(x_k, x_2) = 1$ by Claim 1(ii). We next show $\phi_{x_2}(x_k, x_2) = 1 - \varepsilon$. Note that there exists an alternative-path in $\mathcal{C}_1$ that connects x_1 and x_k, and excludes

we can identify two distinct alternative-paths in M which connect $a_{\underline{k}}$ and $a_{\overline{k}}$. From these two alternative-paths, we can elicit a cycle in M that includes $a_{\underline{k}}$.

x_2. Then, according to this alternative-path, the uncompromising property implies $\phi_{x_2}(x_k, x_2) = \phi_{x_2}(x_1, x_2) = 1 - \varepsilon$, as required.

In case (iii), we first know either $x_k \notin \{x_1, x_2\}$ or $x_{k'} \notin \{x_1, x_2\}$. Assume w.l.o.g. that $x_k \notin \{x_1, x_2\}$. Then, by the verification of cases (i), from (x_1, x_2) to (x_k, x_2), we have $\phi(x_k, x_2) = \varepsilon e_{x_k} + (1-\varepsilon)e_{x_2}$. Furthermore, by case (ii), from (x_k, x_2) to $(x_k, x_{k'})$, we eventually have $\phi(x_k, x_{k'}) = \varepsilon e_{x_k} + (1-\varepsilon)e_{x_{k'}}$.

Last, in case (iv), since the cycle $\mathcal{C}_1$ contains at least three alternatives, we first consider the profile (x_3, x_2) and have $\phi(x_3, x_2) = \varepsilon e_{x_3} + (1-\varepsilon)e_{x_2}$ by the verification of case (i). Next, according to the verification of case (iii), from (x_3, x_2) to (x_2, x_1), we induce $\phi(x_2, x_1) = \varepsilon e_{x_2} + (1-\varepsilon)e_{x_1}$. This completes the verification of the claim.

Symmetrically, according to the circle $\mathcal{C}_2$ of Observation 6.13, ϕ also mimics a random dictatorship on the subdomain $\mathbb{D}^{\mathcal{C}_2} = \{P_i \in \mathbb{D} : r_1(P_i) \in \mathcal{C}_2\}$, i.e., there exists $0 \leq \varepsilon' \leq 1$ such that $\phi(y_k, y_{k'}) = \varepsilon' e_{y_k} + (1-\varepsilon')e_{y_{k'}}$ for all $y_k, y_{k'} \in \mathcal{C}_2$.

Claim 3: We have (i) $\varepsilon = \varepsilon'$, (ii) $\phi(a_{\underline{k}}, a_{\overline{k}}) = \varepsilon\, e_{a_{\underline{k}}} + (1-\varepsilon)e_{a_{\overline{k}}}$, and (iii) $\phi(a_{\overline{k}}, a_{\underline{k}}) = \varepsilon\, e_{a_{\overline{k}}} + (1-\varepsilon)e_{a_{\underline{k}}}$.

According to the graph $G_{\mathbb{D}}$ and the two cycles $\mathcal{C}_1$ and $\mathcal{C}_2$, we can construct an alternative-path $\mathcal{P} = \{z_1, z_2, \ldots, z_{l-1}, z_l\} \subseteq M$ such that (i) $l \geq 3$, (ii) $z_1, z_2 \in \mathcal{C}_1$ and $a_{\underline{k}} \in \{z_1, z_2\}$, and (iii) $z_{l-1}, z_l \in \mathcal{C}_2$ and $a_{\overline{k}} \in \{z_{l-1}, z_l\}$. First, Claim 2 and the uncompromising property imply $\varepsilon = \phi_{z_1}(z_1, z_2) = \phi_{z_1}(z_1, z_l)$ and $1 - \varepsilon = \phi_{z_1}(z_2, z_1) = \phi_{z_1}(z_l, z_1)$. Symmetrically, we have $1 - \varepsilon' = \phi_{z_l}(z_{l-1}, z_l) = \phi_{z_l}(z_1, z_l)$ and $\varepsilon' = \phi_{z_l}(z_l, z_{l-1}) = \phi_{z_l}(z_l, z_1)$. Thus, $\varepsilon + 1 - \varepsilon' = \phi_{z_1}(z_1, z_l) + \phi_{z_l}(z_1, z_l) \leq 1$ which implies $\varepsilon \leq \varepsilon'$, and $1 - \varepsilon + \varepsilon' = \phi_{z_1}(z_l, z_1) + \phi_{z_l}(z_l, z_1) \leq 1$ which implies $\varepsilon \geq \varepsilon'$. Therefore, $\varepsilon = \varepsilon'$. This completes the verification of statement (i).

Since statements (ii) and (iii) are symmetric, we focus on showing statement (ii) and omit the consideration of statement (iii). First, by the verification of statement (i), we have $\phi(z_1, z_l) = \varepsilon\, e_{z_1} + (1-\varepsilon)e_{z_l}$. Second, according to $\mathcal{P}$, the uncompromising property implies $\phi_{z_l}(z_2, z_l) = \phi_{z_l}(z_1, z_l) = 1 - \varepsilon$ and $\phi_{z_k}(z_2, z_l) = \phi_{z_k}(z_1, z_l) = 0$ for all $2 < k < l$. Moreover, since $\sum_{k=2}^{l} \phi_{z_k}(z_2, z_l) = 1$ by Claim 1(i), we have $\phi_{z_2}(z_2, z_l) = 1 - \phi_{z_l}(z_2, z_l) = \varepsilon$, and hence $\phi(z_2, z_l) = \varepsilon\, e_{z_2} + (1-\varepsilon)e_{z_l}$. Symmetrically, we also have $\phi(z_1, z_{l-1}) = \varepsilon\, e_{z_1} + (1-\varepsilon)e_{z_{l-1}}$. Recall that $a_{\underline{k}} \in \{z_1, z_2\}$ and $a_{\overline{k}} \in \{z_{l-1}, z_l\}$. We hence conclude that when $a_{\underline{k}} = z_1$ or $a_{\overline{k}} = z_l$, $\phi(a_{\underline{k}}, a_{\overline{k}}) = \varepsilon\, e_{a_{\underline{k}}} + (1-\varepsilon)e_{a_{\overline{k}}}$. Last, we show that when $a_{\underline{k}} = z_2$ and $a_{\overline{k}} = z_{l-1}$, $\phi(a_{\underline{k}}, a_{\overline{k}}) = \varepsilon\, e_{a_{\underline{k}}} + (1-\varepsilon)e_{a_{\overline{k}}}$. According to $\mathcal{P}$, the uncompromising property implies $\phi_{a_{\underline{k}}}(a_{\underline{k}}, a_{\overline{k}}) = \phi_{z_2}(z_2, z_{l-1}) = \phi_{z_2}(z_2, z_l) = \varepsilon$ and $\phi_{a_{\overline{k}}}(a_{\underline{k}}, a_{\overline{k}}) = \phi_{z_{l-1}}(z_2, z_{l-1}) = \phi_{z_{l-1}}(z_1, z_{l-1}) = 1 - \varepsilon$, as required. This completes the verification of statement (ii), and hence proves the claim.

Claim 4: Given distinct $a_s, a_t \in M$, $\phi(a_s, a_t) = \varepsilon\, e_{a_s} + (1-\varepsilon)e_{a_t}$.

First, consider the situation that there exists $\mathcal{P}_l \in \Pi(a_1, a_m)$ such that $a_s, a_t \in \mathcal{P}_l$. Since $a_s, a_t \in M$, the interval $[a_{\underline{k}}, a_{\overline{k}}]^{\mathcal{P}_l} \equiv \{x_k\}_{k=1}^{l} \subseteq M$ must include a_s and a_t. By Claim 3, we have $\phi(x_1, x_l) = \varepsilon\, e_{x_1} + (1-\varepsilon)e_{x_l}$ and $\phi(x_l, x_1) = \varepsilon\, e_{x_l} + (1-\varepsilon)e_{x_1}$. Then, according to the alternative-path $\{x_k\}_{k=1}^{l}$, by repeatedly applying Claim 1(i) and the uncompromising property, we have $\phi(x_k, x_{k'}) = \varepsilon\, e_{x_k} + (1-\varepsilon)e_{x_{k'}}$ for all distinct $1 \leq k, k' \leq l$. Hence, $\phi(a_s, a_t) = \varepsilon\, e_{a_s} + (1-\varepsilon)e_{a_t}$.

Next, consider the situation that there exists no $\mathcal{P}_l \in \Pi(a_1, a_m)$ that includes both a_s and a_t. According to Observation 6.13, it must be the case that $a_s \notin \{a_{\underline{k}}, a_{\overline{k}}\}$ and $a_t \notin \{a_{\underline{k}}, a_{\overline{k}}\}$. Moreover, by Observation 6.13, let $\{b_k\}_{k=1}^{l} \subseteq M$ be an alternative-path that connects $a_{\underline{k}}$ and $a_{\overline{k}}$, and includes a_s, and let $\{c_k\}_{k=1}^{u} \subseteq M$ be an alternative-path that connects $a_{\underline{k}}$ and $a_{\overline{k}}$, and includes a_t. Evidently, $a_s \notin \{c_k\}_{k=1}^{n}$ and $a_t \notin \{b_k\}_{k=1}^{l}$. Let $a_s = b_p$ and $a_t = c_q$ for some $1 < p < l$ and $1 < q < u$. According to the sub-alternative-paths $\{b_1, b_2, \ldots, b_p\}$ and $\{c_1, c_2, \ldots, c_q\}$, since $b_1 = c_1 = a_{\underline{k}}$, $b_p \notin \{c_k\}_{k=1}^{u}$ and $c_q \notin \{b_k\}_{k=1}^{l}$, we identify $1 \leq \eta < p$ and $1 \leq \nu < q$ such that $b_\eta = c_\nu$ and $\{b_{\eta+1}, \ldots, b_p\} \cap \{c_{\nu+1}, \ldots, c_q\} = \emptyset$. Then, we have the concatenated alternative-path $\mathcal{P} = \{a_s = b_p, \ldots, b_\eta = c_\nu, \ldots, c_q = a_t\} \subseteq M$ which connects a_s and a_t. By the verification in the first situation, we have $\phi_{b_p}(b_p, b_\eta) = \varepsilon$ and $\phi_{c_q}(c_\nu, c_q) = 1-\varepsilon$. Furthermore, according to $\mathcal{P}$, the uncompromising property implies
$\phi_{a_s}(a_s, a_t) = \phi_{b_p}(b_p, c_q) = \phi_{b_p}(b_p, c_\nu) = \phi_{b_p}(b_p, b_\eta) = \varepsilon$ and
$\phi_{a_t}(a_s, a_t) = \phi_{c_q}(b_p, c_q) = \phi_{c_q}(b_\eta, c_q) = \phi_{c_q}(c_\nu, c_q) = 1-\varepsilon$. Therefore, $\phi(a_s, a_t) = \varepsilon\, e_{a_s} + (1-\varepsilon)e_{a_t}$.
This completes the verification of the claim.

In conclusion, every two-voter unanimous and strategy-proof RSCF behaves like a random dictatorship on the subdomain $\overline{\mathbb{D}}$. For the general case $n > 2$, we adopt an induction argument.

Induction Hypothesis: Given $n \geq 3$, for all $2 \leq n' < n$, every unanimous and strategy-proof $\psi : \mathbb{D}^{n'} \to \Delta(A)$ behaves like a random dictatorship on the subdomain $\overline{\mathbb{D}}$.

Given a unanimous and strategy-proof RSCF $\phi : \mathbb{D}^n \to \Delta(A)$, $n > 2$, we show that it behaves like a random dictatorship on the subdomain $\overline{\mathbb{D}}$. If $n \geq 4$, the verification follows exactly from Propositions 5 and 6 of [35]. Therefore, we focus on the case $n = 3$, i.e., $N = \{1, 2, 3\}$. Analogous to Propositions 4 and 6 of [35], we split the verification into the following two parts:

1. There exists $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$ with $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$ such that for all $P \in \overline{\mathbb{D}}^3$, we have $\left[P_i = P_j \text{ for some distinct } i, j \in N\right] \Rightarrow \left[\phi(P) = \varepsilon_1\, e_{r_1(P_1)} + \varepsilon_2\, e_{r_1(P_2)} + \varepsilon_3\, e_{r_1(P_3)}\right]$.
2. For all $P \in \overline{\mathbb{D}}^3$, we have $\phi(P) = \varepsilon_1\, e_{r_1(P_1)} + \varepsilon_2\, e_{r_1(P_2)} + \varepsilon_3\, e_{r_1(P_3)}$.

The second part follows exactly from Proposition 6 of [35]. Therefore, we focus on showing the first part.[17]

[17]Proposition 4 of [35] is not applicable for the verification of the first part since they impose an additional domain condition (see their Definition 18) which cannot be confirmed on domain $\mathbb{D}$.

According to ϕ, we first induce three two-voter RSCFs by merging two voters respectively: For all $P_1, P_2, P_3 \in \mathbb{D}$, let $\psi^1(P_1, P_2) = \phi(P_1, P_2, P_2)$, $\psi^2(P_1, P_2) = \phi(P_1, P_2, P_1)$ and $\psi^3(P_1, P_3) = \phi(P_1, P_1, P_3)$. It is easy to verify that all ψ^1, ψ^2 and ψ^3 are unanimous and strategy-proof on $\mathbb{D}$. Therefore, the induction hypothesis implies that there exist $0 \leq \varepsilon_1, \varepsilon_2, \varepsilon_3 \leq 1$ such that for all $P_1, P_2, P_3 \in \overline{\mathbb{D}}$, $\psi^1(P_1, P_2) = \varepsilon_1\, e_{r_1(P_1)} + (1 - \varepsilon_1)e_{r_1(P_2)}$, $\psi^2(P_1, P_2) = (1 - \varepsilon_2)e_{r_1(P_1)} + \varepsilon_2\, e_{r_1(P_2)}$ and $\psi^3(P_1, P_3) = (1 - \varepsilon_3)e_{r_1(P_1)} + \varepsilon_3\, e_{r_1(P_3)}$. Note that to show the first part holds, it suffices to prove $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$.

Recall the cycle $\mathcal{C}_1 = \{x_k\}_{k=1}^p \subseteq M$ in Observation 6.13. First, according to the three alternative-paths $\{x_2, x_3\}$, $\{x_1, x_2\}$ and $\{x_1, x_p, \ldots, x_4, x_3\}$ in $\mathcal{C}_1$, the uncompromising property implies respectively that (i) $\phi_{x_1}(x_1, x_2, x_3) = \phi_{x_1}(x_1, x_2, x_2) = \psi^1_{x_1}(x_1, x_2) = \varepsilon_1$ and $\phi_{a_s}(x_1, x_2, x_3) = \phi_{a_s}(x_1, x_2, x_2) = \psi^1_{a_s}(x_1, x_2) = 0$ for all $a_s \notin \{x_1, x_2, x_3\}$, (ii) $\phi_{x_3}(x_1, x_2, x_3) = \phi_{x_3}(x_2, x_2, x_3) = \psi^3_{x_3}(x_2, x_3) = \varepsilon_3$, and (iii) $\phi_{x_2}(x_1, x_2, x_3) = \phi_{x_2}(x_3, x_2, x_3) = \psi^2_{x_2}(x_3, x_2) = \varepsilon_2$. Then, we have $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \phi_{x_1}(x_1, x_2, x_3) + \phi_{x_2}(x_1, x_2, x_3) + \phi_{x_3}(x_1, x_2, x_3) + \sum_{a_s \notin \{x_1, x_2, x_3\}} \phi_{a_s}(x_1, x_2, x_3) = \sum_{a_s \in A} \phi_{a_s}(x_1, x_2, x_3) = 1$, as required. This completes the verification of the induction hypothesis, and hence proves Lemma 6.13.10. ■

Lemma 6.13.11 *Let $\phi : \mathbb{D}^n \to \Delta(A)$ be a unanimous and strategy-proof RSCF. Given distinct $a_s, a_t \in M$ and $P_{-i} \in \mathbb{D}^{n-1}$, we have $\phi_{a_k}(a_s, P_{-i}) = \phi_{a_k}(a_t, P_{-i})$ for all $a_k \notin \{a_s, a_t\}$.*

Proof: First, Lemma 6.13.8 implies that ϕ satisfies the tops-only property, and Lemma 6.13.10 implies that ϕ mimics a random dictatorship on the subdomain $\overline{\mathbb{D}} = \{P_i \in \mathbb{D} : r_1(P_i) \in M\}$.

Claim 1: The two statements hold: (i) $[a_{\underline{k}} \notin \{a_s, a_t\}] \Rightarrow [\phi_{a_{\underline{k}}}(a_s, P_{-i}) = \phi_{a_{\underline{k}}}(a_t, P_{-i})]$, and (ii) $[a_{\overline{k}} \notin \{a_s, a_t\}] \Rightarrow \left[\phi_{a_{\overline{k}}}(a_s, P_{-i}) = \phi_{a_{\overline{k}}}(a_t, P_{-i})\right]$.

By symmetry, we focus on showing statement (i) and omit the consideration of statement (ii). Note that if there exists an alternative-path that connects a_s and a_t and excludes $a_{\underline{k}}$, then the uncompromising property implies $\phi_{a_{\underline{k}}}(a_s, P_{-i}) = \phi_{a_{\underline{k}}}(a_t, P_{-i})$. Therefore, to complete the verification, we will construct such an alternative-path.

If $a_s \neq a_{\overline{k}}$, we pick an alternative-path $\langle a_{\underline{k}}, a_{\overline{k}}\rangle$ that includes a_s by Observation 6.13, and elicit the sub-alternative-path $\langle a_s, a_{\overline{k}}\rangle$. If $a_s = a_{\overline{k}}$, we refer to $\langle a_s, a_{\overline{k}}\rangle = \{a_s\}$. Thus, $a_{\underline{k}} \notin \langle a_s, a_{\overline{k}}\rangle$. Similarly, we have an alternative-path $\langle a_{\overline{k}}, a_t\rangle$ which excludes $a_{\underline{k}}$. According to $\langle a_s, a_{\overline{k}}\rangle$ and $\langle a_{\overline{k}}, a_t\rangle$, we construct an alternative-path which connects a_s and a_t, and excludes $a_{\underline{k}}$, as required. This completes the verification of the claim.

Since $a_s, a_t \in M$, by the verification of Claim 4 in the proof of Lemma 6.13.10, there exists an alternative-path $\{x_k\}_{k=1}^p \subseteq M$ connecting a_s and a_t. The uncompromising property first implies

$\phi_{a_k}(a_s, P_{-i}) = \phi_{a_k}(a_t, P_{-i})$ for all $a_k \notin \{x_k\}_{k=1}^{p}$. Therefore, to complete the proof of the lemma, it suffices to show that $\phi_{x_k}(a_s, P_{-i}) = \phi_{x_k}(a_t, P_{-i})$ for all $k = 2, \ldots, p-1$. If $x_k \in \{a_{\underline{k}}, a_{\overline{k}}\}$, it follows immediately from Claim 1 that $\phi_{x_k}(a_s, P_{-i}) = \phi_{x_k}(a_t, P_{-i})$. Hence, we let $\Theta = \{x_2, \ldots, x_{p-1}\} \backslash \{a_{\underline{k}}, a_{\overline{k}}\}$ and show $\phi_z(a_s, P_{-i}) = \phi_z(a_t, P_{-i})$ for all $z \in \Theta$.

For notational convenience, let $i = n$. We partition $\{1, \ldots, n-1\}$ into three parts: $\underline{I} = \{1, \ldots, j\}$, $\bar{I} = \{j+1, \ldots, l\}$ and $\hat{I} = \{l+1, \ldots, n-1\}$, and assume w.l.o.g that $r_1(P_1), \ldots, r_1(P_j) \in L \backslash \{a_{\underline{k}}\}$, $r_1(P_{j+1}), \ldots, r_1(P_l) \in R \backslash \{a_{\overline{k}}\}$ and $r_1(P_{l+1}), \ldots, r_1(P_{n-1}) \in M$. Note that if $l = 0$, Lemma 6.13.10 implies $\phi_z(a_s, P_{-n}) = \phi_z(a_t, P_{-n})$ for all $z \in \Theta$. Next, assume $l > 0$. We construct the following preference profiles: $P^{(\eta)} = \big(P_1, \ldots, P_\eta, \frac{a_{\underline{k}}}{\{\eta+1, \ldots, j\}}, \frac{a_{\overline{k}}}{\bar{I}}, P_{\hat{I}}, a_s\big), \eta = 0, 1, \ldots, j$, and $P^{(\nu)} = \big(P_{\underline{I}}, P_{j+1}, \ldots, P_\nu, \frac{a_{\overline{k}}}{\{\nu+1, \ldots, l\}}, P_{\hat{I}}, a_s\big), \nu = j+1, \ldots, l$. Note that $P^{(0)} = \big(\frac{a_{\underline{k}}}{\underline{I}}, \frac{a_{\overline{k}}}{\bar{I}}, P_{\hat{I}}, a_s\big)$ and $P^{(l)} = (a_s, P_{-n})$.

Given an arbitrary $0 \leq \eta < j$, consider $P^{(\eta)}$ and $P^{(\eta+1)}$. Note that voter $\eta + 1$ has the preference peak $a_{\underline{k}}$ at $P^{(\eta)}$, and has the preference peak $r_1(P_{\eta+1}) = a_k \prec a_{\underline{k}}$ at $P^{(\eta+1)}$. By Lemma 6.13.6, $\{a_k, a_{k+1}, \ldots, a_{\underline{k}}\} \subseteq L$ is the unique alternative-path that connects a_k and $a_{\underline{k}}$, and hence excludes all alternatives of Θ. Then, the uncompromising property implies $\phi_z(P^{(\eta)}) = \phi_z(P^{(\eta+1)})$ for all $z \in \Theta$. Therefore, we have $\phi_z(P^{(0)}) = \cdots = \phi_z(P^{(j)})$ for all $z \in \Theta$. Next, given an arbitrary $j \leq \nu < l$, consider $P^{(\nu)}$ and $P^{(\nu+1)}$. Note that voter $\nu + 1$ has the preference peak $a_{\overline{k}}$ at $P^{(\nu)}$, and has the preference peak $r_1(P_{\nu+1}) = a_k \succ a_{\overline{k}}$ at $P^{(\nu+1)}$. By Lemma 6.13.6, $\{a_{\underline{k}}, \ldots, a_{k-1}, a_k\} \subseteq R$ is the unique alternative-path that connects $a_{\overline{k}}$ and a_k, and hence excludes all alternatives of Θ. Then, the uncompromising property implies $\phi_z(P^{(\nu)}) = \phi_z(P^{(\nu+1)})$ for all $z \in \Theta$. Therefore, we have $\phi_z(P^{(j)}) = \cdots = \phi_z(P^{(l)})$ for all $z \in \Theta$. In conclusion, $\phi_z\big(\frac{a_{\underline{k}}}{\underline{I}}, \frac{a_{\overline{k}}}{\bar{I}}, P_{\hat{I}}, a_s\big) = \phi_z(P^{(0)}) = \cdots = \phi_z(P^{(l)}) = \phi_z(a_s, P_{-n})$ for all $z \in \Theta$.

Symmetrically, we also derive $\phi_z\big(\frac{a_{\underline{k}}}{\underline{I}}, \frac{a_{\overline{k}}}{\bar{I}}, P_{\hat{I}}, a_t\big) = \phi_z(a_t, P_{-n})$ for all $z \in \Theta$. Last, since Lemma 6.13.10 implies $\phi_z\big(\frac{a_{\underline{k}}}{\underline{I}}, \frac{a_{\overline{k}}}{\bar{I}}, P_{\hat{I}}, a_s\big) = \phi_z\big(\frac{a_{\underline{k}}}{\underline{I}}, \frac{a_{\overline{k}}}{\bar{I}}, P_{\hat{I}}, a_t\big)$ for all $z \in \Theta$, we have $\phi_z(a_s, P_{-n}) = \phi_z(a_t, P_{-n})$ for all $z \in \Theta$, as required. ∎

Now, fixing a unanimous and strategy-proof RSCF $\phi : \mathbb{D}^n \to \Delta(A)$, we are ready to show that the corresponding random voting scheme $\phi : A^n \to \Delta(A)$ is locally strategy-proof on $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$.

Fix $i \in N$, $P_i, P_i' \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ with $P_i \sim P_i'$, and $P_{-i} \in \big[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\big]^{n-1}$. For notational convenience, let $r_1(P_i) = a_s$, $r_1(P_i') = a_t$ and $r_1(P_j) = x_j$ for all $j \neq i$. Let $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. We show that $\phi(a_s, x_{-i})$ stochastically dominates $\phi(a_t, x_{-i})$ according to P_i. If $a_s = a_t$, $\phi(a_s, x_{-i}) = \phi(a_t, x_{-i})$, as required. Next, assume $a_s \neq a_t$. Then, $P_i \sim P_i'$ implies $r_1(P_i) = r_2(P_i') = a_s$, $r_1(P_i') = r_2(P_i) = a_t$ and $r_k(P_i) = r_k(P_i')$ for all $k = 3, \ldots, m$. To complete the verification, it suffices to show $\phi_{a_s}(a_s, x_{-i}) \geq \phi_{a_s}(a_t, x_{-i})$ and $\phi_{a_k}(a_s, x_{-i}) = \phi_{a_k}(a_t, x_{-i})$ for all $a_k \notin \{a_s, a_t\}$. Since $r_1(P_i) = a_s$, $r_1(P_i') = a_t$ and $P_i \sim P_i'$, we know $a_s \sim a_t$ in $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$. Then, there are three possible cases: (i) $a_s, a_t \in L$ and $|s - t| = 1$, (ii) $a_s, a_t \in R$ and $|s - t| = 1$, and (iii) $a_s, a_t \in M$. The first two cases are symmetric, and hence we focus on the verification of the first case and omit the consideration of the second case. In the first case, since $|s-t|=1$, it is also true that $a_s \sim a_t$ in $\mathbb{D}$. Hence, we have $\bar{P}_i, \bar{P}'_i \in \mathbb{D}$ such that $r_1(\bar{P}_i) = a_s$, $r_1(\bar{P}'_i) = a_t$ and $\bar{P}_i \sim \bar{P}'_i$. Then, the tops-only property and strategy-proofness of ϕ on $\mathbb{D}$ imply $\phi_{a_s}(a_s, x_{-i}) = \phi_{a_s}(\bar{P}_i, x_{-i}) \geq \phi_{a_s}(\bar{P}'_i, x_{-i}) = \phi_{a_s}(a_t, x_{-i})$, and $\phi_{a_k}(a_s, x_{-i}) = \phi_{a_k}(\bar{P}_i, x_{-i}) = \phi_{a_k}(\bar{P}'_i, x_{-i}) = \phi_{a_k}(a_t, x_{-i})$ for all $a_k \notin \{a_s, a_t\}$, as required. Last, assume $a_s, a_t \in M$. Fixing $\bar{P}_i, \bar{P}'_i \in \mathbb{D}$ with $r_1(\bar{P}_i) = a_s$ and $r_1(\bar{P}'_i) = a_t$ by minimal richness, we have $\phi_{a_s}(a_s, x_{-i}) = \phi_{a_s}(\bar{P}_i, x_{-i}) \geq \phi_{a_s}(\bar{P}'_i, x_{-i}) = \phi_{a_s}(a_t, x_{-i})$ by the tops-only property and strategy-proofness of ϕ on $\mathbb{D}$, and $\phi_{a_k}(a_s, x_{-i}) = \phi_{a_k}(a_t, x_{-i})$ for all $a_k \notin \{a_s, a_t\}$ by Lemma 6.13.11, as required. Therefore, ϕ is locally strategy-proof on $\mathbb{D}_{\mathrm{H}}(\underline{k}, \bar{k})$. This completes the verification of the second part of Theorem 6.7.2 in the case $\bar{k} - \underline{k} > 1$, and hence completely proves Theorem 6.7.2.

6.14 Proof of Fact 6.8

We first introduce some new notation and the formal definition of the no-restoration property of [95]. Let $aP_i!b$ denote that a is *contiguously* preferred to b in P_i, i.e., aP_ib and there exists no $c \in A$ such that aP_ic and cP_ib. Recall the notions of adjacency and path in the beginning of Section 6.2. A domain $\mathbb{D}$ satisfies the **no-restoration property** if for all distinct $P_i, P'_i \in \mathbb{D}$, there exists a path $\{P_i^k\}_{k=1}^t \subseteq \mathbb{D}$ connecting P_i and P'_i such that for all $a_p, a_q \in A$, we have

$$[a_p P_i^{k^*} a_q \text{ and } a_q P_i^{k^*+1} a_p \text{ for some } 1 \leq k^* < t] \Rightarrow [a_p P_i^k a_q \text{ for all } k = 1, \ldots, k^*, \text{ and } a_q P_i^l a_p \text{ for all } l = k^*+1, \ldots, t].$$

By Theorem 1 of [38], to prove Fact 6.8, it suffices to show that $\mathbb{D}_{\mathrm{H}}(\underline{k}, \bar{k})$ satisfies the no-restoration property. Before proceeding the proof, we introduce an important observation on $\mathbb{D}_{\mathrm{H}}(\underline{k}, \bar{k})$. Given $P_i \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \bar{k})$, let $r_1(P_i) = a_s$ and $a_pP_i!a_q$ (it is possible that $a_s = a_p$). Let P''_i be a preference such that $P_i \sim P''_i$ and $a_qP''_i!a_p$. If one of the three conditions is satisfied: (i) $r_1(P_i) = r_1(P''_i)$, and $a_p \prec a_s \prec a_q$ or $a_q \prec a_s \prec a_p$, (ii) $r_1(P_i) = r_1(P''_i) \in M$ and neither both $a_p, a_q \in L$ nor both $a_p, a_q \in R$, and (iii) $r_1(P_i) \neq r_1(P''_i)$, and either $a_p, a_q \in L$ and $|p-q| = 1$, or $a_p, a_q \in R$ and $|p-q| = 1$, or $a_p, a_q \in M$, then $P''_i \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \bar{k})$. □

To show that $\mathbb{D}_{\mathrm{H}}(\underline{k}, \bar{k})$ satisfies the no-restoration property, it suffices to show that for every pair of distinct preferences $P_i, P'_i \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \bar{k})$, there exist $a_p, a_q \in A$ and $P''_i \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \bar{k})$ such that $P_i \sim P''_i$, $a_pP_i!a_q$, $a_qP''_i!a_p$ and $a_qP'_ia_p$. Henceforth, we fix distinct $P_i, P'_i \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \bar{k})$, and let $r_1(P_i) = a_s$ and $r_1(P'_i) = a_t$.

We first assume $a_s = a_t$. We identify $1 < k \leq m$ such that $r_l(P_i) = r_l(P'_i)$ for all $l = 1, \ldots, k-1$, and $r_k(P_i) \neq r_k(P'_i)$. Let $r_k(P'_i) = a_q$ and $a_q = r_v(P_i)$ for some $k < v \leq m$. Meanwhile, let $r_{v-1}(P_i) = a_p$. We generate a preference P''_i by locally switching a_p and a_q in P_i. Thus, $P_i \sim P''_i$, $a_pP_i!a_q$, $a_qP''_i!a_p$ and $a_qP'_ia_p$.

Note that $r_1(P_i) = r_1(P_i'') = r_1(P_i')$. We next show $P_i'' \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$. Suppose not, i.e., $P_i'' \notin \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$. On the one hand, since P_i and P_i'' share the same peak and differ exactly on the relative rankings of a_p and a_q, $P_i \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ and $P_i'' \notin \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ imply that $a_q P_i'' a_p$ must violate Definition 6.3.1. On the other hand, since P_i'' and P_i' share the same peak and the same relative ranking of a_p and a_q, $P_i' \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ implies that $a_q P_i'' a_p$ does not violate Definition 6.3.1. Contradiction! Therefore, $P_i'' \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$.

Next, we assume $a_s \prec a_t$. The verification related to the situation $a_t \prec a_s$ is symmetric, and we hence omit it. We consider the four possible cases: (1) $a_s \prec a_{\underline{k}}$, (2) $a_{\overline{k}} \preceq a_s$, (3) $a_{\underline{k}} \preceq a_s \prec a_{\overline{k}} \preceq a_t$ and (4) $a_{\underline{k}} \preceq a_s \prec a_t \prec a_{\overline{k}}$.

In case (1), we notice $a_s \prec a_{s+1} \preceq a_{\underline{k}}$ and $a_s \prec a_{s+1} \preceq a_t$. Let $a_{s+1} = r_k(P_i)$ for some $1 < k \leq m$ and $r_{k-1}(P_i) = a_p$. Thus, $a_p P_i ! a_{s+1}$. Since $r_1(P_i) = a_s \in L$, $a_p P_i a_{s+1}$ implies $a_p \preceq a_s$ by Definition 6.3.1. Hence, we know $a_p \preceq a_s \prec a_{s+1} \preceq a_{\underline{k}}$ and $a_p \preceq a_s \prec a_{s+1} \prec a_t$, which imply $a_{s+1} P_i' a_p$ by Definition 6.3.1. By locally switching a_p and a_{s+1} in P_i, we generate a preference P_i''. Thus, $P_i \sim P_i''$, $a_p P_i ! a_{s+1}$, $a_{s+1} P_i'' ! a_p$ and $a_{s+1} P_i' a_p$. We last show $P_i'' \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$. If $r_1(P_i'') = r_1(P_i) = a_s$, it is true that $a_p \prec a_s \prec a_{s+1}$, and Observation 6.14(i) then implies $P_i'' \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$. If $r_1(P_i'') \neq r_1(P_i)$, it is true that $r_1(P_i) = a_s = a_p$ and $r_1(P_i'') = a_{s+1}$, and Observation 6.14(iii) then implies $P_i'' \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$.

The verification of case (2) is similar to that of case (1), and we hence omit it.

In case (3), let $a_{\overline{k}} = r_k(P_i)$ for some $1 < k \leq m$ and $r_{k-1}(P_i) = a_p$. Thus, $a_p P_i ! a_{\overline{k}}$. Since $a_{\underline{k}} \preceq a_s \prec a_{\overline{k}}$, $a_p P_i a_{\overline{k}}$ implies $a_p \prec a_{\overline{k}}$ by Definition 6.3.1. Thus, we know either $a_p \prec a_{\underline{k}} \prec a_{\overline{k}} \preceq a_t$ which implies $a_{\overline{k}} P_i' a_{\underline{k}}$ and $a_{\underline{k}} P_i' a_p$ by Definition 6.3.1, or $a_{\underline{k}} \preceq a_p \prec a_{\overline{k}} \preceq a_t$ which implies $a_{\overline{k}} P_i' a_p$ by Definition 6.3.1. Overall, $a_{\overline{k}} P_i' a_p$. By locally switching a_p and $a_{\overline{k}}$ in P_i, we generate a preference P_i''. Thus, $P_i \sim P_i''$, $a_p P_i ! a_{\overline{k}}$, $a_{\overline{k}} P_i'' ! a_p$ and $a_{\overline{k}} P_i' a_p$. We last show $P_i'' \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$. If $r_1(P_i'') = r_1(P_i) = a_s$, Observation 6.14(ii) implies $P_i'' \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$. If $r_1(P_i'') \neq r_1(P_i)$, it is true that $r_1(P_i) = a_s = a_p$ and $r_1(P_i'') = a_{\overline{k}}$, and Observation 6.14(iii) then implies $P_i'' \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$.

In case (4), let $a_t = r_k(P_i)$ for some $1 < k \leq m$ and $r_{k-1}(P_i) = a_p$. By locally switching a_p and a_t in P_i, we generate a preference P_i''. Thus, $P_i \sim P_i''$, $a_p P_i ! a_t$, $a_t P_i'' ! a_p$ and $a_t P_i' a_p$ (recall $r_1(P_i') = a_t$). We last show $P_i'' \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$. If $r_1(P_i'') = r_1(P_i) = a_s$, Observation 6.14(ii) implies $P_i'' \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$. If $r_1(P_i'') \neq r_1(P_i)$, it is true that $r_1(P_i) = a_s = a_p$ and $r_1(P_i'') = a_t$, and Observation 6.14(iii) implies $P_i'' \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$.

In conclusion, domain $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ satisfies the no-restoration condition of [95], as required. □

7

Unanimous and strategy-proof probabilistic rules for single-peaked preference profiles on graphs

7.1 Introduction

Finitely many agents have preferences over a finite set of alternatives. The alternatives are the vertices in a connected graph, and the preferences of an agent are linear orderings which are single-peaked with respect to some spanning tree of the graph: there is a single top alternative, the peak, and preference decreases along the paths in this tree away from the peak. The objective is to choose an alternative based on these preferences, or rather – in this paper – a probability distribution over the alternatives.

An example of such a situation is a road or railroad network, where the vertices (junctions) are also the locations of villages or cities. The objective is to locate a public good (shopping mall, museum, hospital, school, etc.) based on the preferences of the agents over these junctions. Distance from one's home or from a nearby bus stop may determine preference, but also the path one has to take. Single-peakedness is then a plausible assumption. Alternatively, the graph may represent a network of personal relations between the agents, and the objective is to distribute a public good – e.g., disperse information – over the vertices in this network. Also here, both the length of a path and the nodes (e.g., friends) to be visited may be important determinants for preference, and single-peakedness along a specific spanning tree captures this. More generally, the graph structure and single-peakedness condition are formal ways to describe restrictions on the set of all preference profiles that enable to avoid (random) dictatorship as in [57] – see below. This is comparable to (e.g.) the domain restriction in [75]; we briefly comment on this in the concluding section of the paper.

We consider probabilistic rules: these assign a probability distribution over the alternatives to every preference profile of single-peaked preferences. An important reason for considering probabilistic rather than deterministic rules is that even a random dictatorship, for instance each agent's peak having an equal chance of being chosen, seems better than a deterministic dictatorship, where one and the same agent's peak is always chosen.

The conditions we impose are unanimity and strategy-proofness. Unanimity means that if all agents have the same peak then probability one is assigned to that alternative. Strategy-proofness means that no agent, by misrepresenting its true preference, can increase the probability on any upper contour set, i.e., any set of alternatives (weakly) preferred to some given alternative. Put differently, the probability distribution attained by reporting truthfully stochastically dominates any probability distribution achievable by misreporting.

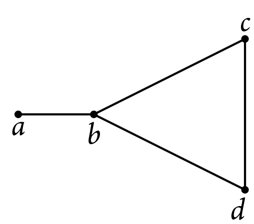
We first consider the case where the graph has no cycles, i.e., is a tree (and thus its own unique spanning tree). For this case, our main result (Theorem 7.3.9) is that a probabilistic rule is unanimous and strategy-proof if and only if it is a 'leaf-peak rule'. In a nutshell, this means that such a rule is uniquely determined by the probability distributions it assigns to the preference profiles with all peaks at the leafs of the tree (i.e., the alternatives with degree one). We show that such a collection of probability distributions has the following properties: (i) a leaf is assigned probability one if all peaks are at this leaf; (ii) if an agent changes its peak from one leaf to another, then (a) probability does not decrease along the path from the former to the latter and (b) probability does not change off this path. These collections of probability distributions are called 'monotonic'. They play a role similar to the collections of 'fixed probabilistic ballots' in [46] – see also below.

Second, for the case where the graph is arbitrary (but connected), we show that every unanimous and strategy-proof probabilistic rule is random dictatorial if and only if the graph has no leafs. In fact, we show this for the case of two agents and then extend the result to more than two agents by using a result of [35] – this is Theorem 7.4.2. Random dictatorship means that each agent is assigned a fixed probability (weight) and every alternative is chosen with probability equal to the sum of the probabilities of the agents having this alternative as their peak. If the graph is not a tree but has a leaf, then indeed unanimous and strategy-proof probabilistic rules exist which are not random dictatorial, as we show by an example, and as follows from the main result of the paper later on (Theorem 7.5.2). In order to prove Theorem

7.4.2 we first consider 2-connected graphs, i.e., graphs in which for every pair of distinct alternatives there is a cycle containing them, and next extend to arbitrary leafless graphs by decomposing the graph in a way analogous to the concept of a 'block tree' ([70]; [104]; or, e.g., [22]).

Third, for the general case, where the graph is not necessarily a tree, can have leafs, but is still connected, we show that every unanimous and strategy-proof probabilistic rule behaves like a leaf-peak rule on the branches of the graph and as a random dictatorial rule on the maximal leafless subgraph of the graph, such that the total probability on each branch is equal to the total weight of the agents who have their peaks on this branch. This is Theorem 7.5.2, which generalizes both Theorems 7.3.9 and 7.4.2.

As a simple example of a unanimous and strategy-proof rule ϕ characterized in Theorem 7.5.2, suppose there are three agents called 1, 2, and 3, and four alternatives called *a*, *b*, *c*, and *d*, structured by the following graph:



Let the agents have equal weights, $\frac{1}{3}$ each. The maximal leafless subgraph is the triangle with vertices *b*, *c*, and *d*. If every agent has one of these points as its peak, then ϕ is just random dictatorship. For instance, if *b* is the peak of 1 and *c* the peak of both 2 and 3, then *b* is assigned $\frac{1}{3}$ and *c* is assigned $\frac{2}{3}$. If all agents have *a* as their peak, then *a* is assigned probability 1. In all other cases, the total weight of the agents with peak *a* is distributed equally between *a* and *b* (*a* and *b* together with their connecting edge form the unique 'branch' in this graph), but on the triangle we have random dictatorship. For instance, if the peak of agent 1 is *a*, the peak of 2 is *b*, and the peak of 3 is *d*, then *a* is assigned probability $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$, *b* is assigned probability $(\frac{1}{2} \cdot \frac{1}{3}) + \frac{1}{3} = \frac{1}{2}$ and *d* is assigned probability $\frac{1}{3}$.

Our first main result, Theorem 7.3.9 on trees, generalizes the case where the alternatives are ordered on a straight line and agents have single-peaked preferences. The latter case has been dealt with in [46]: they consider the whole real line, but their characterization remains valid on a finite or discrete set of alternatives. In [81] it is shown that, for the version with finitely many alternatives, all probabilistic rules are convex combinations of deterministic rules. In the tree case it turns out that this no longer holds – see the concluding Section 7.6.1 for an example of a unanimous and strategy-proof probabilistic rule which is not a convex combination of deterministic rules with these properties. This supports the fact that the general tree case is not a straightforward generalization of the straight line case.

A consequence of Theorem 7.3.9 is a characterization of all unanimous and strategy-proof deterministic rules if agents have single-peaked preferences on a tree, which to the best of our knowledge is new as well (see Section 7.6.1). [97] also consider this issue but their setting is different: a graph is a subset of some

Euclidean space (each of its points is an alternative, not only the vertices, and so there are infinitely many alternatives), and preferences are uniquely determined by their peaks by considering Euclidean distance along the paths in the graph. Nevertheless, their results are roughly in line with ours: if the graph is a tree, then strategy-proof and onto deterministic rules (unanimity is implied) are characterized by so-called extended generalized median voter schemes ([72]); for other graphs, there is dictatorship on cycles but if a graph has a leaf then other rules are possible. For earlier work concerning social choice for single-peaked preferences on trees see [59] and [40]. More recently, see [75] – cf. Section 7.6.2.

Our results show that unanimity and strategy-proofness of probabilistic rules for single-peaked preferences on graphs imply that these rules are tops-only – they depend only on the peaks of the preferences. In fact, we start out by deriving this result using Theorem 1 in [31], see Lemma 7.2.1. From this lemma we then easily obtain that our rules are uncompromising on trees (cf.[24]): if an agent changes its peak, then probabilities assigned to alternatives off the path between the old peak and the new peak remain unaltered (Lemma 7.3.1).[1]

The literature on strategy-proof probabilistic social choice functions or rules started with the paper of [57], who showed that without restrictions on preferences the conditions of unanimity and strategy-proofness result in random dictatorship. The single-peaked domain restriction (which dates back at least to [20]) allows for other rules, which can be seen as probabilistic extensions of the generalized median rules ([72]; [12]; and others): as already mentioned see [46] and [81] for the case with finitely many agents who have single-peaked preferences on the real line or a finite subset of the real line. [43] show that even under single-peaked preferences, every unanimous and strategy-proof probabilistic rule is a random dictatorship if the dimension is higher than one.[2] [36] show a kind of converse to (among others) our results: a domain has to be single-peaked in order to allow for the existence of unanimous and strategy-proof probabilistic rules satisfying two additional conditions.[3] See also [29] for a similar result in the deterministic case. For unanimous and strategy-proof probabilistic rules when preferences are cardinal see the seminal work of [61], and further [44] and [73].

The paper is organized as follows. After preliminaries in Section 7.2, including the result that a unanimous and strategy-proof rule is tops-only, we consider the tree case in Section 7.3 and the leafless graph case in Section 7.4. Our main and most general result is derived in Section 7.5. In the concluding Section 7.6 we show that in this context a probabilistic rule on a tree is not necessarily a convex combination of deterministic rules; we also briefly discuss possible domain variations. An appendix

[1] In an earlier version of the paper ([80]) uncompromisingness on trees was derived independently for a smaller set of single-peaked preferences.

[2] In spirit, this result is in line with our result on leafless graphs (Theorem 7.4.2).

[3] Namely, tops-onliness and a 'compromise' property. Under the assumptions in our paper tops-onliness follows from the other conditions. The 'compromise' property is not necessarily satisfied by a leaf-peak probabilistic rule.

presents the proof of Lemma 7.2.1 on tops-onliness.

7.2 Preliminaries

Let A be a finite set of at least two *alternatives* and let $N = \{1, \ldots, n\}$ with $n \geq 2$ be a finite set of *agents*. A complete, reflexive, antisymmetric, and transitive binary relation on A is called a *preference*. For a preference P and $a, b \in A$, we write aPb instead of $(a, b) \in P$. For distinct $a, b \in A$, aPb is interpreted as a being strictly preferred to b by an agent with preference P. A tuple of preferences $P_N = (P_1, \ldots, P_n)$ is called a *preference profile*.

We denote the top-ranked alternative of a preference P by $t(P)$, i.e., $t(P) = a$ if and only if aPx for all $x \in A$. The *upper contour set* of an alternative a at a preference P is the set $U(a, P) = \{x \in A \colon xPa\}$.[4]

For a preference profile P_N and an agent $i \in N$, P_{-i} denotes the restriction of P_N to $N \setminus \{i\}$, that is, $P_{-i} = (P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_N)$.

7.2.1 Single-peaked preferences

The notion of a single-peaked preference was introduced in [20] and [62]. Here, we consider a generalization.

First, we introduce a graph structure on the set of alternatives. A pair $G = (A, E)$, where $E \subseteq \{\{a, b\} : a, b \in A,\ a \neq b\}$, is a(n *undirected*) *graph*. The elements of E are called *edges*. The *degree* of an alternative $a \in A$ is the number $|\{\{x, y\} \in E \colon a \in \{x, y\}\}|$, that is, the number of edges containing a. A *leaf* is an alternative with degree one. We denote the set of all leafs by $L(G)$.

For $a, b \in A$ with $a \neq b$, a *path* from a to b in G is a sequence of distinct alternatives $a_1, \ldots, a_k$ such that $a_1 = a$, $a_k = b$, and $\{a_i, a_{i+1}\} \in E$ for all $i = 1, \ldots, k-1$. If it is clear which path is meant, we also denote it by $[a, b]$. In this case, by $(a, b]$ we denote the sequence $a_2, \ldots, a_k$, and by (a, b) the sequence $a_2, \ldots, a_{k-1}$. Whenever it is clear from the context, the notations $[a, b]$, $(a, b]$, and (a, b) will also be used to denote the sets of alternatives (instead of the sequences) that appear in the corresponding path. When $a = b$, the notation $[a, b]$ simply denotes the alternative a, $x \in [a, b]$ means $x = a$, and $x \notin [a, b]$ means $x \neq a$.

Throughout this paper we assume that G is *connected*, i.e., there is a path from a to b for all distinct $a, b \in A$. If this path is unique for all $a, b \in A$, then G is called a *tree*. A *spanning tree* of G is a tree $T = (A, E_T)$ where $E_T \subseteq E$. In other words, spanning tree of G is a tree that can be obtained by deleting some edges of G.

[4] Observe that $a \in U(a, P)$ by reflexivity.

For a path $[x_1, x_\ell]$ with sequence $x_1, \ldots, x_\ell$, we write $P = [x_1, x_l] \cdots$ to denote a preference P such that $x_1 P x_2 P \ldots P x_\ell \, P x$ for all $x \in A \setminus [x_1, x_\ell]$. For instance, if the path is $[x_1, x_3]$ with sequence x_1, x_2, x_3, then $P = [x_1, x_3] \cdots$ means that the top-ranked, second-ranked, and the third-ranked alternatives of P are x_1, x_2, and x_3, respectively. Note that this notation does not impose any restriction on the ordering of alternatives that lie outside the path, except that they are all less preferred to the alternatives on the path. Similarly, we use the notation $P = \cdots [x_1, x_\ell] \cdots$ to mean that the alternatives $x_1, \ldots, x_\ell$ are consecutively ranked in P with $x_1 P x_2 \ldots P x_\ell$. Again, as before, this notation does not put any restriction on the ordering of the alternatives that do not lie on the path $[x_1, x_\ell]$, except that they cannot be ranked in-between the alternatives on the path. Combinations of these notations have similar meanings. Also, brackets are sometimes left out if confusion is unlikely.

We are now ready to introduce the notion of single-peaked preferences. A preference is single-peaked if there is a spanning tree of G so that as one moves away from the top-ranked alternative of the preference in any particular direction along the tree, preference decreases.

Definition 7.2.1 A preference P is *single-peaked* if there is a spanning tree T of G such that for all distinct $x, y \in A$ with $t(P) \neq y$,

$$x \in [t(P), y] \implies xPy,$$

where $[t(P), y]$ is the path from $t(P)$ to y in T.

We denote the set of all single-peaked preferences by . For a single-peaked preference, the top alternative is also called the *peak*.

In Section 7.6.2 we briefly further discuss this preference domain choice.

7.2.2 Probabilistic rules

By ΔA, we denote the set of all probability distributions on A. A *probabilistic rule* (PR) is a function $\phi :^N \to \Delta A$. For $a \in A$ and $P_N \in^N$, we denote the probability of a at $\phi(P_N)$ by $\phi_a(P_N)$, and for $B \subseteq A$, we denote the total probability of the alternatives in B by $\phi_B(P_N)$, i.e., $\phi_B(P_N) = \sum_{a \in B} \phi_a(P_N)$.

We proceed by defining the main properties of PRs that are of interest in this paper. The first property is unanimity. It says that if all the agents have the same top-ranked alternative, then that alternative is chosen with probability 1.

Definition 7.2.2 A PR ϕ is *unanimous* if $\phi_a(P_N) = 1$ for all $a \in A$ and all $P_N \in^N$ with $t(P_i) = a$ for all $i \in N$.

The second property is strategy-proofness, introduced in Gibbard (1977). It says that reporting a preference different from the sincere (true) one cannot increase the probability on any sincere upper contour set. In other words, the probability distribution over the alternatives induced by reporting truthfully stochastically dominates any probability distribution induced by reporting differently.

Definition 7.2.3 A PR ϕ is *strategy-proof* if for all $i \in N$, all $P_N \in^N$, all $P_i' \in$, and all $x \in A$,

$$\phi_{U(x,P_i)}(P_i, P_{-i}) \geq \phi_{U(x,P_i)}(P_i', P_{-i}).$$

It is not hard to see that under strategy-proofness the unanimity condition could be weakened to requiring $\phi_a(P_N) = 1$ for all $a \in A$ and all $P_N \in^N$ with $P_i = P_j$ and $t(P_i) = a$ for all $i, j \in N$. For later reference we include the following (straightforward) observation.

Remark 7.2.4 Let $L, L' \in \Delta A$ and let $P \in \mathbb{L}(A)$. Suppose $L_{U(x,P)} = L'_{U(x,P)}$ for all $x \in A$, where $L_{U(x,P)}$ denotes the total probability on the upper contour set $U(x, P)$. Then $L = L'$.

Two profiles $P_N, P_N' \in^N$ are *tops-equivalent* if $t(P_i) = t(P_i')$ for all $i \in N$. A PR is called tops-only if its outcomes do not change over top-equivalent profiles. In other words, the outcome of such a PR depends only on the top-ranked alternatives at a preference profile.

Definition 7.2.5 A PR ϕ is *tops-only* if $\phi(P_N) = \phi(P_N')$ for all tops-equivalent $P_N, P_N' \in^N$.

In our model, unanimity and strategy-proofness of a PR imply tops-onliness. This can be proved by using the main result in Chatterji and Zeng (2018), as we show in the Appendix.

Lemma 7.2.1 *Let $G = (A, E)$ be a connected graph and let a PR $\phi :^N \rightarrow \Delta A$ be unanimous and strategy-proof. Then, ϕ is tops-only.*

Proof: See Appendix 7.7. ■

7.3 Trees

Throughout this section the graph $G = (A, E)$ is a tree. We will characterize all unanimous and strategy-proof probabilistic rules for this case. First, we define the notion of uncompromisingness, introduced by [24] for deterministic rules. It says that if an agent unilaterally changes its preference from P_i to P_i', then the probabilities of the alternatives off the path $[t(P_i), t(P_i')]$, do not change. Uncompromisingness is closely related to strategy-proofness but often is easier to work with. Clearly, an uncompromising PR is tops-only.

Definition 7.3.1 Let $G = (A, E)$ be a tree. A PR $\phi :^N \to \Delta A$ is *uncompromising* if $\phi_d(P_N) = \phi_d(P'_i, P_{-i})$ for all $i \in N$, all $P_N \in^N$, all $P'_i \in$ and all $d \in A$ such that $d \notin [t(P_i), t(P'_i)]$.

Recall that by Lemma 7.2.1 every unanimous and strategy-proof PR is tops-only. In the following lemma we show that, by using tops-onliness, uncompromisingness can easily be derived from unanimity and strategy-proofness.

Lemma 7.3.1 *Let $G = (A, E)$ be a tree and let $\phi :^N \to \Delta A$ be a unanimous and strategy-proof PR. Then ϕ is uncompromising.*

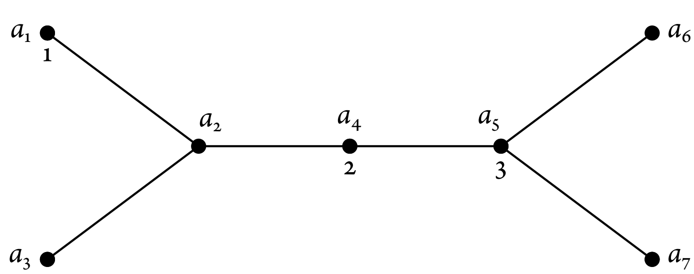
Proof: Let $P_N, P'_N \in^N$ and $i \in N$ be such that $P_{-i} = P'_{-i}$. In order to prove that $\phi_x(P_N) = \phi_x(P'_N)$ for all $x \notin [t(P_i), t(P'_i)]$, it is without loss of generality to assume $\{t(P_i), t(P'_i)\} \in E$. Then, by tops-onliness (Lemma 7.2.1), we may assume that $P_i = t(P_i)t(P'_i)\cdots$ and $P'_i = t(P'_i)t(P_i)\cdots$ such that $zP_iz' \Leftrightarrow zP'_iz'$ for all $z, z' \in A \setminus \{t(P_i), t(P'_i)\}$. Now the lemma follows directly from strategy-proofness. ■

In what follows we show that a unanimous and strategy-proof PR is completely determined by its values at profiles where the peaks of the agents are located at the leafs of the tree. We need the following definitions to formulate this property.

Definition 7.3.2 A *leaf assignment* is a function $\mu : N \to L(G)$. The set of all leaf assignments is denoted by . For $a \in A$ and $P_N \in^N$, a leaf assignment μ *respects* (a, P_N) if for all $i \in N$ and $b \in L(G)$, $\mu(i) = b$ implies $t(P_i) \in [a, b]$. The set of leaf assignments that respect (a, P_N) is denoted by (a, P_N).

Thus, a leaf assignment assigns to each agent a leaf of the tree. Consider an alternative a and a preference profile P_N. A leaf assignment respecting (a, P_N) is obtained as follows. If the top-ranked alternative $t(P_i)$ of agent i is a, then assign i to an arbitrary leaf. Otherwise, assign i to some leaf b such that $t(P_i)$ is on the path $[a, b]$. Clearly, if $P_N, P'_N \in^N$ are tops-equivalent, then $(a, P_N) = (a, P'_N)$. The following example illustrates Definition 7.3.2.

Example 7.3.3 Let $A = \{a_1, \ldots, a_7\}$ and consider the following tree.



Let $N = \{1, 2, 3\}$, and let P_N be a preference profile with $(t(P_1), t(P_2), t(P_3)) = (a_1, a_4, a_5)$, as illustrated in the figure. Then

$$
\begin{aligned}
\mu \in (a_1, P_N) &\quad\Leftrightarrow\quad \mu(1) \in \{a_1, a_3, a_6, a_7\},\ \mu(2), \mu(3) \in \{a_6, a_7\} \\
\mu \in (a_2, P_N) &\quad\Leftrightarrow\quad \mu(1) = a_1,\ \mu(2), \mu(3) \in \{a_6, a_7\} \\
\mu \in (a_3, P_N) &\quad\Leftrightarrow\quad \mu(1) = a_1,\ \mu(2), \mu(3) \in \{a_6, a_7\} \\
\mu \in (a_4, P_N) &\quad\Leftrightarrow\quad \mu(1) = a_1,\ \mu(2) \in \{a_1, a_3, a_6, a_7\},\ \mu(3) \in \{a_6, a_7\} \\
\mu \in (a_5, P_N) &\quad\Leftrightarrow\quad \mu(1) = a_1,\ \mu(2) \in \{a_1, a_3\},\ \mu(3) \in \{a_1, a_3, a_6, a_7\} \\
\mu \in (a_6, P_N) &\quad\Leftrightarrow\quad \mu(1) = a_1,\ \mu(2) \in \{a_1, a_3\},\ \mu(3) \in \{a_1, a_3, a_7\} \\
\mu \in (a_7, P_N) &\quad\Leftrightarrow\quad \mu(1) = a_1,\ \mu(2) \in \{a_1, a_3\},\ \mu(3) \in \{a_1, a_3, a_6\}
\end{aligned}
$$

describes the leaf assignments respecting (a, P_N) for each $a \in A$. ◁

With each $\mu \in$ we associate a probability distribution $_\mu$ over A. We introduce the notion of monotonicity for such a collection of probability distributions.

Definition 7.3.4 A collection of probability distributions $(_\mu)_{\mu\in}$ over A is *monotonic* if

(i) for every $b \in L(G)$ and $\mu \in$, if $\mu(i) = b$ for all $i \in N$, then $_\mu(b) = 1$,

(ii) for all $\mu, \hat{\mu} \in$ and $i \in N$ such that $\mu(j) = \hat{\mu}(j)$ for all $j \in N \setminus \{i\}$,

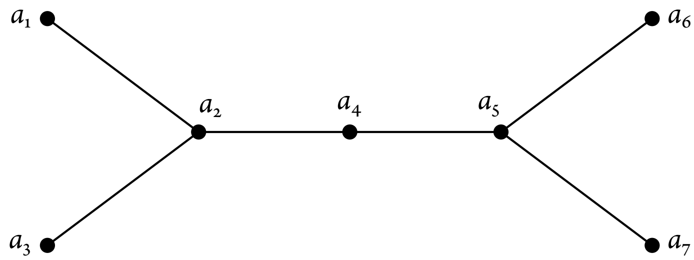
  (a) $_{\hat{\mu}}([c, \hat{\mu}(i)]) \geq_\mu ([c, \hat{\mu}(i)])$ for all $c \in [\mu(i), \hat{\mu}(i)]$, and

  (b) $_\mu(c) =_{\hat{\mu}} (c)$ for all $c \in A \setminus [\mu(i), \hat{\mu}(i)]$.

Part (i) in this definition says that if in a leaf assignment μ, all agents are assigned to the same leaf, then that leaf obtains probability one in the corresponding probability distribution β_μ. Part (ii) says that if an agent i moves from one leaf (at μ) to another (at $\hat{\mu}$), then, roughly speaking, probability increases along the path from the former to the latter leaf (part (a)), whereas off this path nothing changes (part (b)). Clearly, the conditions (i), (ii)(a), and (ii)(b) are related to unanimity, strategy-proofness, and uncompromisingness of a PR, respectively.

The following example illustrates the notion of monotonic probability distributions.

Example 7.3.5 Consider again the tree of Example 7.3.3, replicated here for convenience.



Consider the probability distributions $(_{\mu})_{\mu\in}$ in the table below. In this example, we assume that the collection $(_{\mu})_{\mu\in}$ is 'anonymous', which means that the probabilities depend only on the numbers of agents on the leafs. The μ-assignments are to the leafs a_1, a_3, a_6, and a_7 consecutively. The probabilities (the numbers in the table divided by 10) are those assigned to $a_1, \ldots, a_7$, consecutively. It is straightforward to verify that $(_{\mu})_{\mu\in}$ in this table satisfies monotonicity.

μ	$_{\mu}$	μ	$_{\mu}$
(3, 0, 0, 0)	(10, 0, 0, 0, 0, 0, 0)	(1, 0, 2, 0)	(1, 3, 0, 2, 2, 2, 0)
(0, 3, 0, 0)	(0, 0, 10, 0, 0, 0, 0)	(0, 1, 2, 0)	(0, 2, 3, 2, 1, 2, 0)
(0, 0, 3, 0)	(0, 0, 0, 0, 0, 10, 0)	(0, 0, 2, 1)	(0, 0, 0, 0, 7, 2, 1)
(0, 0, 0, 3)	(0, 0, 0, 0, 0, 0, 10)	(1, 0, 0, 2)	(1, 3, 0, 2, 2, 0, 2)
(2, 1, 0, 0)	(4, 3, 3, 0, 0, 0, 0)	(0, 1, 0, 2)	(0, 2, 3, 2, 1, 0, 2)
(2, 0, 1, 0)	(4, 2, 0, 2, 1, 1, 0)	(0, 0, 1, 2)	(0, 0, 0, 0, 7, 1, 2)
(2, 0, 0, 1)	(4, 2, 0, 2, 1, 0, 1)	(1, 1, 1, 0)	(1, 2, 3, 2, 1, 1, 0)
(1, 2, 0, 0)	(1, 5, 4, 0, 0, 0, 0)	(1, 1, 0, 1)	(1, 2, 3, 2, 1, 0, 1)
(0, 2, 1, 0)	(0, 2, 4, 2, 1, 1, 0)	(1, 0, 1, 1)	(1, 3, 0, 3, 1, 1, 1)
(0, 2, 0, 1)	(0, 2, 4, 2, 1, 0, 1)	(0, 1, 1, 1)	(0, 3, 1, 3, 1, 1, 1)

◁

In what follows, we associate a PR with each monotonic collection of probability distributions. As a preparation we need Lemma 7.3.2 below.

In this lemma the leaf assignments μ_b and $\hat{\mu}_b$ are considered for an alternative a, a leaf b, and a preference profile P_N. Leaf assignment μ_b respects (a, P_N) and has the (additional) property that an agent i is assigned to b if and only if its peak $t(P_i)$ lies on the path $[a, b]$. Leaf assignment $\hat{\mu}_b$ has the same properties as μ_b except that an agent i is now assigned to b if its peak $t(P_i)$ lies on the path $(a, b]$, but is not assigned to b if its peak is a. Thus, the agents who are assigned to b by μ_b are those who are assigned to b by $\hat{\mu}_b$ plus those with peak a (i.e., $\mu_b^{-1}(b) = \hat{\mu}_b^{-1}(b) \cup \{i\colon t(P_i) = a\}$). Note that there is no restriction on how μ_b and $\hat{\mu}_b$ assign agents to the leafs other than b except that they both respect (a, P_N).

Lemma 7.3.2 now says that for any monotonic collection $(_{\mu})_{\mu\in}$, the total probability assigned to the alternatives in $[a, b]$ by β_{μ_b} is at least as high as the total probability assigned to the alternatives in $(a, b]$ by $\beta_{\hat{\mu}_b}$, that is, $\beta_{\mu_b}[a, b] - \beta_{\hat{\mu}_b}(a, b] \geq 0$. Lemma 7.3.2 further says that this quantity $\beta_{\mu_b}[a, b] - \beta_{\hat{\mu}_b}(a, b]$ does not depend on the choice of the leaf b, nor on the exact specification of μ_b and $\hat{\mu}_b$ for agents with peaks not on $[a, b]$. Thus, for a given monotonic collection $({}_{\mu})_{\mu\in}$, the quantity $\beta_{\mu_b}[a, b] - \beta_{\hat{\mu}_b}(a, b]$ depends only on the alternative a and the profile P_N. Later, we will associate a PR with a given monotonic collection $({}_{\mu})_{\mu\in}$ such that the probability of a at a profile P_N is given by this quantity.

Lemma 7.3.2 *Let $({}_{\mu})_{\mu\in}$ be a monotonic collection of probability distributions. Let $a \in A$, $b, c \in L(G)$, $P_N \in^N$, and $\mu_b, \hat{\mu}_b, \mu_c, \hat{\mu}_c \in (a, P_N)$ be such that for each $x \in \{b, c\}$ and all $i \in N$, $\mu_x(i) = x$ if and only if $t(P_i) \in [a, x]$ and $\hat{\mu}_x(i) = x$ if and only if $t(P_i) \in (a, x]$. Then*

$$ {}_{\mu_b}([a, b]) -_{\hat{\mu}_b} ((a, b]) =_{\mu_c} ([a, c]) -_{\hat{\mu}_c} ((a, c]) \geq 0. \tag{7.1}$$

Proof: First, we prove that the amount ${}_{\mu_b}([a, b])$ does not depend on the further specification of μ_b. That is, if $\mu \in (a, P_N)$ also satisfies $\mu(i) = b$ if and only if $t(P_i) \in [a, b]$ for all $i \in N$, then ${}_{\mu}([a, b]) =_{\mu_b} ([a, b])$. To see this, suppose that for some $j \in N$ we have $t(P_j) \notin [a, b]$ and $\mu(j) \neq \mu_b(j)$. Hence, $\mu(j), \mu_b(j) \neq b$. We prove that $d \notin [a, b]$ for all $d \in [\mu(j), \mu_b(j)]$. Suppose to the contrary that there is $d \in A$ with $d \in [a, b] \cap [\mu(j), \mu_b(j)]$. The path from a to $\mu_b(j)$ consists of the subpath $[a, d] \subseteq [a, b]$ followed by the path $(d, \mu_b(j)]$, with $t(P_j) \in (d, \mu_b(j)]$. This implies that $t(P_j) \notin [a, d] \cup (d, \mu(j)] = [a, \mu(j)]$, which contradicts the assumption that $\mu \in (a, P_N)$. The desired result follows from repeating this argument for all j with $\mu(j) \neq \mu_b(j)$ and each time applying condition (ii)(b) in Definition 7.3.4.

Similarly, one proves that the amount ${}_{\hat{\mu}_b}((a, b])$ does not depend on the further specification of $\hat{\mu}_b$, i.e., if $\mu \in (a, P_N)$ also satisfies $\mu(i) = b$ if and only if $t(P_i) \in (a, b]$ for all $i \in N$, then ${}_{\mu}((a, b]) =_{\mu_b} ((a, b])$.

We now prove (7.1). It is sufficient to prove this for the case where $a \in [b, c]$. Otherwise, there is a $d \in L(G)$ such that both $a \in [d, b]$ and $a \in [d, c]$. Then, if we show (7.1) for the pairs of leafs b, d and c, d, then (7.1) for the pair b, c follows by combining the two equations. Thus, we assume $a \in [b, c]$. Moreover, by the first two paragraphs of the proof we may assume that $\mu_b = \hat{\mu}_c$ and $\mu_c = \hat{\mu}_b$. For the equality in (7.1), it is then sufficient to show that

$$ {}_{\mu_b}([a, b]) -_{\mu_c} ((a, b]) =_{\mu_c} ([a, c]) -_{\mu_b} ((a, c]). $$

We have

$$ {}_{\mu_b}([b, c]) =_{\hat{\mu}_c} ([b, c]) =_{\mu_c} ([b, c]), \tag{7.2}$$

where the second equality follows from condition (ii)(b) in Definition 7.3.4. Therefore,

$$\begin{aligned}{}_{\mu_b}([a,b]) -_{\mu_c}((a,b]) &=_{\mu_b}([b,c]) -_{\mu_b}((a,c]) -_{\mu_c}((a,b]) \\ &=_{\mu_c}([b,c]) -_{\mu_c}((a,b]) -_{\mu_b}((a,c]) \\ &=_{\mu_c}([a,c]) -_{\mu_b}((a,c])\end{aligned}$$

where the second equality follows from (7.2).

Finally, by condition (ii)(a) in Definition 7.3.4 we have

$${}_{\mu_b}([a,b]) \geq_{\hat{\mu}_b}([a,b]),$$

which implies the nonnegativity of the expressions in (7.1) and completes the proof of the lemma. ■

With every monotonic collection of probability distributions we associate a probabilistic rule, as follows.

Definition 7.3.6 Let $B = ({}_{\mu})_{\mu \in}$ be a monotonic collection of probability distributions over A. We define $\phi^B :^N \rightarrow \Delta A$ as follows. For each $a \in A$ and $P_N \in_N$

$$\phi^B_a(P_N) =_{\mu_b}([a,b]) -_{\hat{\mu}_b}((a,b]) \tag{7.3}$$

for some $b \in L(G)$ and $\mu_b, \hat{\mu}_b \in (a, P_N)$ such that $\mu_b(i) = b$ if and only if $t(P_i) \in [a,b]$ and $\hat{\mu}_b(i) = b$ if and only if $t(P_i) \in (a,b]$.

Note that by Lemma 7.3.2, ϕ^B is well-defined: it does not depend on the particular choice of b, μ_b, or $\hat{\mu}_b$. Moreover we have:

Lemma 7.3.3 *ϕ^B defined by (7.3) is a PR.*

Proof: Let $P_N \in^N$. By Lemma 7.3.2, $\phi^B_a(P_N) \geq 0$ for every $a \in A$. We still have to prove that $\sum_{a \in A} \phi^B_a(P_N) = 1$.

Let $a \in A$, $b \in L(G)$, and let $\mu \in (a, P_N)$ be such that $\mu(i) = b$ if and only if $t(P_i) \in [a,b]$, for all $i \in N$. We claim that $\phi^B_{[a,b]}(P_N) =_{\mu}([a,b])$. To show this, let $[a,b]$ be the sequence $a_1, \ldots, a_k$ with $a = a_1$ and $b = a_k$. For every $j = 2, \ldots, k$ let $\mu_j, \hat{\mu}_j \in (a_j, P_N)$ be such that for all $i \in N$ we have $\mu_j(i) = b \Leftrightarrow t(P_i) \in [a_j, b]$ and $\hat{\mu}(j) = b \Leftrightarrow t(P_i) \in (a_j, b]$; and let $\hat{\mu} \in (a, P_N)$ such that for all $i \in N$ we have $\hat{\mu}_j(i) = b \Leftrightarrow t(P_i) \in (a, b]$. Then

$$\begin{aligned}
\phi^B_{[a,b]}(P_N) &= {}_{\mu}([a,b]) -_{\hat{\mu}} ((a,b]) \\
&\quad +_{\mu_2}([a_2,b]) -_{\hat{\mu}_2} ((a_2,b]) \\
&\quad +_{\mu_3}([a_3,b]) -_{\hat{\mu}_3} ((a_3,b]) \\
&\quad \vdots \\
&\quad +_{\mu_k}(\{b\}) -_{\hat{\mu}_k} (\emptyset) \\
&= {}_{\mu}([a,b])
\end{aligned}$$

where the second equality follows since ${}_{\hat{\mu}}((a,b]) =_{\hat{\mu}} ([a_2,b]) =_{\mu_2} ([a_2,b])$ and ${}_{\hat{\mu}_j}((a_j,b]) =_{\hat{\mu}_j} ([a_{j+1},b]) =_{\mu_{j+1}} ([a_{j+1},b])$ for every $j = 2, \ldots, k-1$ by condition (ii)(b) in Definition 7.3.4.

We partition A into subsets $A^1, \ldots, A^k$, such that the alternatives in A^ℓ form a path $[a^\ell, \ldots, b^\ell]$ for some $a^\ell \in A$ and $b^\ell \in L(G)$ (possibly $a^\ell = b^\ell$). We define the leaf assignment μ as follows: (i) for each $\ell = 1, \ldots, k$, $\mu^{-1}(b^\ell) = \{i \in N : t(P_i) \in A^\ell\}$, and (ii) for each $b \in L(G) \setminus \{b^1, \ldots, b^k\}$, $\mu^{-1}(b) = \emptyset$ (case (ii) occurs if $b = a^\ell$ for some ℓ). By the previous part of the proof, for each $\ell = 1, \ldots, k$, we have $\phi^B_{A^\ell}(P_N) =_{\mu_\ell} (A^\ell)$ for (any) $\mu_\ell \in (a^\ell, P_N)$ such that $\mu_\ell(i) = b^\ell \Leftrightarrow t(P_i) \in A^\ell$ for all $i \in N$. By definition of μ and condition (ii)(b) in Definition 7.3.4, ${}_{\mu_\ell}(A^\ell) =_{\mu} (A^\ell)$ for every $\ell = 1, \ldots, k$. Hence, $\sum_{a \in A} \phi^B_a(P_N) = \sum_{\ell=1}^{k} {}_{\mu_\ell}(A^\ell) = \sum_{\ell=1}^{k} {}_{\mu}(A^\ell) =_{\mu} (A) = 1$. ■

Definition 7.3.7 A PR ϕ is a *leaf-peak rule* if there is a monotonic collection of probability distributions $B = ({}_{\mu})_{\mu \in}$ such that $\phi = \phi^B$.

An example of a leaf-peak rule is the following.

Example 7.3.8 Consider the tree of Example 7.3.5. Let $N = \{1, 2, 3\}$. Let ϕ be the (anonymous, i.e., invariant under any permutation of the agents) leaf-peak rule with respect to $({}_{\mu})_{\mu \in}$ as in Example 7.3.5. Consider the preference profile P_N with $(t(P_1), t(P_2), t(P_3)) = (a_1, a_4, a_5)$ as in Example 7.3.3. We take the fixed leaf a_1 for the computations in the following table, which provides the outcome of the leaf-peak rule ϕ at P_N.

a	b	$_{\mu}([a,b]) -_{\mu'} ((a,b])$	$\phi_a(P_N)$
a_1	a_1	$_{(1,0,2,0)}([a_1,a_1]) -_{(0,0,3,0)} ((a_1,a_1])$	.1
a_2	a_1	$_{(1,0,2,0)}([a_2,a_1]) -_{(1,0,2,0)} ((a_2,a_1])$	.3
a_3	a_1	$_{(1,0,2,0)}([a_3,a_1]) -_{(1,0,2,0)} ((a_3,a_1])$	0
a_4	a_1	$_{(2,0,1,0)}([a_4,a_1]) -_{(1,0,2,0)} ((a_4,a_1])$	.4
a_5	a_1	$_{(3,0,0,0)}([a_5,a_1]) -_{(2,0,1,0)} ((a_5,a_1])$	.2
a_6	a_1	$_{(3,0,0,0)}([a_6,a_1]) -_{(3,0,0,0)} ((a_6,a_1])$	0
a_7	a_1	$_{(3,0,0,0)}([a_7,a_1]) -_{(3,0,0,0)} ((a_7,a_1])$	0

$\triangleleft$

Our main result shows that leaf-peak rules are exactly the unanimous and strategy-proof PRs for single-peaked preferences on trees. We prove this by means of the following two lemmas.

Lemma 7.3.4 *Let $B = (_{\mu})_{\mu\in}$ be a monotonic collection of probability distributions. Then ϕ^B is unanimous and strategy-proof.*

Proof: In this proof we write ϕ instead of ϕ^B. Unanimity follows directly from the definition of ϕ.

We next argue that ϕ is uncompromising. Let $P_N \in^N$, $i \in N$, $P'_i \in$, and $d \in A \setminus [t(P_i), t(P'_i)]$. Take $b \in L(G)$ such that $[d,b] \cap [t(P_i), t(P'_i)] = \emptyset$. Then, by definition of ϕ, in particular (7.3), we obtain $\phi_d(P_N) = \phi_d(P_{-i}, P'_i)$. This shows that ϕ is uncompromising.

In order to prove strategy-proofness, assume for contradiction that there exists $i \in N$, $P_N \in^N$, and $P'_i \in$ such that $\phi_{U(c,P_i)}(P_N) < \phi_{U(c,P_i)}(P'_i, P_{-i})$ for some $c \in A$. Since ϕ is uncompromising and thus tops-only, we may assume without loss of generality that $P_i = [t(P_i), \ldots, t(P'_i)] \cdots$ and $P'_i = [t(P'_i), \ldots, t(P_i)] \cdots$, and such that $zP_iz' \Leftrightarrow zP'_iz'$ for all $z, z' \notin [t(P_i), t(P'_i)]$. By uncompromisingness we also have $\phi_z(P_N) = \phi_z(P'_i, P_{-i})$ for all $z \notin [t(P_i), t(P'_i)]$. Therefore, $c \in [t(P_i), t(P'_i))$ and thus

$$\phi_{[t(P_i),c]}(P_N) < \phi_{[t(P_i),c]}(P'_i, P_{-i}). \tag{7.4}$$

Let d appear just after c on the path $[t(P_i), t(P'_i)]$. Let $P^c \in$ with $t(P^c) = c$ and $P^d \in$ with $t(P) = d$. By uncompromisingness, $\phi_{[t(P_i),c]}(P_N) = \phi_{[t(P_i),c]}(P^c, P_{-i})$ and $\phi_{[t(P_i),c]}(P^d, P_{-i}) = \phi_{[t(P_i),c]}(P'_i, P_{-i})$. By (7.4), this yields $\phi_{[t(P_i),c]}(P^c, P_{-i}) < \phi_{[t(P_i),c]}(P^d, P_{-i})$. Since by uncompromisingness $\phi_z(P^c, P_{-i}) = \phi_z(P^d, P_{-i})$ for all $z \notin \{c, d\}$, this implies

$$\phi_c(P^c, P_{-i}) < \phi_c(P^d, P_{-i}). \tag{7.5}$$

Now take $b, b' \in L(G)$ such that $\{c, d\} \subseteq [b, b']$ and $d \notin [b, c]$. By (7.3),

$$\phi_c(P^c, P_{-i}) =_{\mu_b} ([c,b]) -_{\hat{\mu}_b} ((c,b]) \tag{7.6}$$

where $\mu_b, \hat{\mu}_b \in (c, (P^c, P_{-i}))$ are such that $\mu_b(j) = b$ if and only if $t(P_j) \in [c, b]$ and $\hat{\mu}_b(j) = b$ if and only if $t(P_j) \in (c, b]$ for all $j \in N$. Let μ'_b be such that $\mu'_b(j) = \mu_b(j)$ for all $j \in N \setminus \{i\}$ and $\mu'_b(i) = b'$; and let

$\hat{\mu}'_b = \hat{\mu}_b$. Note that $\mu'_b, \hat{\mu}'_b \in (c, (P^d, P_{-i}))$. Also, writing $\hat{P}_N = (P^d, P_{-i})$, we have $\mu'_b(j) = b$ if and only if $t(\hat{P}_j) \in [c, b]$ and $\hat{\mu}'_b(j) = b$ if and only if $t(\hat{P}_j) \in (c, b]$ for all $j \in N$. Therefore, by (7.3),

$$\phi_c(P^d, P_{-i}) =_{\mu'_b} ([c, b]) -_{\hat{\mu}'_b} ((c, b]). \tag{7.7}$$

By (7.5), (7.6), (7.7), and the fact that $\hat{\mu}'_b = \hat{\mu}_b$, we obtain

$$_{\mu_b}([c, b]) <_{\mu'_b} ([c, b]). \tag{7.8}$$

However, as (i) $\mu_b^{-1}(\hat{b}) = \mu_b'^{-1}(\hat{b})$ for all $\hat{b} \in L(G) \setminus \{b, b'\}$ and (ii) $\mu_b'^{-1}(b) \subseteq \mu_b^{-1}(b)$, this contradicts condition (ii)(a) in Definition 7.3.4. ∎

Next we show the converse of Lemma 7.3.4.

Lemma 7.3.5 *Let ϕ be a unanimous and strategy-proof PR. Then there is a monotonic collection of probability distributions $B = (_\mu)_{\mu\in}$ such that $\phi = \phi^B$.*

Proof: By Lemma 7.3.1, ϕ is uncompromising. For every $\mu \in$ define $_\mu = \phi(P_N)$, where $P_N \in^N$ satisfies $t(P_i) = \mu(i)$ for all $i \in N$.

We first show that $B = (_\mu)_{\mu\in}$ thus defined, is a monotonic collection. Clearly, since ϕ is unanimous, condition (i) in Definition 7.3.4 is satisfied. For condition (ii), let $\mu, \hat{\mu} \in$ and $i \in N$ be such that $\mu(j) = \hat{\mu}(j)$ for all $j \in N \setminus \{i\}$ and let $P_N, \hat{P}_N$ be such that $t(P_k) = \mu(k)$ and $t(\hat{P}_k) = \hat{\mu}(k)$ for all $k \in N$. Since ϕ is uncompromising, $\phi_c(P_N) = \phi_c(\hat{P}_N)$ for all $c \notin [t(P_i), t(\hat{P}_i)]$, hence $_\mu(c) =_{\hat{\mu}} (c)$ for all $c \notin [\mu(i), \hat{\mu}(i)]$, i.e., condition (ii)(b) is satisfied. Moreover, by strategy-proofness of ϕ we have for all $c \in [t(P_i), t(\hat{P}_i)]$ that $\phi_{U(c,\hat{P}_i)}(\hat{P}_N) \geq \phi_{U(c,\hat{P}_i)}(P_N)$. Since $\phi_z(P_N) = \phi_z(\hat{P}_N)$ for all $z \notin [t(P_i), t(\hat{P}_i)]$, this implies $\phi_{[c,t(\hat{P}_i)]}(\hat{P}_N) \geq \phi_{[c,t(\hat{P}_i)]}(P_N)$, and therefore $_{\hat{\mu}}([c, \hat{\mu}(i)]) \geq_\mu ([c, \hat{\mu}(i)])$ for all $c \in [\mu(i), \hat{\mu}(i)]$. This proves condition (ii)(a).

Finally, we show that $\phi = \phi^B$. Let $P_N \in^N$ and $a \in A$. Let $\mu', \mu'' \in (a, P_N)$ and $b \in L(G)$ be such that, for all $i \in N$, $\mu'(i) = b$ if and only if $t(P_i) \in [a, b]$ and $\mu''(i) = b$ if and only if $t(P_i) \in (a, b]$. Also, let $P'_N \in^N$ be such that $t(P'_i) = \mu'(i)$ for all $i \in N$ and $P''_N \in^N$ be such that $t(P''_i) = \mu''(i)$ for all $i \in N$. Then

$$\begin{aligned} \phi^B_a(P^N) &= {}_{\mu'}([a, b]) -_{\mu''} ((a, b]) \\ &= \phi_{[a,b]}(P'_N) - \phi_{(a,b]}(P''_N) \\ &= \phi_a(P_N) \end{aligned}$$

where the last equality follows by uncompromisingness of ϕ. We conclude that $\phi = \phi^B$. ∎

Lemmas 7.3.4 and 7.3.5 now imply the main result of this section.

Theorem 7.3.9 *Let $G = (A, E)$ be a tree. Then a PR $\phi :^N \to \Delta A$ is unanimous and strategy-proof if and only if it is a leaf-peak rule.*

A characterization of unanimous and strategy-proof deterministic rules follows as a corollary of Theorem 7.3.9. In Section 7.6.1, we show that the probabilistic rules with these properties are not necessarily convex combinations of deterministic rules satisfying the same properties.

7.4 Leafless graphs

In this section, $G = (A, E)$ is a connected graph without leafs. The main result will be that every unanimous and strategy-proof PR is random dictatorial, to be defined below. We will derive this result for the case of two agents, and then use Theorem 5 in [35] to extend it to more than two agents.

Our notational conventions about preferences as introduced in Section 7.2 will still be used. Additionally, for a path $\pi = [x_1, x_\ell]$ with sequence $x_1, \ldots, x_\ell$ we denote by $\pi^{-1} = [x_\ell, x_1]$ the path in reverse direction, i.e., with sequence $x_\ell, \ldots, x_1$, and use this in notations for preferences such as $P = \pi \cdots, P = \pi^{-1} \cdots$, etc., with obvious meaning.

A *cycle* in G is a sequence of distinct alternatives $x_1, \ldots, x_k \in A$ for some $k \geq 3$ such that $\{\{x_i, x_{i+1}\}, \{x_k, x_1\} : i = 1, \ldots, k-1\} \subseteq E$.

The following lemma considers unanimous and strategy-proof PRs for the case of two agents. Consider two alternatives a and b that are contained in some cycle. In words, Lemma 7.4.1 says that in all profiles where the peak of agent 1 is a and that of agent 2 is b, a receives some fixed probability ε and b receives the rest of the probability $1 - \varepsilon$; thus, no alternative other a and b receives any positive probability. Moreover, suppose that there is another alternative c such that there is a cycle through a and c and there is a path from b to c that does not contain a. Then Lemma 7.4.1 says that the same as for a and b holds for a and c, i.e., at all profiles where the peak of agent 1 is a and that of agent 2 is c, a receives (the same) probability ε and c receives the rest of the probability $1 - \varepsilon$.

Lemma 7.4.1 *Let $n = 2$ and let $\phi :^N \to \Delta(A)$ be a unanimous and strategy-proof PR.*

(*i*) *Let $a, b \in A$, $a \neq b$, be such that there is a cycle containing a and b. Then there exists $\varepsilon \in [0, 1]$ such that for all $P_1, P_2 \in$ with $t(P_1) = a$ and $t(P_2) = b$ we have $\phi_a(P_1, P_2) = \varepsilon$ and $\phi_b(P_1, P_2) = 1 - \varepsilon$.*

(*ii*) *Let, additionally, $c \notin \{a, b\}$ be such that there is a cycle containing a and c, and a path from b to c not containing a. Then $\phi_a(P_1, P_2) = \varepsilon$ and $\phi_c(P_1, P_2) = 1 - \varepsilon$ for all $P_1, P_2 \in$ with $t(P_1) = a$ and $t(P_2) = c$, where ε is as in (i).*

Proof: (i) Since there is a cycle containing both a and b, there exist two paths π and $\hat{\pi}$ from a to b in G such that $\pi \cap \hat{\pi} = \{a, b\}$. Hence, there are $P, Q \in$ such that $P = \pi \cdots$ and $Q = \hat{\pi}^{-1} \cdots$

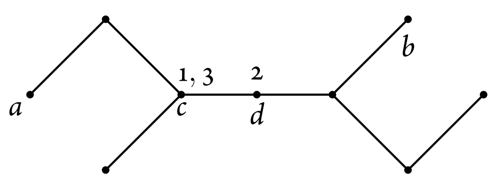
Suppose that $\phi_x(P, Q) > 0$ for some $x \in A \setminus \{a, b\}$. Since $U(b, P) \cap U(a, Q) = \{a, b\}$, we have $x \notin U(b, P)$ or $x \notin U(a, Q)$. By unanimity, in the first case agent 1 can manipulate by changing to Q and in the second case agent 2 can manipulate by changing to P. This contradicts strategy-proofness, and therefore we have $\phi_x(P, Q) = 0$ for all $x \in A \setminus \{a, b\}$. Thus, there exists $\varepsilon \in [0, 1]$ such that $\phi_a(P, Q) = \varepsilon$ and $\phi_b(P, Q) = 1 - \varepsilon$. Statement (i) now follows from tops-onliness of ϕ (Lemma 7.2.1).

(ii) Let $P_1, P_2 \in$ with $t(P_1) = a$ and $t(P_2) = c$. Assume that $\phi_a(P_1, P_2) = \varepsilon'$. By a similar argument as in step (i), this implies $\phi_c(P_1, P_2) = 1 - \varepsilon'$. Thus, it is sufficient to show that $\varepsilon = \varepsilon'$. Suppose not. Assume without loss of generality that $\varepsilon > \varepsilon'$. Let π now be a path from b to c such that $a \notin \pi$, and consider associated preferences $P = \pi \cdots, P' = \pi^{-1} \cdots \in$. By part (i), $\phi_{U(c,P)}(P_1, P) = 1 - \varepsilon < 1 - \varepsilon' = \phi_{U(c,P)}(P_1, P')$. This violates strategy-proofness and, hence, $\varepsilon = \varepsilon'$. ■

A PR ϕ is *random-dictatorial* if there are ${}_1, \ldots, {}_n \in [0, 1]$ with $\sum_{i \in N} {}_i = 1$, such that for every $P_N \in^N$ and $a \in A$ we have $\phi_a(P_N) = \sum_{i \in N: t(P_i) = a} {}_i$.

Clearly, a random dictatorial rule is unanimous and strategy-proof. Indeed, when G is a tree, a random dictatorial rule is a leaf-peak rule. To see this note that, if ϕ is random dictatorial with weights ${}_1, \ldots, {}_n$, then the collection $B = ({}_\mu)_{\mu \in}$ given by ${}_\mu(a) = \sum_{i \in N: \mu(i) = a} {}_i$ for each $\mu \in$ and every $a \in L(G)$, is monotonic. It is easy to verify that $\phi = \phi^B$. The following example provides an illustration of this.

Example 7.4.1 Consider the following tree:



Let $N = \{1, 2, 3\}$ and let ϕ be random dictatorial with weights $({}_1, {}_2, {}_3) = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$. The peaks of the agents in the preference profile P_N are indicated in the figure. Hence $\phi_c(P_N) = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$ and $\phi_d(P_N) = \frac{1}{3}$. With the collection B defined as above, we obtain

$$\begin{aligned}\phi^B_c(P_N) &= {}_\mu([c, a]) - {}_{\hat{\mu}}((c, a]) \\ &= \frac{1}{6} + \frac{1}{2} - 0 = \frac{2}{3} \\ &= \phi_c(P_N),\end{aligned}$$

where $\mu(1) = \mu(3) = a$, $\mu(2) = b$, and $\hat{\mu}(1) = \hat{\mu}(2) = \hat{\mu}(3) = b$. Similarly,

$$\begin{aligned} \phi_d^B(P_N) &= {}_{\mu'}([d,a]) -_{\hat{\mu}'} ((d,a]) \\ &= 1 - \frac{1}{6} - \frac{1}{2} = \frac{1}{3} \\ &= \phi_d(P_N), \end{aligned}$$

where $\mu'(1) = \mu'(2) = \mu'(3) = a$, $\hat{\mu}'(1) = \hat{\mu}'(3) = a$, and $\hat{\mu}'(2) = b$. ◁

A graph G is 2-*connected* if for all distinct $x, y \in A$ there is a cycle in G containing x and y. We can now state the following consequence of Lemma 7.4.1.

Lemma 7.4.2 *Let $n = 2$ and let $\phi :^N \to \Delta(A)$ be a unanimous and strategy-proof PR. Assume that the graph G is 2-connected. Then ϕ is random dictatorial.*

Proof: Let $a \in A$. By Lemma 7.4.1 there is an $\in [0,1]$ such that for all $x \in A$ and $P_1, P_2 \in^N$ with $t(P_1) = a$ and $t(P_2) = x$ we have $\phi_a(P_1, P_2) =$ and $\phi_x(P_1, P_2) = 1-$. Now let $b \in A$, $b \neq a$. Then similarly one proves that there is $' \in [0,1]$ such that for all $x \in A$ and $Q_1, Q_2 \in^N$ with $t(Q_1) = x$ and $t(Q_2) = b$ we have $\phi_b(Q_1, Q_2) =' $ and $\phi_x(Q_1, Q_2) = 1-'$. Since the latter holds for $x = a$ in particular, we have $1-' =$. This implies that for all $x, y \in A$ and $Z_1, Z_2 \in^N$ with $t(Z_1) = x$ and $t(Z_2) = y$ we have $\phi_x(Z_1, Z_2) =$ and $\phi_y(Z_1, Z_2) = 1-$. Hence, ϕ is random dictatorial. ■

The following lemma shows that random dictatorship for $n = 2$ still holds if the graph G has no leaf.

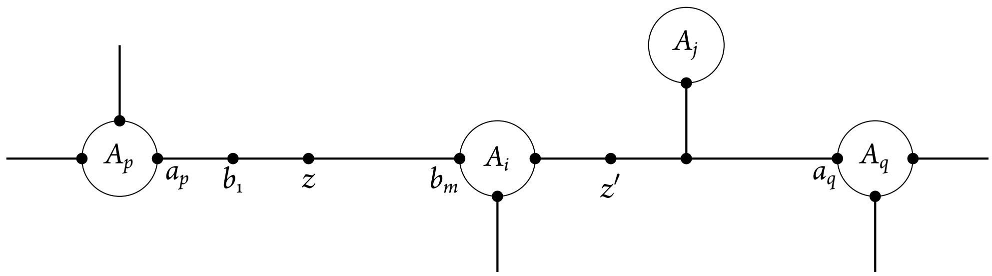
Lemma 7.4.3 *Let $n = 2$, and let G have no leaf. Suppose $\phi :^N \to \Delta(A)$ is a unanimous and strategy-proof PR. Then ϕ is random dictatorial.*

Proof: If G is 2-connected then the result follows from Lemma 7.4.2. Now assume that G is not 2-connected. Since G is connected we can decompose it into 2-connected subgraphs $(A_1, E_1), \ldots, (A_\ell, E_\ell)$, the set of remaining alternatives $B = A \setminus \cup_{i=1}^{\ell} A_i$ and the set of remaining edges $E' = E \setminus \cup_{i=1}^{\ell} E_i$.[5] (We visualize these subgraphs as ordered from left to right, see below.)

For any distinct $1 \leq p, q \leq \ell$ there are $a_p \in A_p$ and $a_q \in A_q$ such that all paths in G from an alternative in A_p to an alternative in A_q leave A_p via a_p and enter A_q via a_q. In this case, with we use the notation $[\![a_p, a_q]\!]$ to denote the set of alternatives containing a_p, a_q, and all x such that there is some path π in G with $x \in \pi$, starting at a_p such that $\pi \cap A_p = \{a_p\}$, and $a_q \notin \pi$; or there is some path π in G with $x \in \pi$, starting at a_q such that $\pi \cap A_q = \{a_q\}$, and $a_p \notin \pi$. Similarly, $[\![a_p, a_q)\!) = [\![a_p, a_q]\!] \setminus \{a_q\}$; $[\![, a_p]\!]$ denotes all

[5]This decomposition is close to the decomposition as a so-called block-tree. See, for instance, [22]. The formal definition of a block-tree is slightly different, but the decomposition here is more convenient for our purposes.

alternatives on paths starting at a_p which have only a_p in common with $[\![a_p, a_q]\!]$; $[\![a_q,]\!]$ denotes all alternatives on paths starting at a_q which have only a_q in common with $[\![a_p, a_q]\!]$; $[\![, a_q]\!]$ denotes all alternatives on paths starting at a_q which have only a_q in common with A_q; and so on and so forth. See the following diagram, which shows a possible part of the decomposition of G, and visualizes parts of rest of the proof.



By (the proof of) Lemma 7.4.2 there are $_1, \ldots, _\ell \in [0, 1]$ such that, for all $i = 1, \ldots, \ell$, $\phi_{t(P_1)}(P_1, P_2) =_i$ and $\phi_{t(P_2)}(P_1, P_2) = 1-_i$ for all $(P_1, P_2) \in^N$ with $t(P_1), t(P_2) \in A_i$. (In words, ϕ induces a random dictatorship on every A_i.) The proof proceeds in three steps.

(a) With notations as above, we first consider a preference profile (P_1, P_2) such that $t(P_1) \in A_p \setminus \{a_p\}$ and $t(P_2) \in A_q \setminus \{a_q\}$ for some $1 \leq p < q \leq \ell$. Since $\phi_{a_q}(P_1', P_2) =_q$ for $P_1' \in$ with $t(P_1') = a_q$, strategy-proofness (considering agent 1) implies that

$$\phi_{[\![, a_q]\!]}(P_1, P_2) \geq_q . \tag{7.9}$$

Similarly,

$$\phi_{[\![a_p,]\!]}(P_1, P_2) \geq 1 -_p . \tag{7.10}$$

Now consider $\tilde{P}_1 \in$ with $t(\tilde{P}_1) \in A_q \setminus \{t(P_2)\}$ and such that $x\tilde{P}_1 t(P_2)$ for all $x \in [\![, a_q]\!]$. Let $y \in [\![, a_q]\!]$ be such that $x\tilde{P}_1 y$ for all $x \in [\![, a_q]\!]$. Since $\phi_{t(\tilde{P}_1)}(\tilde{P}_1, P_2) =_q$, strategy-proofness (considering agent 1) requires that $\phi_{U(y, \tilde{P}_1)}(P_1, P_2) \leq_q$, hence:

$$\phi_{[\![, a_q]\!]}(P_1, P_2) \leq_q . \tag{7.11}$$

Similarly,

$$\phi_{[\![a_p,]\!]}(P_1, P_2) \leq 1 -_p . \tag{7.12}$$

Combining (7.9) and (7.11) we obtain $\phi_{[\![, a_q]\!]}(P_1, P_2) =_q$, and combining (7.10) and (7.12) we obtain $\phi_{[\![a_p,]\!]}(P_1, P_2) = 1-_p$. By adding up these two equalities it follows that $\phi_{[\![a_p, a_q]\!]}(P_1, P_2) =_q -_p$. Similarly one proves $\phi_{[\![a_p, a_q]\!]}(P_1, P_2) =_p -_q$. Hence, $_p =_q$ and $\phi_{[\![a_p, a_q]\!]}(P_1, P_2) = 0$. Now writing for $_1, \ldots, _\ell$, we obtain by (7.9) and (7.10) that $\phi_{[\![, a_p))}(P_1, P_2) =$ and $\phi_{((a_q,]\!]}(P_1, P_2) = 1-$.

We next show that $\phi_{t(P_1)}(P_1, P_2) =$. Consider two paths π and in G from $t(P_1)$ to a_p with all alternatives in A_p and which only have $t(P_1)$ and a_p in common, and let $P'_1 \in$ with $P'_1 = \pi \cdots$. By strategy-proofness[6] (considering agent 1), it is sufficient to prove that

$$\phi_{t(P'_1)}(P'_1, P_2) = . \tag{7.13}$$

Since $\phi_{[\![,a_p))}(P_1, P_2) =$ and $\phi_{a_q}(\hat{P}_1, P_2) =$ for $\hat{P}_1 \in$ with $t(\hat{P}_1) = a_q$, by strategy-proofness we have $\phi_\pi(P'_1, P_2) =$. Suppose that there is a $v \in \pi$, $v \neq t(P'_1)$, $v \neq a_p$, such that

$$\phi_v(P'_1, P_2) > 0. \tag{7.14}$$

Consider $P'_2 \in$ with $t(P'_2) \in A_q$ and with $P'_2 = \cdots x \cdots (\pi^{-1} \setminus \{a_p, t(P_1)\})(\setminus\{a_p\}) \cdots$ for all $x \in [\![a_p,]\!]$. (Hence, P'_2 orders all alternatives 'to the right' of a_p before a_p, then the alternatives on path π in reverse order, next the alternatives on path up to but not including a_p, and finally all remaining alternatives.) By (7.14) and strategy-proofness,

$$\phi_{[v,a_p)}(P'_1, P'_2) > 0 \tag{7.15}$$

where $[v, a_p)$ denotes the part of path π from v up to but excluding the end point a_p. Next consider $P''_1 \in$ with $P''_1 = \cdots$. Then by strategy-proofness $\phi_{(}(P''_1, P'_2) =$ (otherwise agent 1 manipulates), which again by strategy-proofness implies $\phi_{t(P'_1)}(P''_1, P'_2) =$ (otherwise agent 2 manipulates). In turn, by strategy-proofness this implies $\phi_{t(P'_1)}(P'_1, P'_2) =$ (otherwise agent 1 manipulates), which contradicts (7.15). Consequently, (7.14) does not hold, which implies (7.13).

Similarly, one proves that $\phi_{t(P_2)}(P_1, P_2) = 1-$.

(b) Second, all paths in G from a_p to a_q have a common initial part which is either (i) only a_p or (ii) $[a_p, b_1, \ldots, b_m]$ for some $m \geq 1$ with $b_1, \ldots, b_{m-1} \in B$. Let now (P_1, P_2) be a preference profile with $t(P_2) \in A_q \setminus \{a_q\}$ and $t(P_1) = z$, where $z = a_p$ in case (i), or $z \in [a_p, b_1, \ldots, b_m)$ in case (ii). By strategy-proofness (considering agent 1) and part (a), we have $\phi_{[\![,a_q]\!]}(P_1, P_2) =$. By strategy-proofness (considering agent 2) and unanimity, $\phi_{[\![z,]\!]}(P_1, P_2) = 1$. Therefore, $\phi_{[\![z,a_q]\!]}(P_1, P_2) =$ and $\phi_{(\!(a_q,]\!]}(P_1, P_2) = 1-$.

Consider $P'_1 \in$ with $P'_1 = [z, a_p] \cdots x \cdots y \cdots$ for all $x \in [\![, a_p)\!)$ and all $y \in (\!(z,]\!]$. Then as before $\phi_{[z,a_q]}(P'_1, P_2) =$, which together with part (a) and strategy-proofness (considering agent 1) implies $\phi_z(P'_1, P_2) =$. In turn, by strategy-proofness (considering agent 1) this implies $\phi_{t(P_1)}(P_1, P_2) = \phi_z(P_1, P_2) =$.

Suppose $\phi_b(P_1, P_2) > 0$ for some $b \in (\!(a_q,]\!]$ with $b \neq t(P_2)$. Then consider $\tilde{P}_1 \in$ with $t(\tilde{P}_1) \in A_p \setminus \{a_p\}$ and $b\tilde{P}_1 t(P_2)$. Then agent 1 with preference $\tilde{P}_1$ manipulates via P_1, a contradiction.

[6] Or by tops-onliness, Lemma 7.2.1.

Hence, $\phi_{t(P_2)}(P_1, P_2) = 1-$.

Similarly one proves $\phi_{t(P_1)}(P_1, P_2) =$ and $\phi_{t(P_2)}(P_1, P_2) = 1-$ if $t(P_1) \in A_p \setminus \{a_p\}$ and $t(P_2) = z'$, where z' is an alternative on the common initial part of all paths from a_q to a_p, analogously as above.

(c) Finally, let (P_1, P_2) be a preference profile with $t(P_1) = z$ and $t(P_2) = z'$ with z and z' as in part (b). By unanimity and strategy-proofness, $\phi_{[z,z']}(P_1, P_2) = 1$. In order to prove that $\phi_z(P_1, P_2) =$ and $\phi_{z'}(P_1, P_2) = 1-$ it is therefore sufficient to prove that $\phi_z(P_1, P_2) \geq$. Consider $P_1' \in$ as in (b), i.e., $P_1' \in$ with $P_1' = [z, a_p] \cdots x \cdots y \cdots$ for all $x \in [\![, a_p))$ and all $y \in ((z,]\!]$. By strategy-proofness (considering agent 1) and part (b), $\phi_z(P_1', P_2) \geq$. By strategy-proofness this implies $\phi_z(P_1, P_2) \geq$, as was to be proved. ■

Theorem 5 in Chatterji et al (2014) states that if, for $n = 2$, every unanimous and strategy-proof PR on a domain satisfying 'Condition α' is random dictatorial, then the same is true for $n > 2$. This Condition α requires that there are distinct alternatives $a, b, c \in A$ and preferences P_1, P_2, and P_3, such that (i) $P_1 = a \cdots b \cdots c \cdots$, $P_2 = b \cdots c \cdots a \cdots$, and $P_3 = c \cdots a \cdots b \cdots$, and (ii) for every $x \in A \setminus \{a, b, c\}$, either bP_1x or cP_2x or aP_3x. It is not hard to verify that Condition α holds if G does not have a leaf.[7] Hence, by Lemma 7.4.3, we have the following result.

Lemma 7.4.4 *Let $\phi :^N \to \Delta(A)$ be a unanimous and strategy-proof PR, and let the graph G have no leaf. Then ϕ is random dictatorial.*

If G has a leaf, then a unanimous and strategy-proof PR is not necessarily random dictatorial, as the following lemma shows.

Lemma 7.4.5 *Let G have a leaf. Then there exists a unanimous and strategy-proof PR which is not random dictatorial.*

Proof: Let $x \in A$ be a leaf and let $y \in A$ with $\{x, y\} \in E$. Let $_1, \ldots, _n \in [0, 1]$ with $\sum_{i \in N} {}_i = 1$. For every $P_N \in^N$ such that $t(P_i) \neq x$ for some $i \in N$ and every $a \in A \setminus \{x, y\}$ define $\phi_a(P_N) = \sum_{i \in N : t(P_i) = a} {}_i$, and define $\phi_y(P_N) = \sum_{i \in N : t(P_i) \in \{x,y\}} {}_i$. For every $P_N \in^N$ such that $t(P_i) = x$ for every $i \in N$ define $\phi_x(P_N) = 1$. Clearly, ϕ is not random dictatorial, and it is straightforward to verify that it is unanimous and strategy-proof.[8] ■

In fact, in the next section, for general connected graphs, all unanimous and strategy-proof PRs are characterized. For now, combining Lemmas 7.4.4 and 7.4.5, we obtain the main result of this section.

[7] If G does not have a leaf, it has a cycle. Take three adjacent alternatives a, b, c on this cycle and take a spanning tree $T = (A, E_T)$ with $\{a, b\}, \{b, c\} \in E_T$. Take preferences $P_1 = abc \cdots$ and $P_2 = bca \cdots$. Take another spanning tree including a path from c to a that does not contain b, and take a preference $P_3 = c \cdots a \cdots b \cdots$. Then a, b, c and $P_1, P_2, P_3 \in^N$ satisfy Condition α.

[8] As to strategy-proofness, an agent with peak unequal to x clearly cannot manipulate. An agent with peak x has y as second best alternative and therefore again cannot manipulate.

Theorem 7.4.2 *Let G be a connected graph. Then every unanimous and strategy-proof PR $\phi :^N \to \Delta A$ is random dictatorial if and only if G has no leaf.*

7.5 General connected graphs

Throughout the section, $G = (A, E)$ is an arbitrary connected graph. Let $\bar{G} = (\bar{A}, \bar{E})$ denote the maximal subgraph[9] of G that has no leaf.[10] Observe that $\bar{G}$ is unique, and $\bar{G} = \emptyset$ (i.e., $\bar{A} = \bar{E} = \emptyset$) if and only if G is a tree.

Let l be a leaf of G. Let $a \in A$ be such that there is a path from l to a that either does not intersect $\bar{A}$ or intersects $\bar{A}$ at exactly one point. The collection of all such alternatives a is defined as $A(l)$. Formally, for each leaf $l \in L(G)$, the set of alternatives $A(l) \subseteq A$ is defined as

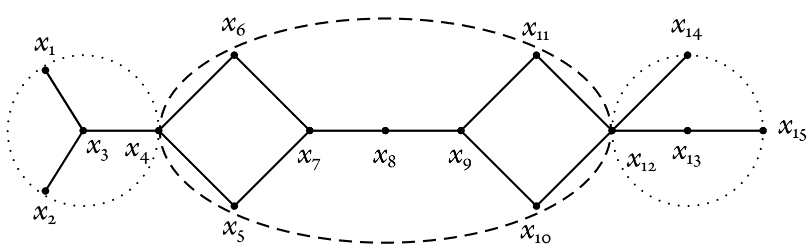
$$A(l) = \{l\} \cup \{a \in A : \text{ there is a path } [a, l] \text{ such that } |[a, l] \cap \bar{A}| \leq 1\}.$$

Observe that $A(l)$ has a unique alternative in common with $\bar{A}$, which we denote by $a(l)$. We also denote $\bar{A}^\circ = \bar{A} \setminus \{a(l) : l \in L(G)\}$. Thus, $\bar{A}^\circ$ together with the sets $A(l)$ for $l \in L(G)$ form a partition of A. We denote the set of edges containing the alternatives in $A(l)$ by $E(l)$, i.e.,

$$E(l) = \{\{a, b\} \in E : a, b \in A(l)\}.$$

The subgraph $(A(l), E(l))$ is called the *branch* of l.

Example 7.5.1 Consider the following graph:



This graph has two branches (within the dotted circles), and the maximal leafless subgraph is the middle part (within the dashed oval). Here, $\bar{A} = A \setminus \{x_1, x_2, x_3, x_{13}, x_{14}, x_{15}\}$, $\bar{A}^\circ = \bar{A} \setminus \{x_4, x_{12}\}$, $A(x_1) = A(x_2) = \{x_1, x_2, x_3, x_4\}$, $A(x_{14}) = A(x_{15}) = \{x_{12}, x_{13}, x_{14}, x_{15}\}$, $a(x_1) = a(x_2) = x_4$, and $a(x_{14}) = a(x_{15}) = x_{12}$. ◁

[9] I.e., $\bar{E} \subseteq E$ and $\bar{A} = \{a \in A : \{a, b\} \in \bar{E}$ for some $b \in A\}$.

[10] This maximal leafless subgraph can be obtained by removing the leafs (and the edges containing these) of G step by step as follows. First, remove the leafs $L(G)$ (and the edges containing these leafs) of G. Let the graph obtained after this be $G \setminus L(G)$. Then, remove all the leafs (if any) of the graph $G \setminus L(G)$. Continue until the remaining graph does not have any leaf.

In this section we characterize all unanimous and strategy-proof PRs. We start with the following auxiliary lemma.

Lemma 7.5.1 *Let $i \in N$, $P_i \in$, $P_{-i} \in^{N\setminus\{i\}}$, and $x, y \in A$ be such that $\{x, y\} \in E$ and $t(P_i) = x$. Let $P_i' = yx \cdots \in$ be such that $aP_i'b \Leftrightarrow aP_ib$ for all $a, b \in A \setminus \{x, y\}$. Let ϕ be a unanimous and strategy-proof PR. Then $\phi_a(P_i, P_{-i}) = \phi_a(P_i', P_{-i})$ for all $a \notin U(y, P_i)$.*

Proof: Write $P_i = xb_1 \cdots b_k ya_1 \cdots a_\ell$, then $P_i' = yxb_1 \cdots b_k a_1 \cdots a_\ell$. By strategy-proofness, $\phi_{U(a_{\ell-1},P_i)}(P_i, P_{-i}) \geq \phi_{U(a_{\ell-1},P_i)}(P_i', P_{-i})$ and $\phi_{U(a_{\ell-1},P_i')}(P_i', P_{-i}) \geq \phi_{U(a_{\ell-1},P_i')}(P_i, P_{-i})$, hence $\phi_{a_\ell}(P_i, P_{-i}) = \phi_{a_\ell}(P_i', P_{-i})$. Repeating this argument we obtain $\phi_{a_j}(P_i, P_{-i}) = \phi_{a_j}(P_i', P_{-i})$ for all $j = 1, \ldots, \ell$. ■

The following lemma shows that a unanimous and strategy-proof PR ϕ is a random dictatorship when restricted to profiles with all peaks in $\bar{A}$.

Lemma 7.5.2 *Let ϕ be a unanimous and strategy-proof PR. Then there exist $_1, \ldots, _n \geq 0$ with $\sum_{i=1}^n {}_i = 1$ such that $\phi_a(P_N) = \sum_{i \in N: t(P_i) = a} {}_i$ for all $a \in \bar{A}$ and all $P_N \in^N$ with $t(P_i) \in \bar{A}$ for all $i \in N$.*

Proof: Let $P_N \in^N$ with $t(P_i) \in \bar{A}$ for all $i \in N$. Suppose that $\phi_{A(l)\setminus\{a(l)\}}(P_N) > 0$ for some $l \in L(G)$. Consider $i \in N$ and let T be a spanning tree of G such that P_i is single-peaked with respect to T. Let $x = t(P_i)$ and suppose that $x \neq a(l)$. Take $y \in \bar{A}$ such that $\{x, y\}$ is an edge of T and y is on the path from x to $a(l)$ in T. Let P_i' be derived from P_i as in Lemma 7.5.1, i.e., $P_i' = yx \cdots a(l) \cdots$, $P_i = x \cdots y \cdots a(l) \cdots$, and P_i and P_i' order all alternatives different from x and y equally. Then Lemma 7.5.1 implies that $\phi_a(P_i', P_{-i}) = \phi_a(P_i, P_{-i})$ in particular for all $a \in A(l)$. By repeatedly applying this argument for player i and for all other players we arrive at a preference profile P_N with $t(P_j) = a(l)$ for every $j \in N$ and still $\phi_{A(l)\setminus\{a(l)\}}(P_N) > 0$, which contradicts unanimity of ϕ. Hence, $\phi_{\bar{A}}(P_N) = 1$.

Next, for all $a(l) \in \bar{A}$, let P^l be a single-peaked preference on $A(l)$ with graph (tree) $(A(l), E(l))$ and peak $a(l)$. For any single-peaked preference $\bar{P}$ on $(\bar{A}, \bar{E})$, construct the single-peaked preference $\bar{P}^e$ on G by substituting, in $\bar{P}$, each $a(l)$ by P^l. Now define the PR $\bar{\phi}$ on $(\bar{A}, \bar{E})$ by

$$\bar{\phi}(\bar{P}_N) = \phi(\bar{P}_N^e) \tag{7.16}$$

for each $\bar{P}_N$ on $\bar{A}$ which is single-peaked with respect to $(\bar{A}, \bar{E})$. By the first part of the proof, $\bar{\phi}$ is well-defined, i.e., $\bar{\phi}_{\bar{A}}(\bar{P}_N) = 1$ for all $\bar{P}_N$. Also, it inherits unanimity and strategy-proofness from ϕ. By Theorem 7.4.2 it follows that there are $_1, \ldots, _n \geq 0$ with $\sum_{i=1}^n {}_i = 1$ such that $\bar{\phi}_a(\bar{P}_N) = \sum_{i \in N: t(\bar{P}_i) = a} {}_i$ for all $a \in \bar{A}$ and each $\bar{P}_N$ consisting of preferences that are single-peaked with respect to $(\bar{A}, \bar{E})$. By (7.16), the proof of the lemma is complete by observing that, due to tops-onliness (Lemma 7.2.1), $\bar{\phi}$ does not depend on the particular extension $\bar{P}^e$ of $\bar{P}$. ■

Our next lemma extends the previous one by additionally including the branches of G.

Lemma 7.5.3 *Let ϕ be a unanimous and strategy-proof PR. Then there exist ${}_1, \ldots, {}_n \geq 0$ with $\sum_{i=1}^{n} {}_i = 1$ such that for all $a \in \bar{A}^\circ$ and all $l \in L(G)$*

$$\phi_a(P_N) = \sum_{i \in N : t(P_i) = a} {}_i$$

and

$$\phi_{A(l)}(P_N) = \sum_{i \in N : t(P_i) \in A(l)} {}_i$$

for every $P_N \in{}_N$.

Proof: Let $P_N \in{}^N$ and suppose that $i \in N$ and $t(P_i) = x \in A(l) \setminus \{a(l)\}$ for some $l \in L(G)$. Consider P_i' with $t(P_i') = y$ such that $\{x, y\} \in E$ and y is on the path from x to $a(l)$, as in Lemma 7.5.1. By this lemma, we obtain that $\phi_a(P_i', P_{-i}) = \phi_a(P_N)$ for all $a \notin A(l) \setminus \{a(l)\}$. The proof is complete by repeating this argument for agent i and all other agents, and next applying Lemma 7.5.2. ■

We now fix a spanning tree $T = (A, E_T)$ of the graph $G = (A, E)$. Clearly, $L(G) \subseteq L(T)$, i.e., each leaf of G is still a leaf of T. For $l \in L(T) \setminus L(G)$ define $A(l) = \{l\}$. The set of preferences on A that are single-peaked with respect to T is denoted by ${}_T$. Let denote the set of leaf assignments with respect to T (cf. Section 7.3).

The next lemma says that the restriction of a unanimous and strategy-proof PR ϕ to profiles that are single-peaked with respect to T, can be written as a leaf-peak rule ϕ^B (cf. Section 7.4), where the monotonic collection of probability distributions $B = ({}_\mu)_{\mu \in}$ associated with T satisfies the following condition: there are non-negative weights $\alpha_1, \ldots, \alpha_n$ of the agents summing to 1 such that for all $l \in L(G)$ and all $\mu \in$, the total probability of the alternatives in $A(l)$ according to β_μ is the total weight of the agents who are assigned to l by μ.

Lemma 7.5.4 *Let $\phi :{}^N \rightarrow \Delta A$ be a unanimous and strategy-proof PR, and let ϕ denote the restriction of ϕ to ${}^N_T$. Then there are ${}_1, \ldots, {}_n \geq 0$ with $\sum_{i=1}^{n} {}_i = 1$ and a monotonic collection of probability distributions $B = ({}_\mu)_{\mu \in}$ with*

$${}_\mu(A(l)) = \sum_{i \in N : \mu(i) \in A(l)} {}_i \text{ for every } l \in L(T) \text{ and } \mu \in \qquad (7.17)$$

such that $\phi = \phi^B$.

Proof: Let the numbers $_1, \ldots, _n$ be as in Lemma 7.5.2. Clearly, ϕ defined on ^N_T is unanimous and strategy-proof, and thus by Lemma 7.3.5 there is a monotonic collection of probability distributions $B = (_\mu)_{\mu \in}$ such that $\phi = \phi^B$. We are left to show (7.17). Let $\mu \in$ and $P_N \in^N_T$ be such that $t(P_i) = \mu(i)$ for every $i \in N$.

(i) First consider $l \in L(G)$, and consider $\hat{\mu} \in$ such that $\hat{\mu}(i) = \mu(i)$ for all $i \in N$ with $\mu(i) \neq l$, and with $\hat{\mu}(i) \neq l$ for all $i \in N$ with $\mu(i) = l$. Then $\mu, \hat{\mu} \in (l, P_N)$ and by (7.3) we obtain

$$\phi^B_l(P_N) =_\mu (\{l\}) -_{\hat{\mu}} (\emptyset) =_\mu (\{l\}). \tag{7.18}$$

Again by (7.3), for $a \in A(l) \setminus L(G)$,

$$\phi^B_a(P_N) =_\mu ([a, l]) -_{\hat{\mu}} ((a, l]) =_\mu (\{a\}), \tag{7.19}$$

where $[a, l]$ and $(a, l]$ are paths in T. By (7.18) and (7.19) we obtain for each $l \in L(G)$

$$_\mu(A(l)) = \sum_{l' \in A(l) \cap L(G)} {}_\mu(\{l'\}) + \sum_{a \in A(l) \setminus L(G)} {}_\mu(\{a\}) = \phi^B_{A(l)}(P_N), \tag{7.20}$$

hence by the definition of $\phi = \phi^B$ and Lemma 7.5.3

$$_\mu(A(l)) = \phi_{A(l)}(P_N) = \sum_{i \in N : \mu(i) \in A(l)} {}_i. \tag{7.21}$$

(ii) Second consider $l \in L(T) \setminus L(G)$. In a similar way as in (i), we obtain $_\mu(A(l)) =_\mu (\{l\}) = \phi^B_l(P_N)$, which by Lemma 7.5.3 implies

$$_\mu(A(l)) = \sum_{i \in N : \mu(i) = l} {}_i. \tag{7.22}$$

Now (7.17) follows from (7.21) and (7.22). ■

We can now state and prove the main and most general result of this paper. It characterizes all unanimous and strategy-proof PRs on ^{N}.

Theorem 7.5.2 *Let $G = (A, E)$ be a connected graph and let T be a spanning tree of G. A PR $\phi :^N \to \Delta A$ is unanimous and strategy-proof if and only if there are $_1, \ldots, _n \geq 0$ with $\sum_{i=1}^n {}_i = 1$ and a monotonic collection of probability distributions $B = (_\mu)_{\mu \in}$ with*

$$_\mu(A(l)) = \sum_{i \in N : \mu(i) \in A(l)} {}_i \text{ for every } l \in L(T) \tag{7.23}$$

such that $\phi(P_N) = \phi^B(P_N)$ *for all tops-equivalent* $P_N \in^N$ *and* $P_N \in^N_T$.

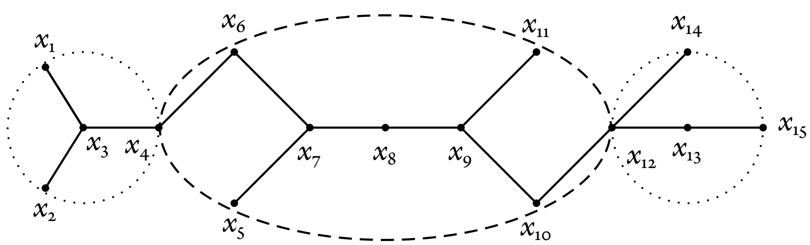
Proof: The only-if direction follows from Lemmas 7.5.4 and 7.2.1. For the if direction, with $(_i)_{i \in N}$ and B as in the statement of the theorem, define the PR ϕ on ^{N} by $\phi(P_N) = \phi^B(P_N)$ for every $P_N \in^N$, where $P_N \in^N_T$ is arbitrary but tops-equivalent to P_N. Clearly, since ϕ^B is tops-only by Lemma 7.2.1 and Theorem 7.3.9, ϕ is well-defined. It is straightforward to check that ϕ is unanimous and strategy-proof. ■

Theorem 7.5.2 indeed generalizes Theorems 7.3.9 and 7.4.2, as we show in the following remark.

Remark 7.5.3 (i) If G is a tree, then $T = G$ and $A(l) = A$ for all $l \in L(G)$. In this case one can take $_1, \ldots, _n$ arbitrary and (7.23) is trivially satisfied. Thus, Theorem 7.5.2 reduces to Theorem 7.3.9. (ii) If G has no leaf, then $A(l) = \{l\}$ for every $l \in L(T)$. Now (7.23) and the definition of ϕ^B imply that ϕ is a random dictatorship with weights $_1, \ldots, _n$. Thus, Theorem 7.5.2 reduces to Theorem 7.4.2.

We conclude the section with a few examples illustrating Theorem 7.5.2.

Example 7.5.4 Consider the graph in Example 7.5.1. We take an arbitrary spanning tree (leaving out the edges $\{x_4, x_5\}$ and $\{x_{11}, x_{12}\}$):



Now every unanimous and strategy-proof probabilistic rule is of the form ϕ^B, where $B = (_\mu)_{\mu \in}$ is a monotonic collection of probability distributions for this spanning tree satisfying, for every $\mu \in$,

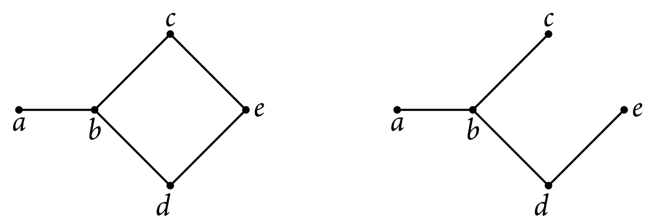
$$_\mu(x) = \sum_{i \in N : \mu(i) = x} {}_i \text{ for } x \in \{x_5, x_{11}\}$$

and

$$_\mu(\{x_1, \ldots, x_4\}) = \sum_{i \in N : \mu(i) \in \{x_1, x_2\}} {}_i \text{ and } _\mu(\{x_{12}, \ldots, x_{15}\}) = \sum_{i \in N : \mu(i) \in \{x_{14}, x_{15}\}} {}_i$$

for weights $_1, \ldots, _n$. ◁

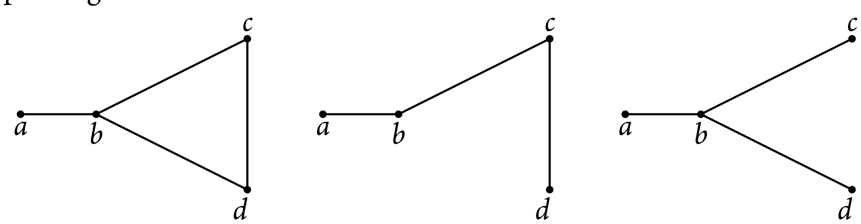
Example 7.5.5 Consider the following graph and (on the right) a spanning tree:



Let $N = \{1, 2, 3\}$, $_1 =_2=_3= \frac{1}{3}$, and let each $_\mu$ assign equal probabilities to a and b if the number of agents assigned to a is below 3. Then, for instance, if $P_N \in^N$ satisfies $t(P_1) = a$, $t(P_2) = c$, and $t(P_3) = d$, then ϕ^B assigns $(\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}, 0)$ to (a, b, c, d, e). ◁

We finally reconsider the example given in the Introduction.

Example 7.5.6 As in the previous example, let $N = \{1, 2, 3\}$, $_1 =_2=_3= \frac{1}{3}$, and let each $_\mu$ assign equal probabilities to a and b if the number of agents assigned to a is below 3. Consider the following graph and two possible spanning trees:



For the left spanning tree, let each $_\mu$ be defined by $_\mu(a) =_\mu (b) = \frac{1}{2} \sum_{i \in N:\mu(i)=a} {}_i$ and $_\mu(d) = \sum_{i \in N:\mu(i)=d} {}_i$.

For the right spanning tree, let each $_\mu$ be defined by $_\mu(a) =_\mu (b) = \frac{1}{2} \sum_{i \in N:\mu(i)=a} {}_i$, $_\mu(c) = \sum_{i \in N:\mu(i)=c} {}_i$, and $_\mu(d) = \sum_{i \in N:\mu(i)=d} {}_i$.

It is straightforward to verify that both choices result in the probabilistic rule described in the Introduction. ◁

7.6 Concluding remarks

The main result in this paper (Theorem 7.5.2) characterizes all unanimous and strategy-proof probabilistic rules for single-peaked preference profiles on a connected but otherwise arbitrary graph of which the nodes are the alternatives. Such a rule is a random dictatorship on the maximal leafless subgraph, and on each branch it is a leaf-peak rule – extending the median-like rules in [72] and the probabilistic rules in [46] on the line graph – such that the total probability on each branch equals the sum of the random dictatorship weights of the agents who have their peaks on this branch.

We conclude with, first, a consideration of probabilistic versus deterministic rules and, second, a few reflections on our domain of single-peaked preferences.

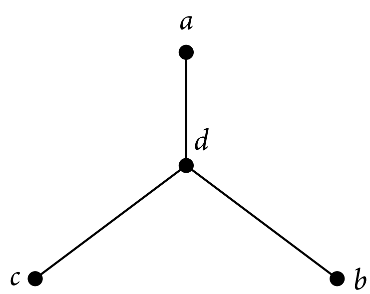
7.6.1 Probabilistic and deterministic rules

Contrary to the line graph case [81] not every probabilistic rule is a convex combination of deterministic rules, as we will show now.

Let $G = (A, E)$ be a tree. The collection of leaf-peak rules characterized in Theorem 7.3.9 contains deterministic rules, i.e., rules that assign probability one to some alternative. It is not difficult to verify that these deterministic rules correspond to monotonic collections $B = (_{\mu})_{\mu \in}$ which are deterministic, that is, for every $\mu \in, _{\mu}(x) = 1$ for some $x \in A$.

The following example shows that, in contrast to the case where the graph is a line graph ([81]), not every leaf-peak rule can be written as a convex combination of deterministic leaf-peak rules.

Example 7.6.1 Let $N = \{1, 2, 3\}$ and $A = \{a, b, c, d\}$, and let $G = (A, E)$ be the tree below. We consider the anonymous leaf-peak rule with monotonic collection of leaf assignments as in the following table, in which $_{(j,k,l)}$ denotes the probabilities assigned by the leaf assignment where j agents are assigned to a, k agents to b, and l agents to c.



	a	b	c	d
(1,1,1)	.5	.3	.2	0
(2,1,0)	.7	.3	0	0
(1,2,0)	.5	.4	0	.1
(2,0,1)	.7	0	.2	.1
(1,0,2)	.5	0	.3	.2
(0,2,1)	0	.4	.2	.4
(0,1,2)	0	.3	.3	.4

Additionally, $_{(3,0,0)}$, $_{(0,3,0)}$, and $_{(0,0,3)}$ assign probability 1 to a, b, and c, respectively. The associated PR is denoted by ψ, and we will show that ψ cannot be written as a convex combination of unanimous and strategy-proof deterministic rules.

Let F be the set of all unanimous and strategy-proof deterministic rules for preference profiles that are single-peaked with respect to the given tree. Further, for an alternative x and a preference profile P_N, let $F(x, P_N)$ be the set of all deterministic rules f such that $f(P_N) = x$. By (S_1, S_2, S_3), where S_1, S_2, S_3 are disjoint with union N, we denote a preference profile where the top-alternatives of the agents in S_1, S_2, and S_3 are a, b, and c, respectively. Let $F_1 = F(a, (\{1, 2\}, \{3\}, \emptyset))$, $F_2 = F(b, (\{1, 3\}, \{2\}, \emptyset))$, $F_3 = F(c, (\{1\}, \{2\}, \{3\}))$, $F_4 = F(b, (\{1, 2\}, \{3\}, \emptyset))$, and $F_5 = F(b, (\{1\}, \{2, 3\}, \emptyset))$. Then, by Theorem 7.3.9, or more directly by uncompromisingness (Lemma 7.3.1), it follows that $F_1 \cap F_3 = \emptyset$ and $F_2 \cap F_3 = \emptyset$. Combining, we have

$$(F_1 \cup F_2) \cap F_3 = \emptyset. \tag{7.24}$$

Assume for contradiction that ψ can be written as $\sum_{f\in F} {}_f f$, where ${}_f \geq 0$ for all $f \in F$ and $\sum_{f\in F} {}_f = 1$. For $G \subseteq F$, let ${}_G = \sum_{f\in G} {}_f$. Then ${}_{F_1\cup F_2} = {}_{F_1} + {}_{F_2} - {}_{F_1\cap F_2}$ together with (7.24), yields ${}_{F_1} + {}_{F_2} - {}_{F_1\cap F_2} + {}_{F_3} \leq 1$. Since $\psi = \sum_{f\in F} {}_f f$, we have ${}_{F_1} = \psi_a(\{1,2\},\{3\},\emptyset)$, ${}_{F_2} = \psi_b(\{1,3\},\{2\},\emptyset)$, ${}_{F_3} = \psi_c(\{1\},\{2\},\{3\})$. Using the values given in the table we obtain

$$ {}_{F_1\cap F_2} \geq 0.2. \tag{7.25}$$

Since the rules in F_1 and F_4 give different outcomes (a and b, respectively) at the same preference profile $(\{1,2\},\{3\},\emptyset)$, we have $F_1 \cap F_4 = \emptyset$. Moreover, by uncompromisingness, $F_2 \subseteq F_5$ and $F_4 \subseteq F_5$, and hence $F_2 \cup F_4 \subseteq F_5$. Because $F_1 \cap F_4 = \emptyset$, we have

$$(F_1 \cap F_2) \cap F_4 = \emptyset. \tag{7.26}$$

Also, because $F_2 \cup F_4 \subseteq F_5$,

$$(F_1 \cap F_2) \cup F_4 \subseteq F_5. \tag{7.27}$$

Combining (7.26) and (7.27), we have ${}_{F_1\cap F_2} + {}_{F_4} \leq {}_{F_5}$. By (7.25) and the table, ${}_{F_1\cap F_2} + {}_{F_4} \geq 0.5$, and hence ${}_{F_5} \geq 0.5$. However, from the table it follows that ${}_{F_5} = 0.4$. This is a contradiction. Thus, ψ cannot be written as a convex combination of deterministic rules. $\lhd$

7.6.2 The domain

An earlier version of the paper ([80]) shows that at least the results in the case where the graph is a tree can be derived for a smaller set of single-peaked preferences.

In the opposite direction, enlarging the set of allowed single-peaked preferences, one could weaken the single-peakedness requirement by demanding that an alternative x is preferred to an alternative y if x is on *every* path from the peak of the preference to y. Then, logically, the collection of all unanimous and probabilistic rules must be a subset of the collection characterized in Theorem 7.5.2, but it can actually be shown that the two are equal.

Finally, if we would require that all preferences are single-peaked with respect to one fixed spanning tree, then our domain would satisfy the 'generalized single-peakedness' condition in [75], who consider deterministic rules. Since we allow that preferences are single-peaked with respect to different spanning trees, our domain for general connected graphs is larger.

Appendix

7.7 Proof of Lemma 7.2.1

The proof of Lemma 7.2.1 will be based on Theorem 1 in [31]. We need to introduce two concepts used there, namely the Interior Property and the Exterior Property.

We say that preferences P, P' are *adjacent* if there are distinct $x, y \in A$ with xPy, $yP'x$, $aPb \Leftrightarrow aP'b$ for all $a, b \in A$ with $\{a, b\} \neq \{x, y\}$, and $xPzPy$, $yP'zP'y$ for no $z \neq x, y$. A set of preferences has the *Interior Property* if for all $a \in A$ and all $P, P' \in$ with $t(P) = t(P') = a$ there are $P^1, \dots, P^k \in$ with $k \geq 1$ and $t(P^j) = a$ for every $j = 1, \dots, k$ such that $P = P^1$, $P' = P^k$, and for each $j = 1, \dots, k-1$ the preferences P^j, P^{j+1} are adjacent.

Lemma 7.7.1 *Let $G = (A, E)$ be a connected graph. Then has the Interior Property.*

Proof: Let $1 \leq k \leq |A| - 2$ and let $a_1, \dots, a_k, a_{k+1}$ be distinct alternatives. Consider a preference P, single-peaked with respect to a spanning tree T of G, such that $t(P) = a_1$ and $a_k P x P a_{k+1}$ such that $xPzPa_{k+1}$ for no $z \neq x, a_{k+1}$; and a preference P' single-peaked with respect to a spanning tree T', such that $t(P') = a_1$ and $a_k P a_{k+1}$ such that $a_k P z P a_{k+1}$ for no $z \neq a_k, a_{k+1}$. (Thus, $a_1, \dots, a_k$ are ranked above all other alternatives at P, and $a_1, \dots, a_{k+1}$ are ranked above all other alternatives at P'.) It is sufficient to prove that there is a spanning tree T with respect to which the preference P obtained by switching x and a_{k+1} in P, is single-peaked. If x is not on the path $\pi = [a_1, a_{k+1}]$ in T, then we can simply take $T = T$. Otherwise, we have $\pi = [a_1, \dots, x, a_{k+1}]$. Let $\pi' = [a_1, \dots, a_\ell, a_{k+1}]$ be the path in T' from a_1 to a_{k+1}; observe that the alternatives in π' are a subset of $\{a_1, \dots, a_{k+1}\}$. Construct T from T as follows. First, delete the edge $\{x, a_{k+1}\}$ from T. This results in two disconnected subtrees with $a_1, \dots, a_k$ and x in one subtree and a_{k+1} in the other: this follows from single-peakedness of P with respect to T (if a_i for some $2 \leq i \leq k$ would be in the same subtree as a_{k+1}, then a_{k+1} would be on the path in T from a_1 to a_i and thus $a_{k+1} P a_i$ by single-peakedness, a contradiction). Therefore, by adding the edge $\{a_\ell, a_{k+1}\}$ we obtain a spanning tree T. The proof of the lemma is complete if we show that P is single-peaked with respect to T.

Suppose this were not the case. Then there are distinct $z, z' \in A$ such that z is on the path $\pi = [a, z']$ in T, but $z'Pz$. If π is also a path in T, then we have zPz', hence $z = x$ and $z' = a_{k+1}$, and in particular $\{x, a_{k+1}\}$ is an edge in T, which is a contradiction. Hence, π is not a path in T, and we can write $\pi = [a_1, a_\ell] \cdot \{a_\ell, a_{k+1}\} \cdot [a_{k+1}, z']$, where $[a_1, a_\ell]$ and $[a_{k+1}, z']$ are also paths in T. If $z \in [a_{k+1}, z']$ then z is on the path $[a_1, x] \cdot \{x, a_{k+1}\} \cdot [a_{k+1}, z']$ in T, hence zPz' and therefore zPz', a contradiction. Therefore, we have that z is on the path $[a_1, a_\ell]$ in T and T, thus $z \in \{a_1, \dots, a_k\}$, and again zPz', a contradiction. ■

For a preference P and a number $\ell \in \{1, \dots, |A|\}$, let $B_\ell(P) \subseteq A$ denote the set of the ℓ highest ranked alternatives according to P, i.e., if $a_1 P a_2 \dots a_\ell P a_{\ell+1} P a_{\ell+2} \dots P a_{|A|}$ then $B_\ell(P) = \{a_1, \dots, a_\ell\}$. A set of preferences has the *Exterior Property* if for all $P, P' \in$ with $t(P) \neq t(P')$ and all distinct $x, y \in A$ with xPy and $xP'y$, there are $P^1, \ldots, P^k \in$, $k \geq 2$, such that $P = P^1$, $P' = P^k$, and for every $j = 1, \ldots, k-1$ there is an $\ell \in \{1, \ldots, |A|\}$ such that $x \in B_\ell(P^j) = B_\ell(P^{j+1})$ and $y \notin B_\ell(P^j)$.

Lemma 7.7.2 *Let $G = (A, E)$ be a connected graph. Then has the Exterior Property.*

Proof: Let $P, P' \in$ with $t(P) = a \neq b = t(P')$ and distinct $x, y \in A$ with xPy and $xP'y$. Let T be a spanning tree of G with respect to which P is single-peaked.

(i) First suppose that bPy. Let the path $[a, b]$ in T consist of the sequence $a, z_1, \ldots, z_k, b$, hence $aPz_1P \ldots Pz_kPb$. Define P'' by $bP''z_kP'' \ldots P''z_1P''a \ldots$ such that $zP''z' \Leftrightarrow zPz'$ for all $z, z' \in A \setminus \{a, z_1, \ldots, z_k, b\}$, and let $\ell = \max\{|U(b, P)|, |U(x, P)|\}$.

We show that P'' is single-peaked with respect to T. To this end, let $[b \cdots z \cdots z']$ be a path in T. We show that $zP''z'$. If $z, z' \in \{a, z_1, \ldots, z_k, b\}$, say $z = z_i$ and $z' = z_j$, then we have $i > j$ and $z_iP''z_j$, hence $zP''z'$. If $z \in \{a, z_1, \ldots, z_k, b\}$ and $z' \notin \{a, z_1, \ldots, z_k, b\}$ then $zP''z'$. If $z \notin \{a, z_1, \ldots, z_k, b\}$ and $z' \in \{a, z_1, \ldots, z_k, b\}$ then $[b \cdots z \cdots z'] \cdot [z' \cdots b]$ contains a cycle, a contradiction. If, finally, $z, z' \notin \{a, z_1, \ldots, z_k, b\}$ then there is a path $[a \cdots z \cdots z']$ in T, hence zPz' and therefore $zP''z'$. This completes the proof of single-peakedness of P'' with respect to T.

Also, $t(P'') = b$, $x \in B_\ell(P) = B_\ell(P'')$, and $y \notin B_\ell(P)$. The proof for this case is then complete by constructing a sequence of adjacent preferences starting from P'' and ending in P' by using the Interior Property (Lemma 7.7.1).

(ii) Second suppose that yPb and y is not on the path $[a, b]$ in T. Construct the preference P as follows. Let $C = \{z \in A : \ y \text{ is on the path } [a, z] \text{ in } T\}$. Then let $z'Pz$ for all $z \in C$ and $z' \in A \setminus C$, and $zPz' \Leftrightarrow zPz'$ for all $z, z' \in C$ and all $z, z' \in A \setminus C$. Then P is still single-peaked with respect to T, and for $\ell = |U(x, P)|$ we have $x \in B_\ell(P) = B_\ell(P)$ and $y \notin B_\ell(P)$. Since $b \notin C$ and therefore bPy we can complete the proof by applying the arguments in (i) now starting from P.

(iii) Third suppose that yPb and y is on the path $[a, b]$ in T. Let the path $[a, b]$ in T be the sequence $a, \ldots, a', y, \ldots, b$. Let, similarly as above, $C = \{z \in A : y \text{ is on the path } [a, z] \text{ in } T\}$. Since $P' \in$ there is a path $\pi = [b, x]$ in G with $y \notin [b, x]$. On this path let $\{c, d\}$ be the first edge with $c \in C$ and $d \in A \setminus C$. Now first delete the edge $\{a', y\}$ from T; next add the part $\pi' = [b \cdots cd]$ of π; and finally delete edges $\{v, w\}$ with $v, w \in C$ but $\{v, w\}$ not in π' such that a spanning tree $\bar{T}$ of G is obtained. Next construct a preference $\bar{P}$, single-peaked with respect to $\bar{T}$, with $z\bar{P}z'$ for all $z \in A \setminus C$ and $z \in C$, $z\bar{P}z' \Leftrightarrow zPz'$ for all $z, z' \in A \setminus C$, $x \in B_p(P) = B_p(\bar{P})$, and $y \notin B_p(P)$, where $p = |U(x, P)| = |U(x, \bar{P})|$. Then either $b\bar{P}y$, and we are back in case (i), or $y\bar{P}b$. In the latter case, since the path $[a, b]$ in $\bar{T}$ is of the form $[a \cdots dc \cdots b]$ where $[dc \cdots b]$ is the converse path of π', y $(\in C)$ is not on this path, and we are back in case (ii). ∎

Lemma 7.2.1 now follows by applying Theorem 1 in [31].

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8
List of Publication(s)/Submitted Article(s)

- **Published/Accepted Papers**

  (1) "An Extreme Point Characterization of Strategy-proof and Unanimous Probabilistic Rules over Binary Restricted Domains" *Journal of Mathematical Economics*, 2017, 69, 84-90.

  (2) "A characterization of random min–max domains and its applications" *Economic Theory*, 2019, 68, 887–906.

  (3) "Formation of Committees Through Random Voting Rules" In: Trockel W. (eds) *Social Design. Studies in Economic Design*, 2019, 219-231.

  (4) "Unanimous and strategy-proof probabilistic rules for single-peaked preference profiles on graphs" accepted in *Mathematics of Operations Research*.

- **Submitted Papers**

  (1) "A unified characterization of the randomized strategy-proof rules" revised and resubmitted in *Journal of Economic Theory*.

- **Completed Papers**

  (1) "Restricted Probabilistic Fixed Ballot Rules and Hybrid Domains".